Matroids and Polymatroids in Congestion Games

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Part I: Congestion Games
  - Existence of Equilibria
  - Computation of Equilibria
  - Matroids

Part II: Integral Splittable Congestion Games
  - Existence and Computation of Equilibria
  - Integral Polymatroids

Part III: Nonatomic Congestion Games
  - Efficiency of Equilibria
  - The Braess Paradox
  - Matroids are Immune to Braess Paradox
Strategic Games

- Strategic game $G = (N, X, \pi)$
- $N = \{1, \ldots, n\}$ set of players
- $X = \times_{i \in N} X_i$ set of pure strategies
- $x = (x_1, \ldots, x_N)$ strategy profile
- $\pi_i(x) : X \to \mathbb{R}, i \in N$ private cost/utility
Strategic Games

- **Strategic game** $G = (N, X, \pi)$
- $N = \{1, \ldots, n\}$ set of players
- $X = \times_{i \in N} X_i$ set of pure strategies
- $x = (x_1, \ldots, x_N)$ strategy profile
- $\pi_i(x) : X \rightarrow \mathbb{R}, i \in N$ private cost/utility
- **Mixed strategy**: for each player a probability distribution over pure strategies
Solution Concept

**Definition**

**Pure Nash equilibrium (PNE):** no player has an incentive to unilaterally deviate.

**Definition**

**Mixed Nash equilibrium (MNE):** no player has an incentive to unilaterally change her mixed strategy.

"Minimax Theorem" John von Neumann (1928), Nash (1950)
Motivation: Party Affiliation Games

- each strategy set $X_i = \{1, -1\}$
- Weight $w_{i,j}$ measures relationship between $i$ and $j$
- payoff $u_i(x) = \sum_{j \in N} x_i x_j w_{i,j} \rightarrow \max$

\[
\begin{align*}
\text{u}_1(x) &= 1 \\
\text{u}_2(x) &= 0 \\
\text{u}_3(x) &= 0 \\
\text{u}_4(x) &= 0 \\
\text{u}_5(x) &= 5
\end{align*}
\]
Definition

Pure Nash equilibrium (PNE): no player has an incentive to unilaterally change his pure strategy.

Definition

Mixed Nash equilibrium (MNE): no player has an incentive to unilaterally change his mixed strategy.
Motivation: Party Affiliation Games

- each strategy set \( X_i = \{1, -1\} \)
- Weight \( w_{i,j} \) measures relationship between \( i \) and \( j \)
- payoff \( u_i(x) = \sum_{j \in N} x_i x_j w_{i,j} \rightarrow \max \)
Nash (1951)

**Theorem**

*Every finite game possesses a mixed Nash equilibrium.*

Pure Nash Equilibrium need not exist!

Example: Assymmetric Party Affiliation Game

In the mixed Nash equilibrium, each player chooses each party with probability $1/2$. 
Part I

Congestion Games
### Congestion Games

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<th>Modell</th>
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<tbody>
<tr>
<td>( N = {1, \ldots, n} ) set of players</td>
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<td>( R = {r_1, \ldots, r_m} ) set of resources</td>
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<tr>
<td>( X = \times_{i \in N} X_i ) set of strategy profiles with ( X_i \subseteq 2^R )</td>
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<tr>
<td>Strategy profile ( x = (x_1, \ldots, x_n) \in X )</td>
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<tr>
<td>Load of a resource ( x_r =</td>
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<tr>
<td>Cost functions ( c_r : \mathbb{N} \to \mathbb{R} ) nondecreasing (convex)</td>
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<tr>
<td>private cost: ( \pi_i(x) = \sum_{r \in x_i} c_r(x_r) )</td>
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Example

\[
\begin{align*}
d_1 &= 1 \\
d_2 &= 1 \\
d_3 &= 1
\end{align*}
\]
1. When do pure Nash equilibria exist?
2. How do players find them?
3. How difficult is it to compute them?
Potential Functions

**Definition (Exact potential function)**

\[ P : X_1 \times \cdots \times X_n \rightarrow \mathbb{R} \]

If a player changes his action, the change in the potential function value is equal to the change in her payoff.

\[ u_i(x_i, x_{-i}) - u_i(y_i, x_{-i}) = P(x_i, x_{-i}) - P(y_i, x_{-i}) \]
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*b–Potential*

\[ u_i(x_i, x_{-i}) - u_i(y_i, x_{-i}) = b_i \left( P(x_i, x_{-i}) - P(y_i, x_{-i}) \right) \]

Monderer and Shapley (1996)
Theorem

Every finite exact potential game (with potential $P$)

- possesses a PNE
- every sequence of improving moves is finite (FIP)
- every local minimum of $P$ is a PNE

Monderer and Shapley (1996)
Proof

- path \( \gamma = (x^0, x^1, \ldots, ) \) sequence of unilateral moves
- improvement path \( \gamma = (x^0, x^1, \ldots, ) \) sequence of unilateral improving moves

\[ \gamma = x_0, x_1, \ldots \text{ improvement path} \]

\[ \Rightarrow P(x_0) > P(x_1) > \cdots \text{ must be finite.} \]
Theorem (Rosenthal ’73)

Every congestion game

- admits an exact potential function
- possesses a PNE
- possesses the Finite Improvement Property, that is, every sequence of improving moves is finite.
Rosenthal's exact potential function $P : X_1 \times \cdots \times X_n \rightarrow \mathbb{R}$ is defined as

$$P(x) := \sum_{r \in R} \sum_{k=1}^{x_r} c_r(k).$$

(1)

Let $x \in X$ and $y_i \neq x_i$ be a unilateral deviation of $i$.

$$u_i(x_{-i}, y_i) - u_i(x) = \sum_{r \in y_i \atop r \notin x_i} c_r(x_r + 1) - \sum_{r \in x_i \atop r \notin y_i} c_r(x_r).$$

The potential of $(x_{-i}, y_i)$ is given by:

$$P(x_{-i}, y_i) = \sum_{r \in R} \sum_{k=1}^{x_r} c_r(k) + \underbrace{\sum_{r \in y_i \atop r \notin x_i} c_r(x_r + 1) - \sum_{r \in x_i \atop r \notin y_i} c_r(x_r)}_{=u_i(x_{-i}, y_i) - u_i(x)} = P(x) + u_i(x_{-i}, y_i) - u_i(x).$$
### Complexity of Computing PNE

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<th>Theorem (Fabrikant et al. ’04, Ackermann et al. ’08)</th>
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<td><em>It is PLS-complete to compute a PNE even for symmetric congestion games with affine costs.</em></td>
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Theorem (Fabrikant et al. ’04, Ackermann et al. ’08)

It is PLS-complete to compute a PNE even for symmetric congestion games with affine costs.

Theorem (Fabrikant et al. ’04)

For symmetric network congestion games, there is a polynomial time algorithm to compute a PNE.

Subdivide each arc $e$ into $n$ parallel arcs with capacity 1 each and assign costs $c_{ei} = c_e(i)$ for $i \in \{1, \ldots, n\}$. 

Remark (Ackermann et al. ’08)

There are instances on which every best response dynamic needs exponential convergence time.
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Remark (Ackermann et al. ’08)

There are instances on which every best response dynamic needs exponential convergence time.

Are there other set systems $X_i$ with efficiently comp. PNE?
A matroid is a pair \( \mathcal{M} = (R, \mathcal{I}) \) where \( R \) is a set of resources, and \( \mathcal{I} \) is a family of subsets of \( S \) such that:

1. \( \emptyset \in \mathcal{I} \).
2. If \( I \subset J \) and \( J \in \mathcal{I} \), then \( I \in \mathcal{I} \).
3. Let \( I, J \in \mathcal{I} \) and \( |I| < |J| \), then there exists an \( x \in J \setminus I \) such that \( I + x \in \mathcal{I} \).

A set system \( R, \mathcal{I} \) that only satisfies (1) and (2) is called an independence system.
Introduction Matroids

Definition (Matroid)

A matroid is a pair $\mathcal{M} = (R, \mathcal{I})$ where $R$ is a set of resources, and $\mathcal{I}$ is a family of subsets of $S$ such that:

1. $\emptyset \in \mathcal{I}$.
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A set system $R, \mathcal{I}$ that only satisfies (1) and (2) is called an independence system.

Bases are sets in $\mathcal{I}$ of maximal cardinality, denoted by $\mathcal{B}$.
The independent sets of a $k$-uniform matroid are the sets that contain at most $k$ elements.

Example

4 resources: $\{1, 2, 3, 4\}$
The independent sets of a $k$-uniform matroid are the sets that contain at most $k$ elements.

Example

4 resources: \{1, 2, 3, 4\}
Independent sets of the 3-uniform matroid:

$I = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\},
\{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}$
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**Example**

4 resources: $\{1, 2, 3, 4\}$

Independent sets of the 3-uniform matroid:

$$I = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}$$

Bases: $B = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}$
Figure: $K_4$ with two bases $B_1$ (red), $B_2$ (blue).
Theorem

Let \((R, \mathcal{I})\) be an independence system. The, the following is equivalent:

\[\begin{align*}
\text{M1} & \quad M = (R, \mathcal{I}) \text{ is a matroid.} \\
\text{M2} & \quad \text{If } I, J \in \mathcal{I} \text{ with } |I| = |J| + 1 \Rightarrow \text{there is } x \in I \setminus J \text{ with } J + x \in \mathcal{I}. \\
\text{M3} & \quad \text{For every } I \subseteq R, \text{ every basis of } I \text{ has the same cardinality (a basis of } I \subseteq E \text{ is an inclusion-wise maximal set in } I).}
\end{align*}\]

Proof.

\((M1) \iff (M2)\) is trivial. \((M1) \Rightarrow (M3)\) is trivial. \((M3) \Rightarrow (M1)\):

Let \(I, J \in \mathcal{I}\) with \(|I| > |J|\). With \((M3)\) we have that \(J\) is no base of \(I \cup J\). Hence, there is \(x \in (I \cup J) \setminus J = I \setminus J \in \mathcal{I}\) with \(J + x \in \mathcal{I}\). \(\square\)
Theorem (Basis exchange theorem)

Let \((R, \mathcal{I})\) be a matroid with basis system \(\mathcal{B}\). Then,

1. \(\mathcal{B} \neq \emptyset\)

2. For every \(B_1, B_2 \in \mathcal{B}\) and \(x \in B_1 \setminus B_2\) there is \(y \in B_2 \setminus B_1\) such that \(B_1 - x + y \in \mathcal{B}\).

Proof.

The bases set of \((R, \mathcal{I})\) satisfies (1) since \(\emptyset \in \mathcal{I}\). For condition (2) let \(B_1, B_2 \in \mathcal{B}\) and \(x \in B_1 \setminus B_2\). Since \(B_1 - x \in \mathcal{I}\) we can use (\(M2\)): \(|B_1 - x| + 1 = |B_2|\) hence there is \(y \in B_2 \setminus (B_1 - x)\) with \(B_1 - x + y \in \mathcal{I}\). As all bases have the same cardinality (see (\(M3\))) we get \(B_1 - x + y \in \mathcal{B}\). \(\square\)
Figure: Two bases $B$ (red) and $B'$ (blue) of a graphic matroid
Figure: Two bases $B$ (red) and $B'$ (blue) of a graphic matroid

$B \setminus B' \quad B' \setminus B$

Figure: Bipartite graph $G(B \triangle B') = (B \triangle B', R)$.

$$(x, y) \in R \iff B - x + y \in \mathcal{B}.$$
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$B \setminus B'$ $B' \setminus B$

Figure: Bipartite graph $G(B \Delta B') = (B \Delta B', R)$.

$$(x, y) \in R \iff B - x + y \in \mathcal{B}.$$
**Theorem (Matching Property)**

Let $\mathcal{M} = (R, I)$ be a matroid. Then, for every pair $(B, B')$ of bases of $\mathcal{M}$ there exists a perfect matching $W(B, B')$ in $G(B \triangle B')$. 
Matroid Congestion Games

- \( N = \{1, \ldots, n\} \) set of players
- \( R = \{r_1, \ldots, r_m\} \) set of resources
- \( X_i \subseteq 2^R \) with \( X_i = \mathcal{B}_i \) for \( M_i = (R, \mathcal{I}_i) \)
- Private cost for \( B = (B_1, \ldots, B_n) \in \mathcal{B} : \)

\[
\pi_i(B) = \sum_{r \in B_i} c_r(x_r)
\]
Theorem (Ackermann, Röglin, Vöcking ’08)

For matroid congestion games, the best-response dynamic converges after at most \( n^2 \cdot m \cdot \max_{i \in \mathbb{N}} r_k_i \leq n^2 m^2 \) steps.
Theorem (Ackermann, Röglin, Vöcking ’08)

For matroid congestion games, the best-response dynamic converges after at most \( n^2 \cdot m \cdot \max_{i \in N} rk_i \leq n^2 m^2 \) steps.

Let \( L \) be a list of all cost values \( c_r(i), r \in R = \{r_1, \ldots, r_m\}, i \in N = \{1, \ldots, n\} \) sorted in non-decreasing order. Define alternative cost function

\[
c'_r : \mathbb{N} \rightarrow \{1, \ldots, n \cdot m\},
\]

where \( i \in [0, n] \) is mapped to list position of \( c_r(i) \) in \( L \) (same costs are mapped to same position).
Lemma

Let $B^*_i$ be a best-response w.r.t. $B \in \mathcal{B}$ with $B^*_i \neq B_i$. Then $B^*_i$ decreases also the private costs w.r.t. $c'$. 
Lemma

Let $B_i^*$ be a best-response w.r.t. $B \in \mathcal{B}$ with $B_i^* \neq B_i$. Then $B_i^*$ decreases also the private costs w.r.t. $c'$.

Consider $G(B_i^* \Delta B_i)$ with perfect matching $N$. Let $(u, v) \in N$, i.e.,

$$v \in B_i^* \setminus B_i, \ u \in B_i \setminus B_i^* \text{ and } B_i^* - v + u \in \mathcal{B}_i.$$  

For $B^* = (B_i^*, B_{-i})$ with load vector $x^*$ we have

$$c_v(x_v^*) \leq c_u(x_u^* + 1) \quad \text{for all } (u, v) \in N,$$

as else $B_i' := B_i^* - v + u$ would give less costs than $B_i^*$. There must be $(v, u) \in N$ with

$$c_v(x_v^*) < c_u(x_u^* + 1),$$

since $B_i^*$ strictly decreases the private cost for $i$. Thus, also the private costs under $c'$ must decrease.
Consider $P'$ w.r.t. $c'$. We get

$$c'_r(i) \leq n \cdot m \text{ for all } r \in R \text{ and all } 1 \leq i \leq n.$$
Proof

Consider $P'$ w.r.t. $c'$. We get

$$c'_r(i) \leq n \cdot m \text{ for all } r \in R \text{ and all } 1 \leq i \leq n.$$ 

Hence,

$$P'(B) = \sum_{r \in R} \sum_{i=1}^{x_r} c'_r(i) \leq \sum_{r \in R} \sum_{i=1}^{x_r} n \cdot m \leq n^2 \cdot m \cdot \max_{i \in N} rk_i.$$
Proof

Consider $P'$ w.r.t. $c'$. We get

\[ c'_r(i) \leq n \cdot m \text{ for all } r \in R \text{ and all } 1 \leq i \leq n. \]

Hence,

\[ P'(B) = \sum_{r \in R} \sum_{i=1}^{x_r} c'_r(i) \leq \sum_{r \in R} \sum_{i=1}^{x_r} n \cdot m \leq n^2 \cdot m \cdot \max_{i \in N} rk_i. \]

Remark (Ackermann, Röglin, Vöcking '08)

For every non-matroid set system $X_i$, $i \in N$, there are isomorphic instances with exponentially long best-response sequences.
Part II

Integral Splittable Congestion Games
Integral Splittable Congestion Games

Integral Congestion Games

- $N = \{1, \ldots, n\}$ set of players
- $R = \{r_1, \ldots, r_m\}$ set of resources
- $X_i \subseteq 2^R$ set of allowable subsets
- Demands $d_i \in \mathbb{N}$
- Strategy corresponds to integral distribution of $d_i$ among the sets in $X_i$
- cost of a resource $c_r(x) = c_r(\sum_{i \in N} x_{i,r})$
- private cost $\pi_i(x) = \sum_{r \in R} c_r(x) x_{i,r}$
Rosenthal (1973b)

- One player from $A$ to $C$ with demand 2 ("two taxicaps")
- One player from $A$ to $B$ with demand 1

Fig. 3
Positive Results

Theorem (Tran-Thanh et al. (2011))

- If $X_i$ consists of *singletons*, then there is a PNE.
What are maximal structures of $X_i$ such that the existence of PNE is guaranteed?
What are maximal structures of $X_i$ such that the existence of PNE is guaranteed?

**Theorem (H, Klimm, Peis (2014))**

*Polymatroids are the maximal structure.*
Submodular Functions

Definition (integral, submodular, monotone, normalized)

- $f : 2^\mathbb{R} \rightarrow \mathbb{N}$ is submodular if
  
  $$f(U) + f(V) \geq f(U \cup V) + f(U \cap V)$$

  for all $U, V \in 2^\mathbb{R}$.

- $f$ is monotone, if $U \subseteq V \Rightarrow f(U) \leq f(V)$

- $f$ is normalized, if $f(\emptyset) = 0$. The rank function $rk : 2^\mathbb{R} \rightarrow \mathbb{N}$ of a matroid $M = (I, R)$ for $F \subseteq R$:
  
  $$rk(F) := \max\{|B|, B \text{ basis of } F\}$$
Submodular Functions

**Definition (integral, submodular, monotone, normalized)**

- **f : 2^R \rightarrow \mathbb{N}** is submodular if
  \[ f(U) + f(V) \geq f(U \cup V) + f(U \cap V) \] for all \( U, V \in 2^R \).
- **f is monotone**, if \( U \subseteq V \Rightarrow f(U) \leq f(V) \)
- **f is normalized**, if \( f(\emptyset) = 0 \).

**Example**
- rank function \( rk : 2^R \rightarrow \mathbb{N} \) of a matroid \( M = (\mathcal{I}, R) \)
- for \( F \subseteq R : rk(F) := \max\{|B|, B \text{ basis of } F\} \)
Integral polymatroid

\[ P_f := \left\{ x \in \mathbb{N}^R \mid \sum_{r \in U} x_r \leq f(U) \text{ for all } U \subseteq R \right\}. \]
integral polymatroid

\[ \mathcal{P}_f := \left\{ x \in \mathbb{N}^R \mid \sum_{r \in U} x_r \leq f(U) \text{ for all } U \subseteq R \right\}. \]

“truncated” integral polymatroid

\[ \mathcal{P}_f(d) := \left\{ x \in \mathbb{N}^R \mid \sum_{r \in U} x_r \leq f(U) \text{ for all } U \subseteq R, \sum_{r \in R} x_r \leq d \right\}. \]
Integral Polymatroids

integral polymatroid

$$\mathbb{P}_f := \left\{ x \in \mathbb{N}^R \mid \sum_{r \in U} x_r \leq f(U) \text{ for all } U \subseteq R \right\}.$$  

“truncated” integral polymatroid

$$\mathbb{P}_f(d) := \left\{ x \in \mathbb{N}^R \mid \sum_{r \in U} x_r \leq f(U) \text{ for all } U \subseteq R, \sum_{r \in R} x_r \leq d \right\}.$$  

integral polymatroid base polytope

$$\mathcal{B}_f(d) := \left\{ x \in \mathbb{N}^R \mid \sum_{r \in U} x_r \leq f(U) \text{ for all } U \subseteq R, \sum_{r \in R} x_r = d \right\}.$$
Congestion Games on Polymatroids

**Strategic Game**

- \( G = (N, X, \pi) \)
- \( X_i = B_{f(i)}(d_i) \)
- \( \pi_i(x) = \sum_{r \in R} c_{i,r}(x)x_{i,r} \)

\[
B_{f(i)}(d_i) = \left\{ x_i \in \mathbb{N}^R \mid \sum_{r \in U} x_{i,r} \leq f^{(i)}(U) \text{ for all } U \subseteq R, \sum_{r \in R} x_{i,r} = d_i \right\}
\]
Singletons

\[ f^{(i)}(\{r\}) = d_i \text{ falls } r \in R_i \text{ und } f^{(i)}(\{r\}) = 0, \text{ sonst.} \]

Bases of Matroids

\( f^{(i)} \) is rank function of some matroid \( \mathcal{M}_i = (R_i, \mathcal{I}_i) \) and 
\[ d_i = \text{rk}_i(R_i). \]

Spanning Trees

- \( R \) edges of a graph
- \( R_i \subset R \) connected subgraph
- Bases of \( \mathcal{M}_i = (R_i, \mathcal{I}_i) \) are spanning trees of \( R_i \)
Existence Proof

- Induction over $d = \sum_{i \in N} d_i$
- $d = 1$ trivial
- $d \rightarrow d + 1$
  - let $x$ be a PNE for a game with demand $d$
  - In step $d \rightarrow d + 1$ exactly for one player $i$: $d_i \rightarrow d_i + 1$. 
Existence Proof (contd.)

Hamming Distance of $x, y \in \mathbb{N}^{\mid R\mid}$: $H(y, x) = \sum_{r \in R} |y_r - x_r|

Sensitivity Lemma - Increased Demand

Let $x$ be given. There exists a best response $y_i \in \text{argmin} \left[ \pi_i(y_i, x_{-i}) \text{ s.t. } y_i \in B_{f(i)}(d_i + 1) \right]$ with $H(x_i, y_i) = 1.$
Hamming Distance of $x, y \in \mathbb{N}^{|R|}$: $H(y, x) = \sum_{r \in R} |y_r - x_r|$

Sensitivity Lemma - Increased Demand

Let $x$ be given. There exists a best response $y_i \in \arg\min \left[ \pi_i(y_i, x_{-i}) \text{ s.t. } y_i \in B_{f(i)}(d_i + 1) \right]$ with $H(x_i, y_i) = 1$. 
Existence Proof (contd.)

Sensitivity Lemma - Increased Load

Let $a_r = \sum_{i \neq j} x_{i,r}, r \in R$ be given and $x_j$ be a best response to $a$. If $a_r \rightarrow a_r + 1$ for one $r \in R$, then there exists a best response $y_j \in \text{argmin} \left[ \pi_j(y_j, a) \text{ s.t. } y_j \in B_{f(j)}(d_j) \right]$ with $H(x_j, y_j) \in \{0, 2\}$. 
Existence Proof (contd.)

**Sensitivity Lemma - Increased Load**

Let $a_r = \sum_{i \neq j} x_{i,r}, r \in R$ be given and $x_j$ be a best response to $a$. If $a_r \to a_r + 1$ for one $r \in R$, then there exists a best response $y_j \in \arg\min [\pi_j(y_j, a) \text{ s.t. } y_j \in B_{f(j)}(d_j)]$ with $H(x_j, y_j) \in \{0, 2\}$. 

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Existence Proof (contd.)

**Sensitivity Lemma - Increased Load**

Let \( a_r = \sum_{i \neq j} x_{i,r}, r \in R \) be given and \( x_j \) be a best response to \( a \). If \( a_r \rightarrow a_r + 1 \) for one \( r \in R \), then there exists a best response \( y_j \in \arg\min \left[ \pi_j(y_j, a) \text{ s.t. } y_j \in \mathcal{B}_{f(j)}(d_j) \right] \) with \( H(x_j, y_j) \in \{0, 2\} \).
Invariant

There is a sequence of best-responses $y^k$ with $H(y^0 = y, y^k) = \sum_{r \in R} |y_r - x_r| = 2$. 
Invariant

There is a sequence of best-responses $y^k$ with

$$H(y^0 = y, y^k) = \sum_{r \in R} |y_r - x_r| = 2.$$  

Convergence

The sequence $y^k$ is finite.

- Define “marginal costs” of every unit of every player
- Sorted vector of marginal costs decreases lexicographically

Remark

For “non-polymatroid” $X_i$, there are counterexamples.
Part III

Nonatomic Congestion Games - The Braess Paradox
Left equilibrium: \[ C(x) = \left( \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot 1 \right) \cdot 2 = \frac{3}{2} \]
Right equilibrium: \[ C(x) = 1 \cdot 1 + 1 \cdot 1 = 2 \]
Left equilibrium: \( C(x) = (\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot 1) \cdot 2 = \frac{3}{2} \)

Right equilibrium: \( C(x) = 1 \cdot 1 + 1 \cdot 1 = 2 \)

\( G \) is immune to BP-free iff \( G \) is series-parallel [Milchtaich, ’06, Chen et al. ’15].
Left equilibrium:  $C(x) = \left(\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot 1\right) \cdot 2 = \frac{3}{2}$
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$G$ is immune to BP-free iff $G$ is series-parallel [Milchtaich, ’06, Chen et al. ’15].

What about other strategy spaces:

- tours in a graph
- scheduling jobs on machines
- spanning trees
- Steiner trees
Braess Paradox via Cost Reductions

Left equilibrium: \( C(x) = \left( \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot 1 \right) \cdot 2 = 3/2 \)
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\( G \) is immune to BP-free iff \( G \) is series-parallel [Milchtaich, '06, Chen et al. '15].

What about other strategy spaces:

- tours in a graph
- scheduling jobs on machines
- spanning trees
- Steiner trees

Q: What is the maximal combinatorial structure of strategy spaces so that there is no Braess paradox?
Nonatomic Congestion Model

Model $\mathcal{M}$

- $N = \{1, \ldots, n\}$ populations, represented by $[0, d_i], i \in N$
- $R = \{r_1, \ldots, r_m\}$ resources
- Strategies $S_i \subseteq 2^R$ with $S = \bigcup_{i \in N} S_i$
- Strategy distribution $(x_S)_{S \in S_i}$ with $\sum_{S \in S_i} x_S = d_i$
- Load of overall distribution $x$: $x_r = \sum_{S \in S: r \in S} x_S$
- Cost function $c_r : \mathbb{R}_+ \to \mathbb{R}_+$ nondecreasing

Example

- $S_i$ corresponds to paths from $s_i$ to $t_i$ in a graph
- tours in a graph
- machines ...
Nonatomic Congestion Model

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If player from population $i$ picks $S$ she gets disutility:

$$
\pi_{i,S}(x) = \sum_{r \in S} c_r(x_r).
$$

**Definition (Wardrop Equilibrium)**

$$
\pi_i(x) := \sum_{r \in S_i} c_r(x_r) \leq \sum_{r \in S'_i} c_r(x_r)
$$

for any $S_i, S'_i \in S_i$ with $x_{S_i} > 0$, for all $i \in N$. 
Braess Paradox via Cost Reductions

Figure: Example of the Braess paradox.
Braess Paradox via Demand Reductions

\[ d_1 = 1, \quad d_2 = 2, \quad d_3 = M \quad \text{with cost} \quad C(x^*) = 1 \cdot 2 + 2 \cdot 2 + M \cdot 0 = 6 \]

Reducing \( d_2 \to 0 \): \( C(x^{\text{new}}) = M + 2 \)
Let $x$ and $\bar{x}$ be Wardrop equilibria before and after a cost/demand reduction, resp.

### Definition

$(S_i)_{i \in N}$ admits the

1. weak BP, if there exists a resource $r \in R$ with $\bar{c}_r(\bar{x}_r) > c_r(x_r)$.
2. strong BP, if there exists a population $i \in N$ with $\pi_i(\bar{x}) > \pi_i(x)$.
Q: What is the maximal combinatorial structure of \((S_i)_{i \in \mathbb{N}}\) which is immune to the weak or strong Braess paradox?
**Main Result**

**Q:** What is the maximal combinatorial structure of \((S_i)_{i \in \mathbb{N}}\) which is immune to the weak or strong Braess paradox?

**A:** Bases of matroids (and only matroids!) (jointly with S. Fujishige, M. Goemans, B. Peis and R. Zenklusen)

**Theorem**

Let \((S_i)_{i \in \mathbb{N}}\) be a family of set systems. Then, the following statements are equivalent.

1. For all \(i \in \mathbb{N}\), the clutter \(\text{cl}(S_i)\) consists of the base sets of a matroid \(M_i = (R, I_i)\).
2. \((S_i)_{i \in \mathbb{N}}\) is immune to the weak (and strong) Braess paradox.

Proof via Sensitivity Theory for Polymatroids.
Q: What is the maximal combinatorial structure of $(S_i)_{i \in \mathbb{N}}$ which is immune to the weak or strong Braess paradox?

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(I) For all $i \in \mathbb{N}$, the clutter $cl(S_i)$ consists of the base sets of a matroid $M_i = (R, \mathcal{I}_i)$.

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Proof via Sensitivity Theory for Polymatroids.
Compact Representations of Strategies

\[ P(\mathcal{M}) := \left\{ x \in \mathbb{R}^S_{\geq 0} \mid \sum_{S \in S_i} x_S = d_i \text{ for all } i \in N \right\} . \]
Compact Representations of Strategies

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Resource-based polytope \( \tilde{P}(\mathcal{M}) \subseteq \mathbb{R}^{R}_{\geq 0} \)

\[ \tilde{P}(\mathcal{M}) := \left\{ \sum_{i \in N} \sum_{S \in S_i} x_S \cdot \chi_S \left| x \in P(\mathcal{M}) \right. \right\} , \]

\( \chi_S \in \{0, 1\}^{R} \) for \( S \subseteq R \) is the characteristic vector of \( S \).
Compact Representations of Strategies

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**Theorem (Beckmann et al. ’56)**

\( x \) is a Wardrop equilibrium if and only if it solves

\[
\min_{x \in \tilde{P}(M)} \left\{ \Phi(x) := \sum_{r \in R} \int_0^{x_r} c_r(t) \, dt \right\}. \tag{2}
\]

We call \( \Phi \) the Beckmann potential.
Definition (submodular, monotone, normalized)

- \( f : 2^R \to \mathbb{R} \) is submodular if
  \( f(U) + f(V) \geq f(U \cup V) + f(U \cap V) \) for all \( U, V \in 2^R \).
- \( f \) is monotone, if \( U \subseteq V \Rightarrow f(U) \leq f(V) \).
- \( f \) is normalized, if \( f(\emptyset) = 0 \).

\[
P_h := \left\{ x \in \mathbb{R}_+^R \mid x(U) \leq h(U) \ \forall U \subseteq R, \ x(R) = h(R) \right\},
\]

for \( U \subseteq R, \ x(U) := \sum_{r \in U} x_r \).
Polymatroids and Submodular Functions

Definition (submodular, monotone, normalized)

- $f : 2^R \rightarrow \mathbb{R}$ is submodular if
  $$f(U) + f(V) \geq f(U \cup V) + f(U \cap V)$$
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$P_h := \left\{ x \in \mathbb{R}^R_+ \mid x(U) \leq h(U) \ \forall U \subseteq R, \ x(R) = h(R) \right\}$,

for $U \subseteq R$, $x(U) := \sum_{r \in U} x_r$.

Matroid Base Polytope

$P_{d_i \cdot \text{rk}_i} = \left\{ x_i \in \mathbb{R}^R_+ \mid x_i(U) \leq d_i \cdot \text{rk}_i(U) \ \forall U \subseteq R, \ x_i(R) = d_i \cdot \text{rk}_i(R) \right\}$

$\tilde{P}(\mathcal{M}) := \sum_{i \in N} P_{d_i \cdot \text{rk}_i} = P\sum_{i \in N} d_i \cdot \text{rk}_i$
Sensitivity in Polymatroid Optimization

\[
\min_{x \in P_h} \left\{ \Phi(x) := \sum_{r \in R} \int_0^{x_r} c_r(t) \, dt \right\},
\]

where \( P_h \) is a polymatroid base polytope with rank function \( h \) and for all \( r \in R \), \( c_r : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \), are non-decreasing and continuous functions.
Sensitivity in Polymatroid Optimization

\[
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\tag{3}
\]

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**Optimality Conditions**

\( x \in P_h \) is optimal if and only if

\[
c_e(x_e) \leq c_f(x_f) \text{ for all } e, f \in R, x_e > 0 \text{ with } x' := x + \epsilon (\chi_f - \chi_e) \in P_h \text{ for some } \epsilon > 0.
\]
To show $\bar{c}_e(\bar{x}_e) \leq c_e(x_e)$ for all $e \in R$.

Assume $\bar{c}_e(\bar{x}_e) > c_e(x_e)$ for some $e \in R$. 

Matroids are Immune to Braess Paradox

To show $\bar{c}_e(\bar{x}_e) \leq c_e(x_e)$ for all $e \in R$.

Assume $\bar{c}_e(\bar{x}_e) > c_e(x_e)$ for some $e \in R$. Then $\bar{x}_e > x_e$. 
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Strong exchange property of polymatroids: $\exists f \in R - e$ with $\bar{x}_f < x_f$ and $\epsilon > 0$ such that

$$x - \epsilon \chi_f + \epsilon \chi_e \in P_h \quad \text{and} \quad \bar{x} - \epsilon \chi_e + \epsilon \chi_f \in P_h.$$
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Wardrop equilibrium condition:

1. $c_f(x_f) \leq c_e(x_e)$, as $x$ WE for $c$. 

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$$x - \epsilon \chi_f + \epsilon \chi_e \in P_h$$ and $$\bar{x} - \epsilon \chi_e + \epsilon \chi_f \in P_h.$$

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2. $\bar{c}_e(\bar{x}_e) \leq \bar{c}_f(\bar{x}_f)$, as $\bar{x}$ WE for $\bar{c}$.

We get the following contradiction

$$c_f(x_f) \leq c_e(x_e) < \bar{c}_e(\bar{x}_e) \leq \bar{c}_f(\bar{x}_f) \leq c_f(x_f).$$
Summary

Part I matroids lead to fast convergence of best-response in congestion games

Part II polymatroids allow PNE in integral splittable congestion games

Part III matroids are immune to Braess Paradox
Part I matroids lead to fast convergence of best-response in congestion games

Part II polymatroids allow PNE in integral splittable congestion games

Part III matroids are immune to Braess Paradox
  ▶ all results are tight (in some sense)!