Matroids and Polymatroids in Congestion Games

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Outline

- Part I: Congestion Games
  - Existence of Equilibria
  - Computation of Equilibria
  - Matroids
- Part II: Integral Splittable Congestion Games
  - Existence and Computation of Equilibria
  - Integral Polymatroids
- Part III: Nonatomic Congestion Games
  - Efficiency of Equilibria
  - The Braess Paradox
  - Matroids are Immune to Braess Paradox

Strategic Games

- Strategic game $G = (N, X, \pi)$
- $N = \{1, \ldots, n\}$ set of players
- $X = \times_{i \in N} X_i$ set of pure strategies
- $x = (x_1, \ldots, x_N)$ strategy profile
- $\pi_i(x) : X \to \mathbb{R}, i \in N$ private cost/utility

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- $\pi_i(x) : X \to \mathbb{R}, i \in N$ private cost/utility
- Mixed strategy: for each player a probability distribution over pure strategies

Solution Concept

Definition

Pure Nash equilibrium (PNE): no player has an incentive to unilaterally deviate.

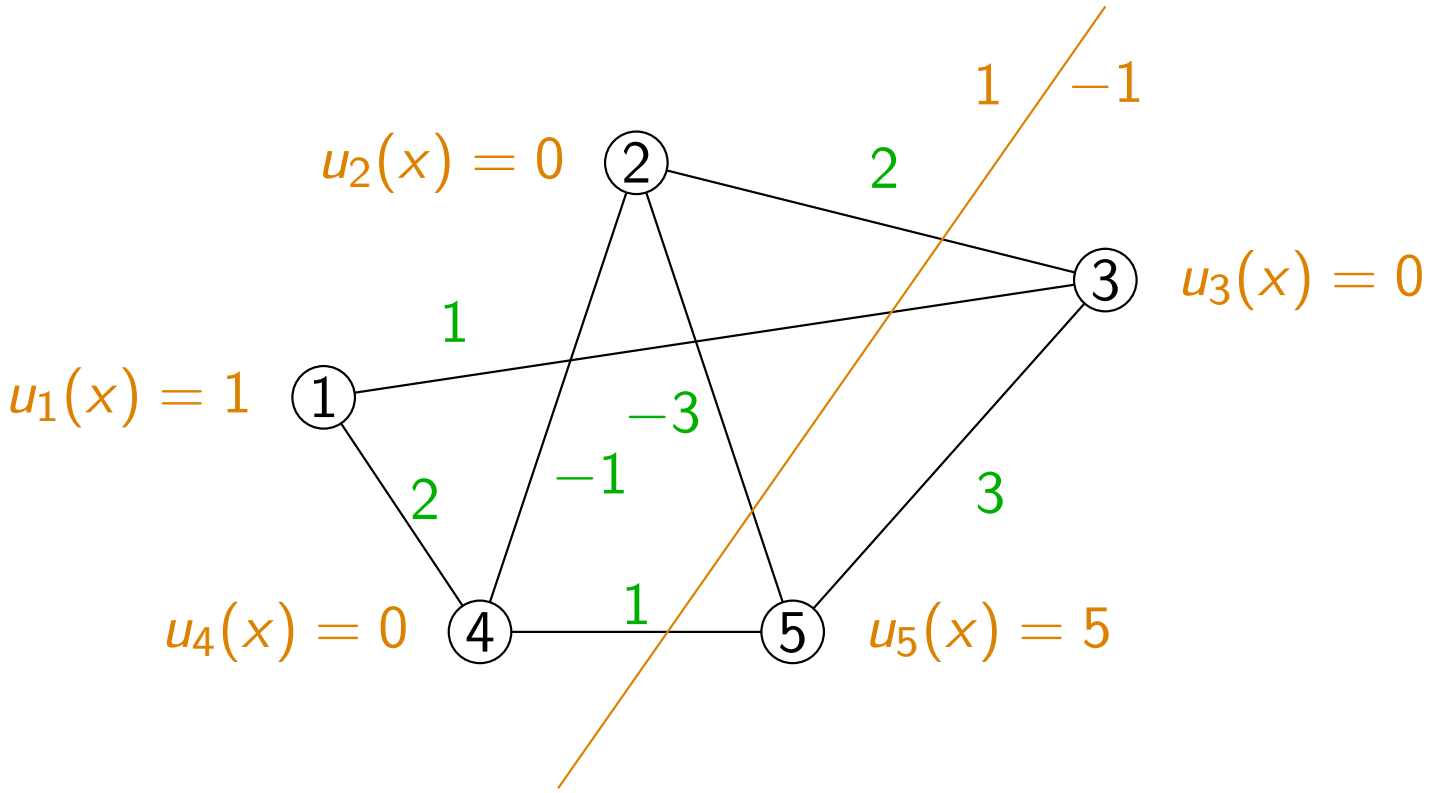
Definition

Mixed Nash equilibrium (MNE): no player has an incentive to unilaterally change her mixed strategy.

"Minimax Theorem" John von Neumann (1928), Nash (1950)

Motivation: Party Affiliation Games

- each strategy set $X_i = \{1, -1\}$
- Weight $w_{i,j}$ measures relationship between i and j
- payoff $u_i(x) = \sum_{j \in N} x_i \, x_j \, w_{i\,j} \to$ max



Solution Concept: Pure Nash Equilibrium

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Pure Nash equilibrium (PNE): no player has an incentive to unilaterally change his pure strategy.

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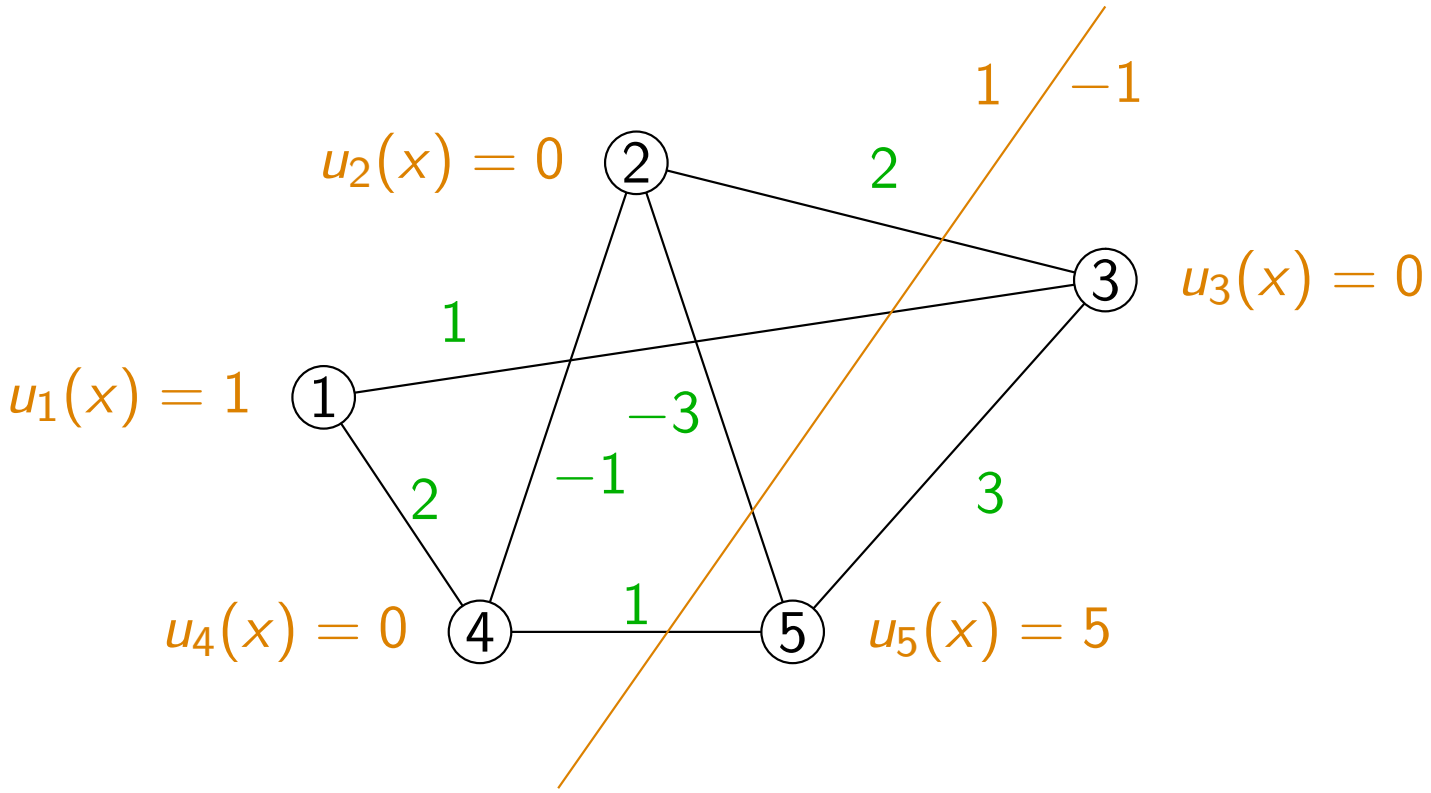


Existence of Nash Equilibria

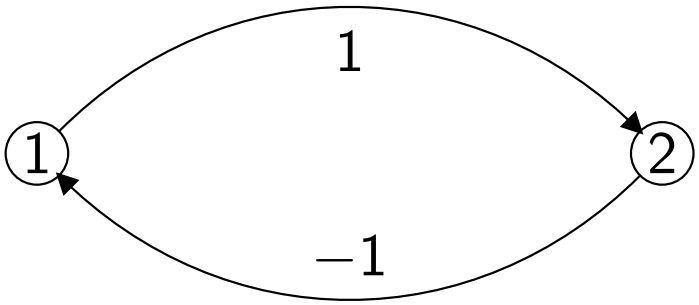
Nash (1951)

Theorem

Every finite game possesses a mixed Nash equilibrium.

Pure Nash Equilibrium need not exist!

Example: Assymmetric Party Affiliation Game



In the mixed Nash equilibrium, each player chooses each party with probability $1/2$.

Part I

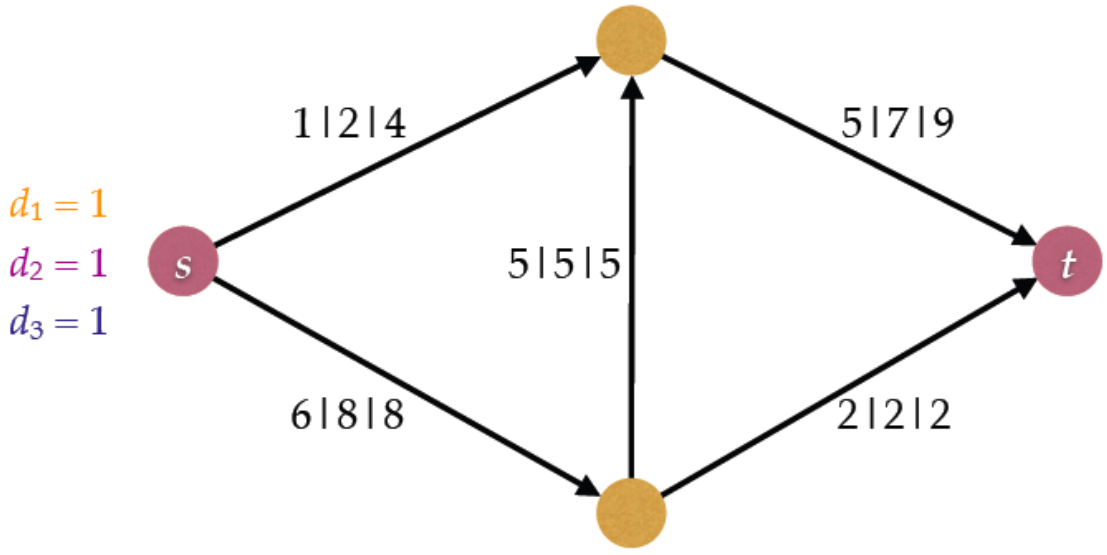
Congestion Games

Congestion Games

Modell

- $N = \{1, \ldots, n\}$ set of players
- $R = \{r_1, \ldots, r_m\}$ set of resources
- $X = \times_{i \in N} X_i$ set of strategy profiles with $X_i \subseteq 2^R$
- Strategy profile $x = (x_1, \ldots, x_n) \in X$
- Load of a resource $x_r = |\{i \in N : r \in x_i\}|$ for $x \in X$
- Cost functions $c_r : \mathbb{N} \to \mathbb{R}$ nondecreasing (convex)
- private cost: $\pi_i(x) = \sum_{r \in x_i} c_r(x_r)$

Example



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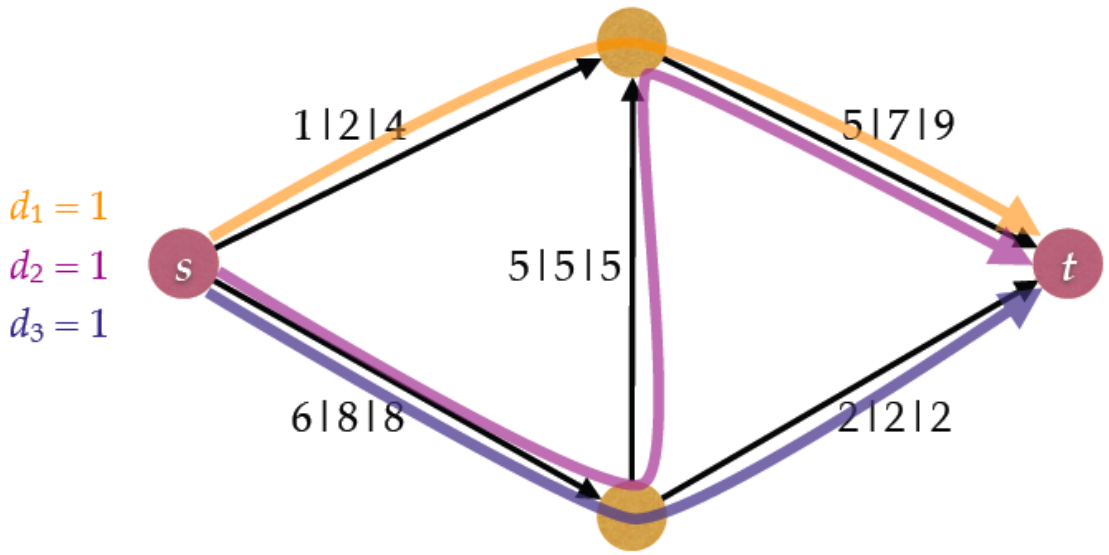

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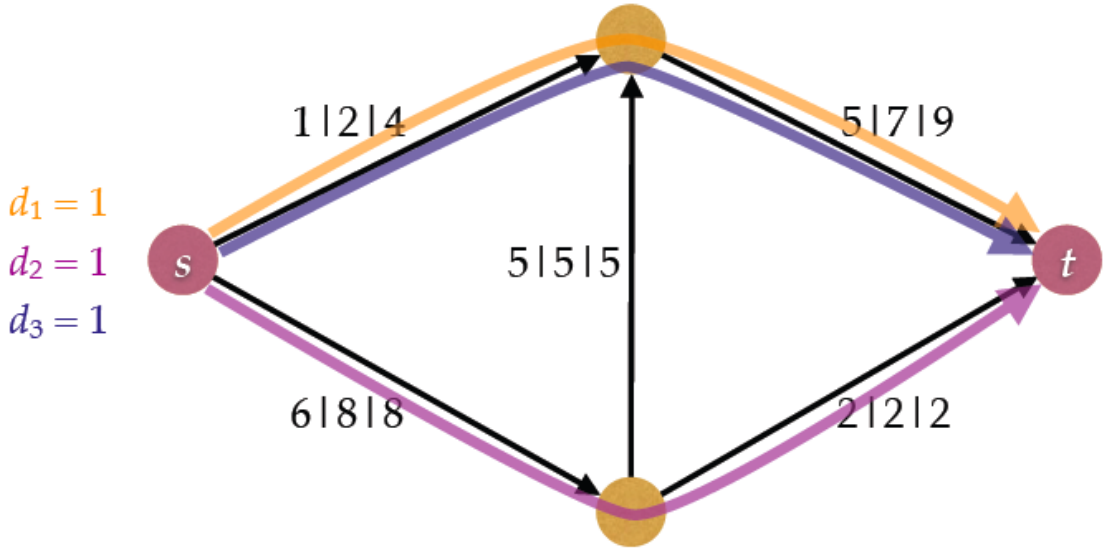

Fundamental Questions

1. When do pure Nash equilibria exist ?
2. How do players find them ?
3. How difficult is it to compute them ?

Potential Functions

Definition (Exact potential function)

$$P : X_1 \times \cdots \times X_n \to \mathbb{R}$$

If a player changes his action, the change in the potential function value is equal to the change in her payoff.

$$u_i(x_i, x_{-i}) - u_i(y_i, x_{-i}) = P(x_i, x_{-i}) - P(y_i, x_{-i})$$

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b–Potential

$$u_i(x_i, x_{-i}) - u_i(y_i, x_{-i}) = b_i \left(P(x_i, x_{-i}) - P(y_i, x_{-i})\right)$$

Monderer and Shapley (1996)

Merits of Potential Games

Theorem

Every finite exact potential game (with potential P)

- *possesses a PNE*
- *every sequence of improving moves is finite (FIP)*
- *every local minimum of P is a PNE*

Monderer and Shapley (1996)

Proof

- path $\gamma = (x^0, x^1, \ldots,)$ sequence of unilateral moves
- improvement path $\gamma = (x^0, x^1, \ldots,)$ sequence of unilateral improving moves

$\gamma = x_0, x_1, \ldots$ improvement path

$$\Rightarrow P(x_0) > P(x_1) > \cdots > \text{must be finite.}$$

Congestion Games are Potential Games

Theorem (Rosenthal '73)

Every congestion game

- *admits an exact potential function*
- *possesses a PNE*
- *possesses the Finite Improvement Property, that is, every sequence of improving moves is finite.*

Proof

Rosenthal's exact potential function $P : X_1 \times \cdots \times X_n \to \mathbb{R}$ is defined as

$$P(x) := \sum_{r \in R} \sum_{k=1}^{x_r} c_r(k). \tag{1}$$

Let $x \in X$ and $y_i \neq x_i$ be a unilateral deviation of i.

$$u_i(x_{-i}, y_i) - u_i(x) = \sum_{\substack{r \in y_i \\ r \notin x_i}} c_r(x_r + 1) - \sum_{\substack{r \in x_i \\ r \notin y_i}} c_r(x_r).$$

The potential of (x_{-i}, y_i) is given by:

$$\begin{aligned} P(x_{-i}, y_i) &= \sum_{r \in R} \sum_{k=1}^{x_r} c_r(k) + \underbrace{\sum_{\substack{r \in y_i \\ r \notin x_i}} c_r(x_r + 1) - \sum_{\substack{r \in x_i \\ r \notin y_i}} c_r(x_r)}_{= u_i(x_{-i}, y_i) - u_i(x)} \\ &= P(x) + u_i(x_{-i}, y_i) - u_i(x). \end{aligned}$$

Complexity of Computing PNE

Theorem (Fabrikant et al. '04, Ackermann et al. '08)

It is PLS-complete to compute a PNE even for symmetric congestion games with affine costs.

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For symmetric network congestion games, there is a polynomial time algorithm to compute a PNE.

Subdivide each arc e into n parallel arcs with capacity 1 each and assign costs $c_{e_i} = c_e(i)$ for $i \in \{1, \ldots, n\}$.

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Are there other set systems X_i with efficiently comp. PNE?

Definition (Matroid)

A matroid is a pair $\mathcal{M} = (R, \mathcal{I})$ where R is a set of resources, and $\mathcal{I}$ is a family of subsets of S such that:

1. $\emptyset \in \mathcal{I}$.
2. If $I \subset J$ and $J \in \mathcal{I}$, then $I \in \mathcal{I}$.
3. Let $I, J \in \mathcal{I}$ and $|I| < |J|$, then there exists an $x \in J \setminus I$ such that $I + x \in \mathcal{I}$.

A set system $R, \mathcal{I}$ that only satisfies (1) and (2) is called an independence system.

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Bases are sets in $\mathcal{I}$ of maximal cardinality, denoted by $\mathcal{B}$

Example: Uniform Matroids

The independent sets of a k-uniform matroid are the sets that contain at most k elements.

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4 resources: $\{1, 2, 3, 4\}$

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Independent sets of the 3-uniform matroid:

$$\mathcal{I} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\},$$

$$\{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}$$

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Bases: $\mathcal{B} = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}$

Graphic Matroid (Cycle Matroid)

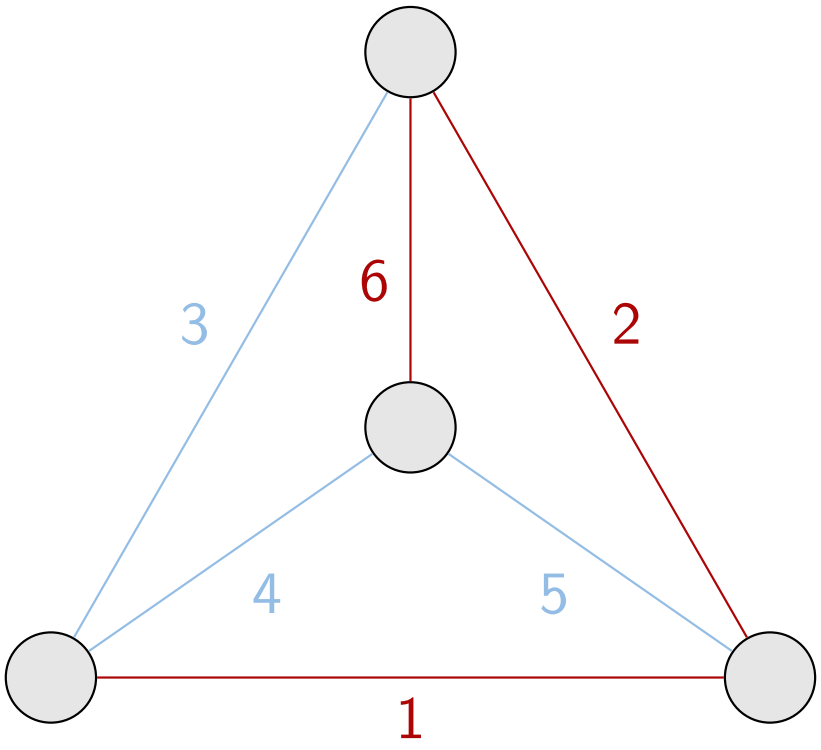


Figure : K_4 with two bases B_1 (red), B_2 (blue).

Fundamental Results

Theorem

Let $(R, \mathcal{I})$ be an independence system. The, the following is equivalent:

M1 $M = (R, \mathcal{I})$ *is a matroid.*

M2 $I, J \in \mathcal{I}$ *with* $|I| = |J| + 1 \Rightarrow$ *there is* $x \in I \setminus J$ *with* $J + x \in \mathcal{I}$.

M3 *For every* $I \subseteq R$, *every basis of* I *has the same cardinality (a basis of* $I \subseteq E$ *is an inclusion-wise maximal set in* I*).*

Proof.

$(M1) \Leftrightarrow (M2)$ is trivial. $(M1) \Rightarrow (M3)$ ist trivial. $(M3) \Rightarrow (M1)$: Let $I, J \in \mathcal{I}$ with $|I| > |J|$. With $(M3)$ we have that J is no base of $I \cup J$. Hence, there is $x \in (I \cup J) \setminus J = I \setminus J \in \mathcal{I}$ with $J + x \in \mathcal{I}$. $\square$

Fundamental Results

Theorem (Basis exchange theorem)

Let $(R, \mathcal{I})$ be a matroid with basis system $\mathcal{B}$. Then,

1. $\mathcal{B} \neq \emptyset$
2. *For every $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \setminus B_2$ there is $y \in B_2 \setminus B_1$ such that $B_1 - x + y \in \mathcal{B}$.*

Proof.

The bases set of $(R, \mathcal{I})$ satisfies (1) since $\emptyset \in \mathcal{I}$. For condition (2) let $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \setminus B_2$. Since $B_1 - x \in \mathcal{I}$ we can use $(M2)$: $|B_1 - x| + 1 = |B_2|$ hence there is $y \in B_2 \setminus (B_1 - x)$ with $B_1 - x + y \in \mathcal{I}$. As all bases have the same cardinality (see $(M3)$) we get $B_1 - x + y \in \mathcal{B}$. $\square$

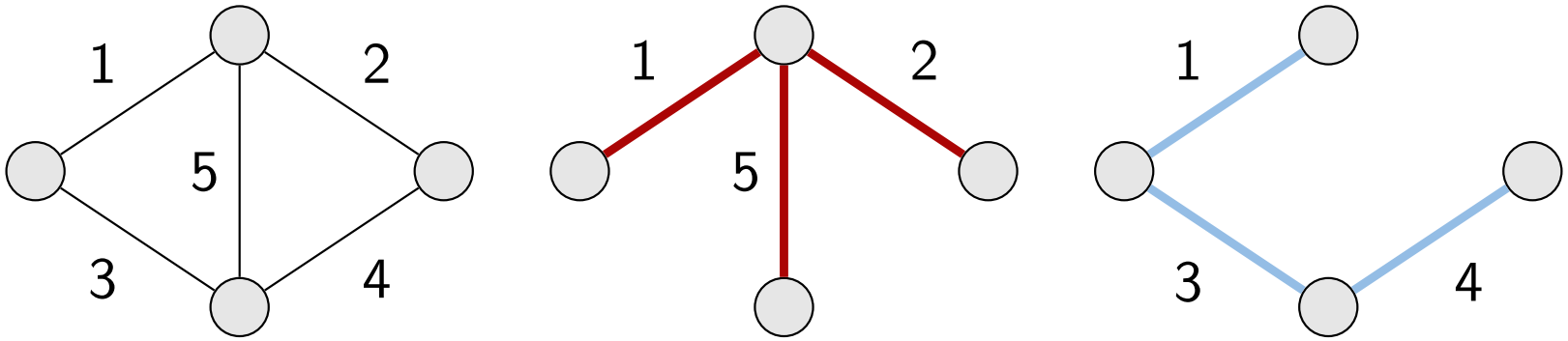
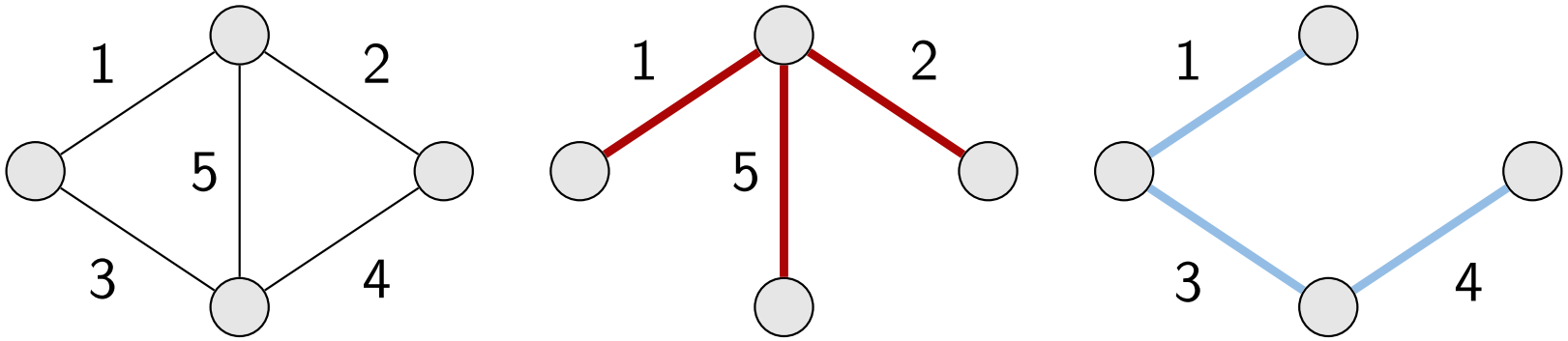


Figure : Two bases B (red) and B' (blue) of a graphic matroid



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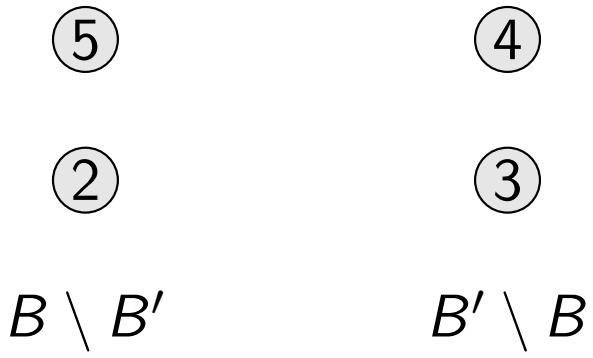
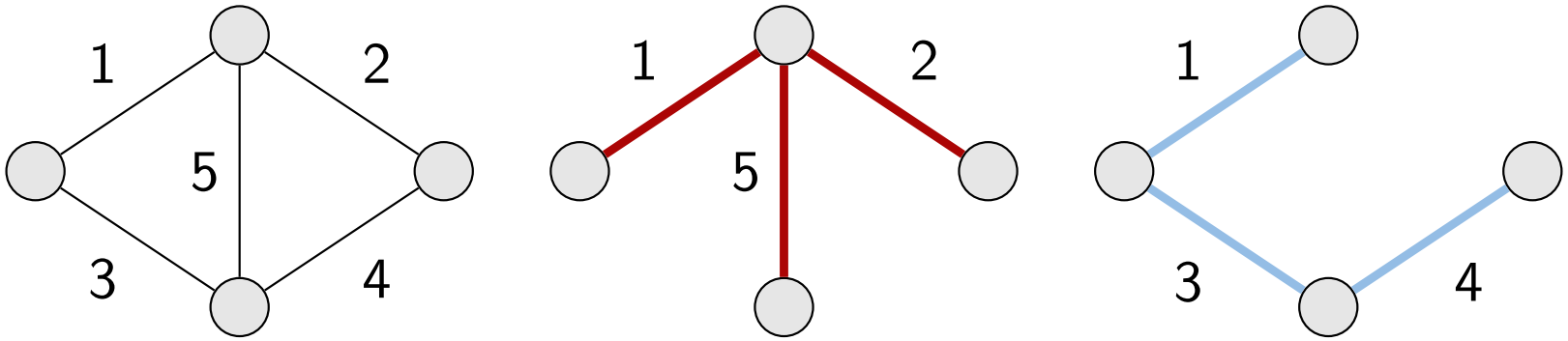


Figure : Bipartite graph $G(B \triangle B') = (B \triangle B', R)$.

$$(x, y) \in R \Leftrightarrow B - x + y \in \mathcal{B}.$$



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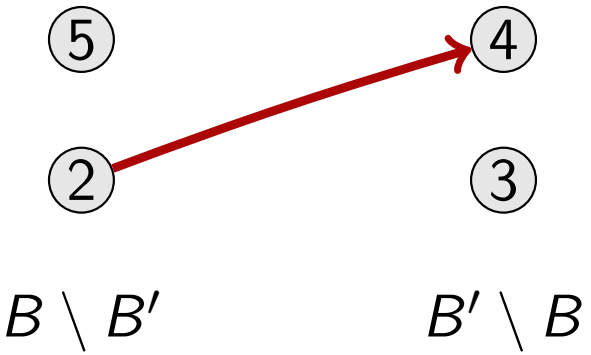

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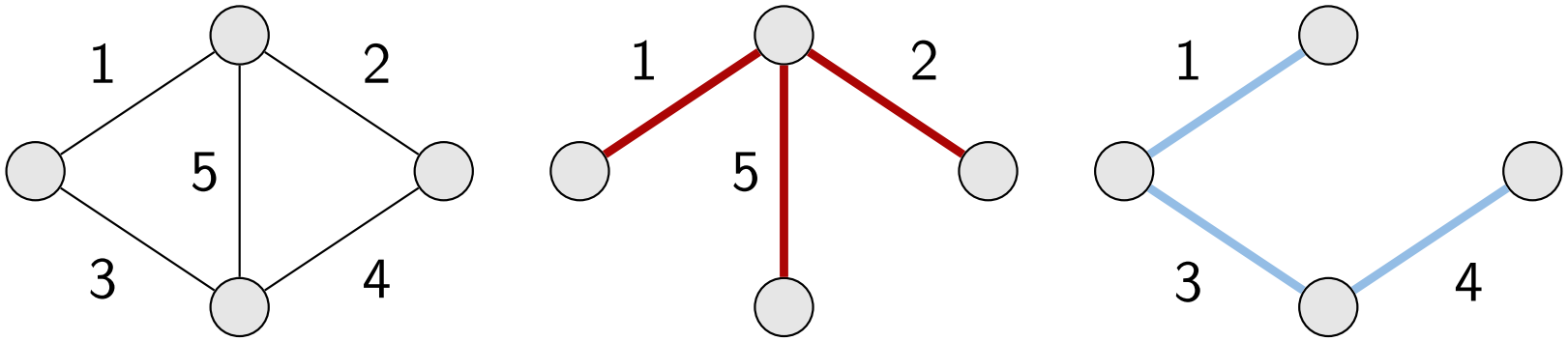

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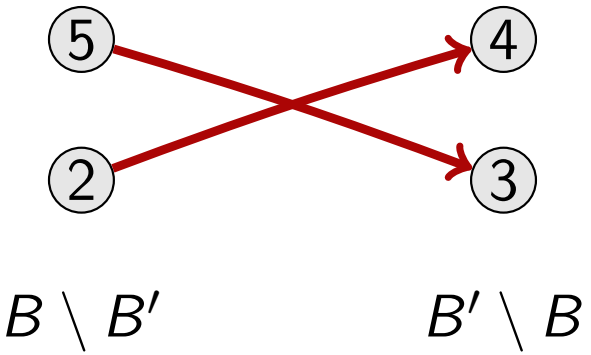

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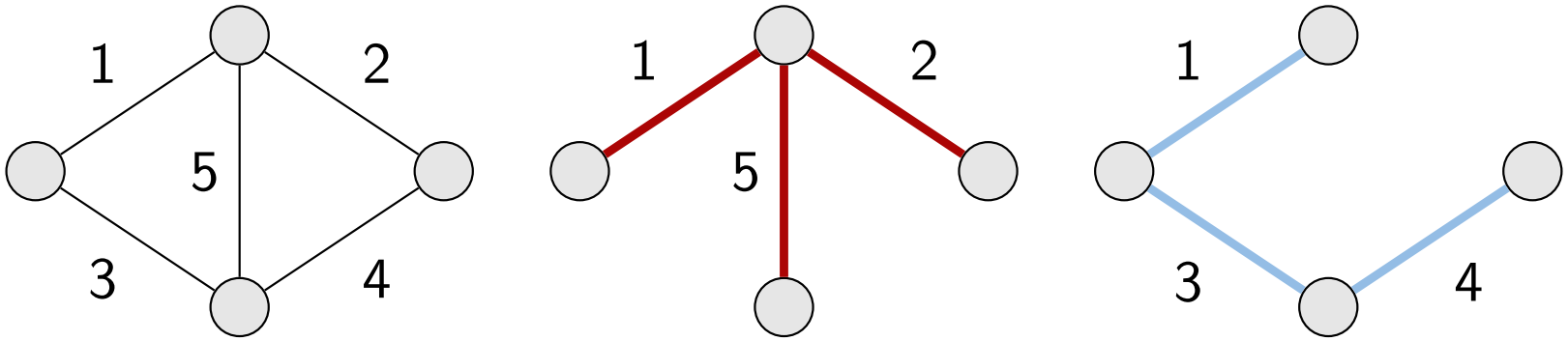

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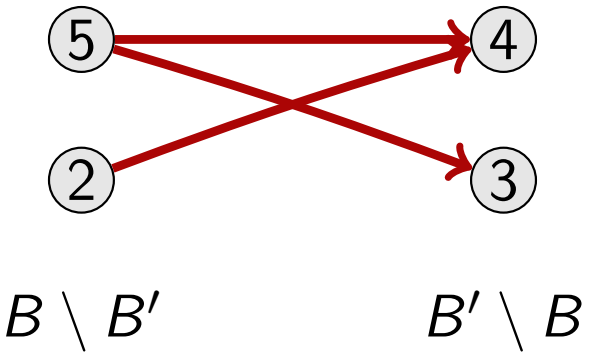

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$$(x, y) \in R \Leftrightarrow B - x + y \in \mathcal{B}.$$

Perfect matching in $G(B \triangle B')$

Theorem (Matching Property)

Let $\mathcal{M} = (R, \mathcal{I})$ be a matroid. Then, for every pair (B, B') of bases of $\mathcal{M}$ there exists a perfect matching $W(B, B')$ in $G(B \triangle B')$.

Matroid Congestion Games

Matroid Congestion Games

- $N = \{1, \ldots, n\}$ set of players
- $R = \{r_1, \ldots, r_m\}$ set of resources
- $X_i \subseteq 2^R$ with $X_i = \mathcal{B}_i$ for $M_i = (R, \mathcal{I}_i)$
- private cost for $B = (B_1, \ldots, B_n) \in \mathcal{B}$:

$$\pi_i(B) = \sum_{r \in B_i} c_r(x_r)$$

Best-Response - Polynomial Time

Theorem (Ackermann, Röglin, Vöcking '08)

For matroid congestion games, the best-response dynamic converges after at most $n^2 \cdot m \cdot \max_{i \in N} rk_i \leq n^2 m^2$ *steps.*

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Let L be a list of all cost values $c_r(i), r \in R = \{r_1, \ldots, r_m\}, i \in N = \{1, \ldots, n\}$ sorted in non-decreasing order. Define alternative cost function

$$c'_r : \mathbb{N} \to \{1, \ldots, n \cdot m\},$$

where $i \in [0, n]$ is mapped to list position of $c_r(i)$in L (same costs are mapped to same position).

Lemma

Let B_i^* be a best-response w.r.t. $B \in \mathcal{B}$ with $B_i^* \neq B_i$. Then B_i^* decreases also the private costs w.r.t. c'.

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Consider $G(B_i^* \Delta B_i)$ with perfect matching N. Let $(u, v) \in N$, i.e.,

$$v \in B_i^* \setminus B_i, u \in B_i \setminus B_i^* \text{ and } B_i^* - v + u \in \mathcal{B}_i.$$

For $B^* = (B_i^*, B_{-i})$ with load vector x^* we have

$$c_v(x_v^*) \leq c_u(x_u^* + 1) \text{ for all } (u, v) \in N,$$

as else $B_i' := B_i^* - v + u$ would give less costs than B_i^*. There must be $(v, u) \in N$ with

$$c_v(x_v^*) < c_u(x_u^* + 1),$$

since B_i^* strictly decreases the private cost for i. Thus, also the private costs under c' must decrease.

Proof

Consider P' w.r.t. c'. We get

$$c'_r(i) \leq n \cdot m \text{ for all } r \in R \text{ and all } 1 \leq i \leq n.$$

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Hence,

$$P'(B) = \sum_{r \in R} \sum_{i=1}^{x_r} c'_r(i) \leq \sum_{r \in R} \sum_{i=1}^{x_r} n \cdot m \leq n^2 \cdot m \cdot \max_{i \in N} rk_i.$$

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Remark (Ackermann, Röglin, Vöcking '08)

For every non-matroid set system $X_i, i \in N$, there are isomorphic instances with exponentially long best-response sequences.

Part II

Integral Splittable Congestion Games

Integral Splittable Congestion Games

Integral Congestion Games

- $N = \{1, \ldots, n\}$ set of players
- $R = \{r_1, \ldots, r_m\}$ set of resources
- $X_i \subseteq 2^R$ set of allowable subsets
- Demands $d_i \in \mathbb{N}$
- Strategy corresponds to integral distribution of d_i among the sets in X_i
- cost of a resource $c_r(x) = c_r(\sum_{i \in N} x_{i,r})$
- private cost $\pi_i(x) = \sum_{r \in R} c_r(x) x_{i,r}$

A Counter-Example

Rosenthal (1973b)

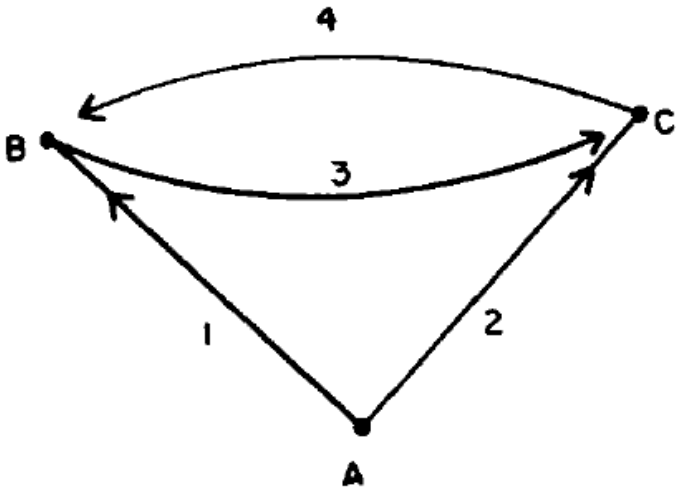


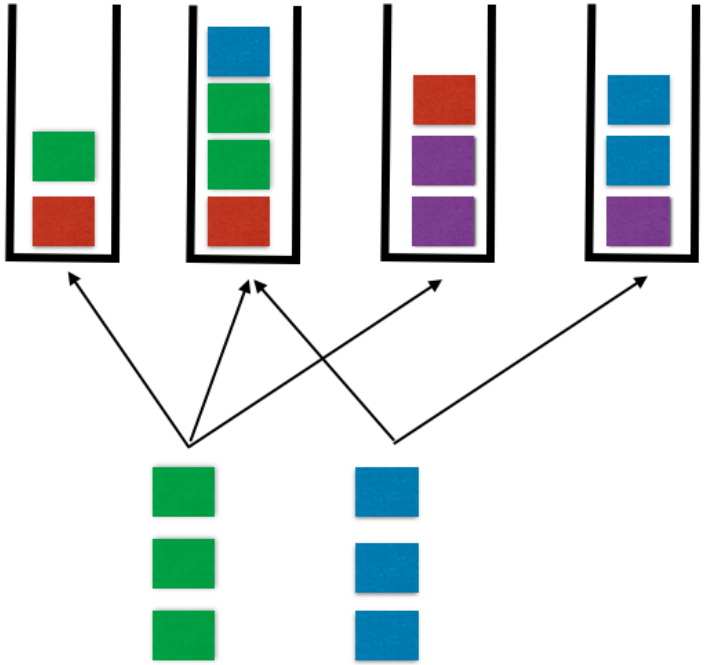
Fig. 3

- One player from A to C with demand 2 ("two taxicaps")
- One player from A to B with demand 1

Positive Results

Theorem (Tran-Thanh et al. (2011))

- *If X_i consists of singletons, then there is a PNE.*



Maximal Structures

What are maximal structures of X_i such that the existence of PNE is guaranteed?

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Theorem (H, Klimm, Peis (2014))

Polymatroids are the maximal structure.

Submodular Functions

Definition (integral, submodular, monotone, normalized)

- $f : 2^R \to \mathbb{N}$ is submodular if $f(U) + f(V) \geq f(U \cup V) + f(U \cap V)$ for all $U, V \in 2^R$.
- f is monotone, if $U \subseteq V \Rightarrow f(U) \leq f(V)$
- f is normalized, if $f(\emptyset) = 0$.

Submodular Functions

Definition (integral, submodular, monotone, normalized)

- $f : 2^R \to \mathbb{N}$ is submodular if $f(U) + f(V) \geq f(U \cup V) + f(U \cap V)$ for all $U, V \in 2^R$.
- f is monotone, if $U \subseteq V \Rightarrow f(U) \leq f(V)$
- f is normalized, if $f(\emptyset) = 0$.

Example

- rank function $rk : 2^R \to \mathbb{N}$ of a matroid $M = (\mathcal{I}, R)$
- for $F \subseteq R$: $rk(F) := \max\{|B|, B \text{ basis of } F\}$

Integral Polymatroids

integral polymatroid

$$\mathbb{P}_f := \left\{ x \in \mathbb{N}^R \mid \sum_{r \in U} x_r \leq f(U) \text{ for all } U \subseteq R \right\}.$$

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"truncated" integral polymatroid

$$\mathbb{P}_f(d) := \{ x \in \mathbb{N}^R \mid \sum_{r \in U} x_r \leq f(U) \text{ for all } U \subseteq R, \sum_{r \in R} x_r \leq d \}.$$

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integral polymatroid base polytope

$$\mathcal{B}_f(d) := \left\{ x \in \mathbb{N}^R \mid \sum_{r \in U} x_r \leq f(U) \text{ for all } U \subseteq R, \sum_{r \in R} x_r = d \right\}.$$

Congestion Games on Polymatroids

Strategic Game

- $G = (N, X, \pi)$
- $X_i = \mathcal{B}_{f^{(i)}}(d_i)$
- $\pi_i(x) = \sum_{r \in R} c_{i,r}(x) x_{i,r}$

$$\mathcal{B}_{f^{(i)}}(d_i) = \left\{ x_i \in \mathbb{N}^R \mid \sum_{r \in U} x_{i,r} \leq f^{(i)}(U) \text{ for all } U \subseteq R, \sum_{r \in R} x_{i,r} = d_i \right\}$$

Examples

Singletons

$$f^{(i)}(\{r\}) = d_i \text{ falls } r \in R_i \text{ und } f^{(i)}(\{r\}) = 0, \text{ sonst.}$$

Bases of Matroids

$f^{(i)}$ is rank function of some matroid $\mathcal{M}_i = (R_i, \mathcal{I}_i)$ and $d_i = rk_i(R_i)$.

Spanning Trees

- R edges of a graph
- $R_i \subset R$ connected subgraph
- Bases of $\mathcal{M}_i = (R_i, \mathcal{I}_i)$ are spanning trees of R_i

Existence Proof

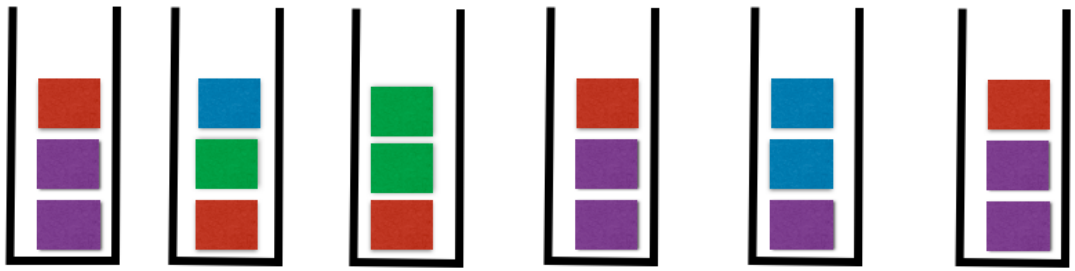
- Induction over $d = \sum_{i \in N} d_i$
- $d = 1$ trivial
- $d \to d + 1$
  - let x be a PNE for a game with demand d
  - In step $d \to d + 1$ exactly for one player i: $d_i \to d_i + 1$.

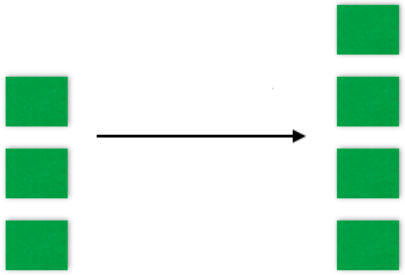
Existence Proof (contd.)

Hamming Distance of $x, y \in \mathbb{N}^{|R|}$: $H(y, x) = \sum_{r \in R} |y_r - x_r|$

Sensitivity Lemma - Increased Demand

Let x be given. There exists a best response
$y_i \in \mathit{argmin}\Big[\pi_i(y_i, x_{-i})$ s.t. $y_i \in \mathcal{B}_{f(i)}(d_i + 1)\Big]$ with $H(x_i, y_i) = 1$.





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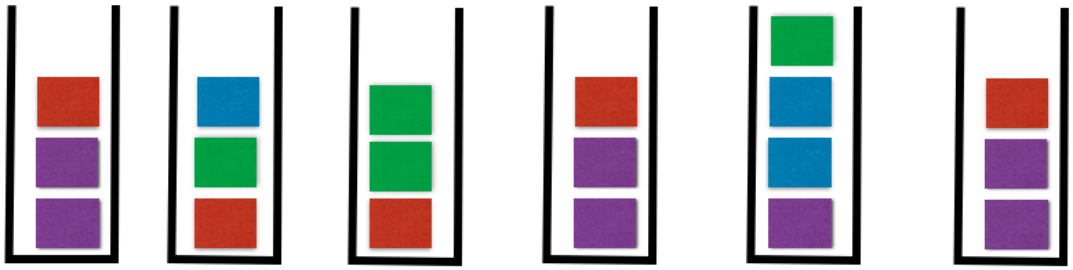
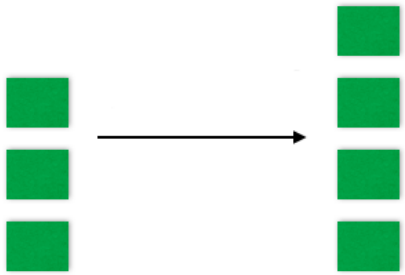
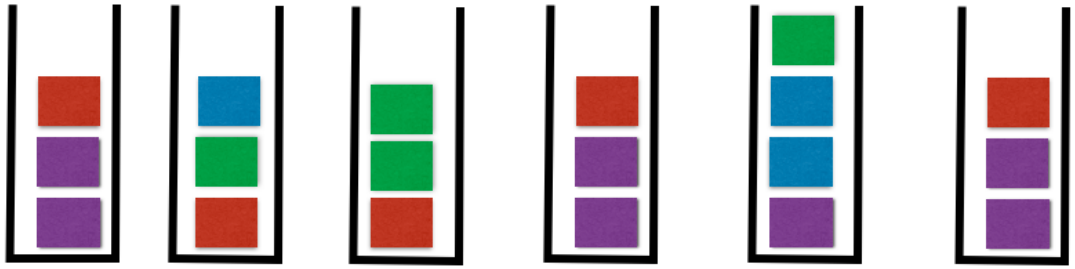




Existence Proof (contd.)

Sensitivity Lemma - Increased Load

Let $a_r = \sum_{i \neq j} x_{i,r}, r \in R$ be given and x_j be a best response to a. If $a_r \to a_r + 1$ for one $r \in R$, then there exists a best response $y_j \in argmin\Big[\pi_j(y_j, a) \text{ s.t. } y_j \in \mathcal{B}_{f(j)}(d_j)\Big]$ with $H(x_j, y_j) \in \{0, 2\}$.



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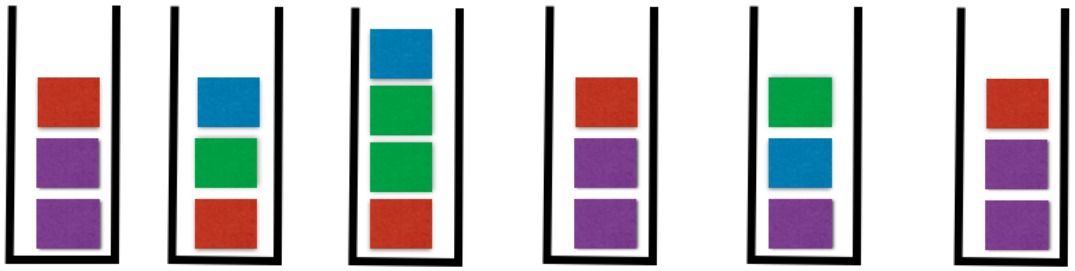

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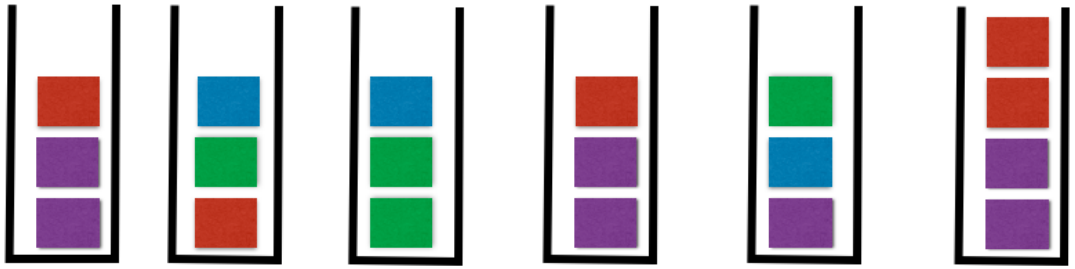

Existence Proof (contd.)

Invariant

There is a sequence of best-responses y^k with $H(y^0 = y, y^k) = \sum_{r \in R} |y_r - x_r| = 2$.

Existence Proof (contd.)

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Convergence

The sequence y^k is finite.

- Define "marginal costs" of every unit of every player
- Sorted vector of marginal costs decreases lexicographically

$\square$

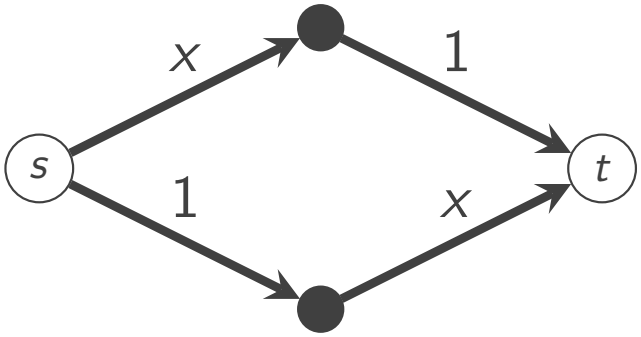
Remark

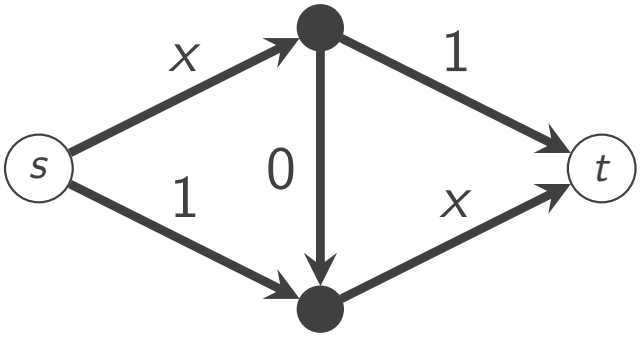
For "non-polymatroid" X_i, there are counterexamples.

Part III

Nonatomic Congestion Games - The Braess Paradox

Braess Paradox via Cost Reductions





Left equilibrium: $C(x) = (\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot 1) \cdot 2 = 3/2$
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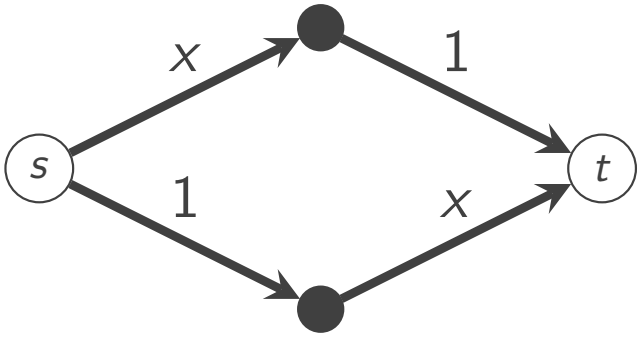

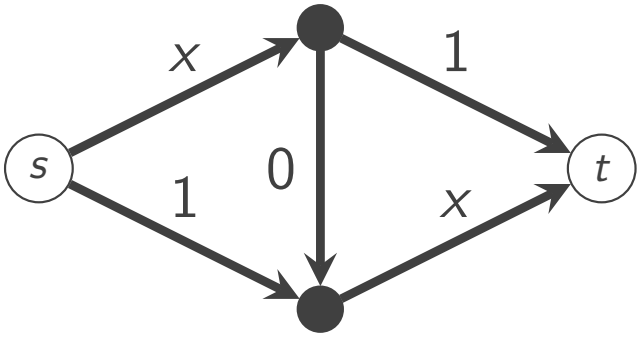

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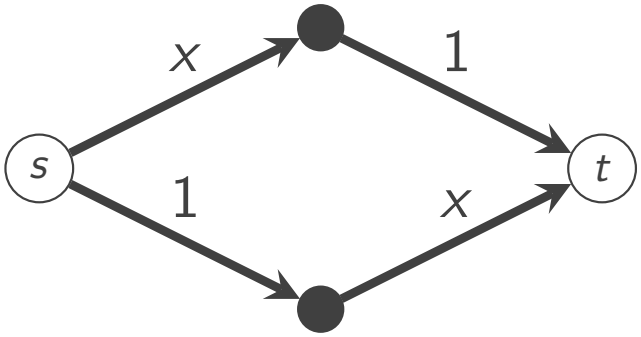

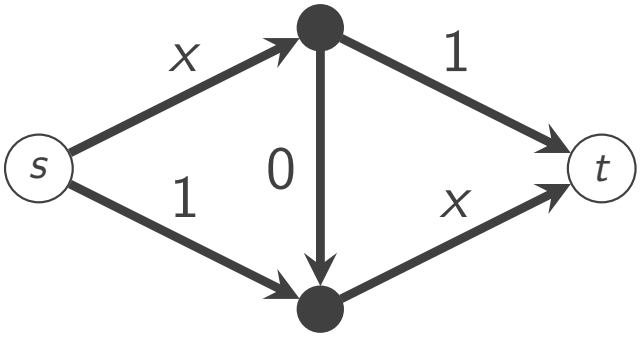

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What about other strategy spaces:

- tours in a graph
- scheduling jobs on machines
- spanning trees
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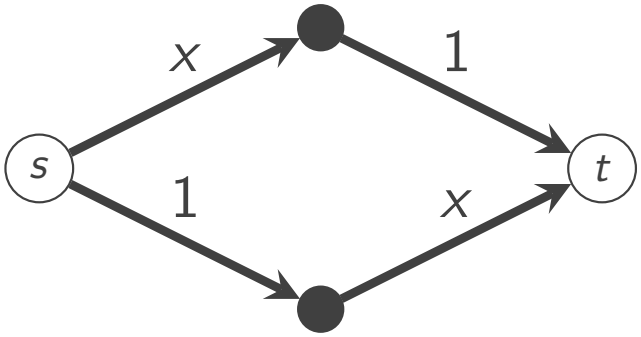

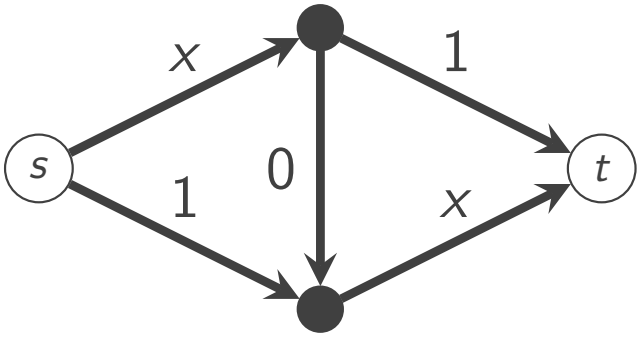

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What about other strategy spaces:

- tours in a graph
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Q: What is the maximal combinatorial structure of strategy spaces so that there is no Braess paradox?

Nonatomic Congestion Model

Model $\mathcal{M}$

- $N = \{1, \ldots, n\}$ populations, represented by $[0, d_i], i \in N$
- $R = \{r_1, \ldots, r_m\}$ resources
- Strategies $\mathcal{S}_i \subseteq 2^R$ with $\mathcal{S} = \cup_{i \in N} \mathcal{S}_i$
- Strategy distribution $(x_S)_{S \in \mathcal{S}_i}$ with $\sum_{S \in \mathcal{S}_i} x_S = d_i$
- Load of overall distribution x: $x_r = \sum_{S \in \mathcal{S}: r \in S} x_S$
- Cost function $c_r : \mathbb{R}_+ \to \mathbb{R}_+$ nondecreasing

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Example

- $\mathcal{S}_i$ corresponds to paths from s_i to t_i in a graph
- tours in a graph
- machines ...

Nonatomic Congestion Games

If player from population i picks S she gets disutility: $\pi_{i,S}(x) = \sum_{r \in S} c_r(x_r)$.

Definition (Wardrop Equilibrium)

$$\pi_i(x) := \sum_{r \in S_i} c_r(x_r) \leq \sum_{r \in S_i'} c_r(x_r)$$

for any $S_i, S_i' \in \mathcal{S}_i$ with $x_{S_i} > 0$, for all $i \in N$.

Braess Paradox via Cost Reductions

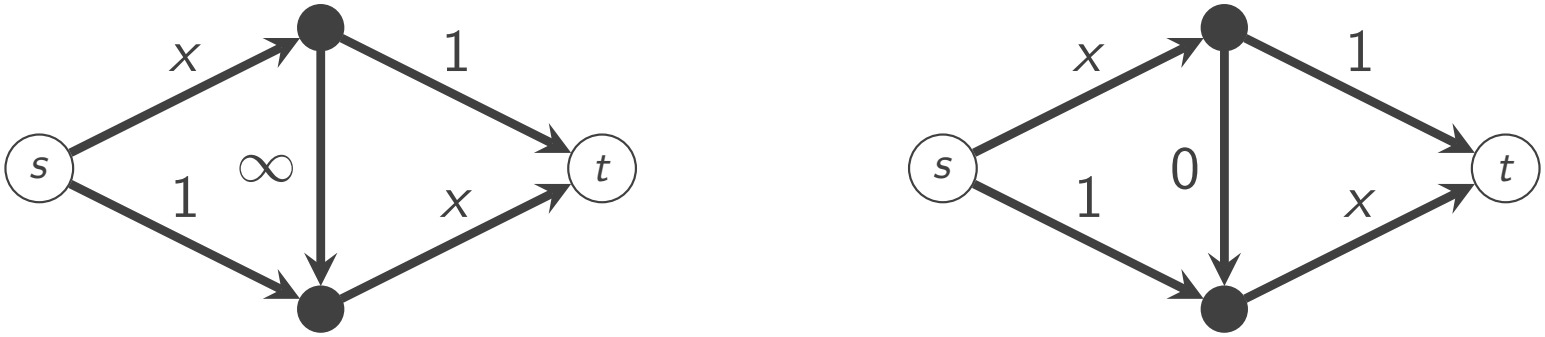
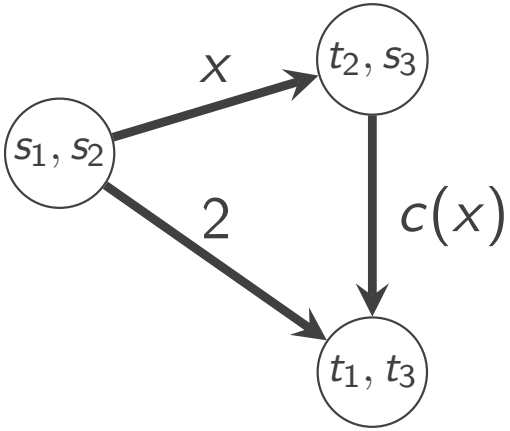
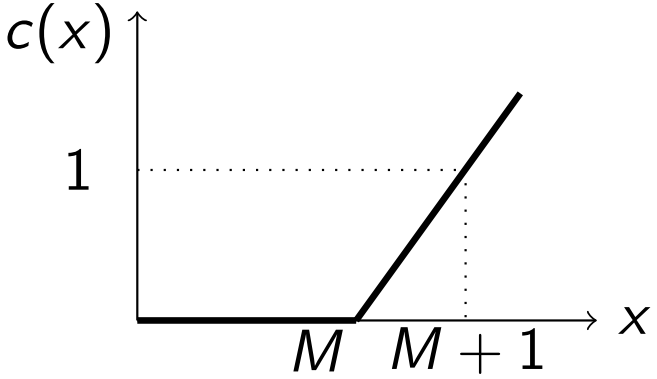


Figure : Example of the Braess paradox.

Braess Paradox via Demand Reductions





$d_1 = 1, d_2 = 2, d_3 = M$ with cost $C(x^*) = 1 \cdot 2 + 2 \cdot 2 + M \cdot 0 = 6$
Reducing $d_2 \to 0$: $C(x^{new}) = M + 2$

Weak/Strong Braess Paradox

Let x and $\bar{x}$ be Wardrop equilibria before and after a cost/demand reduction, resp.

Definition

$(\mathcal{S}_i)_{i \in N}$ admits the

1. weak BP, if there exists a resource $r \in R$ with $\bar{c}_r(\bar{x}_r) > c_r(x_r)$.
2. strong BP, if there exists a population $i \in N$ with $\pi_i(\bar{x}) > \pi_i(x)$.

Main Result

Q: What is the maximal combinatorial structure of $(\mathcal{S}_i)_{i \in N}$ which is immune to the weak or strong Braess paradox?

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Theorem

Let $(\mathcal{S}_i)_{i \in N}$ be a family of set systems. Then, the following statements are equivalent.

(I) *For all $i \in N$, the clutter $cl(\mathcal{S}_i)$ consists of the base sets of a matroid $M_i = (R, \mathcal{I}_i)$.*

(II) *$(\mathcal{S}_i)_{i \in N}$ is immune to the weak (and strong) Braess paradox.*

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Proof via Sensitivity Theory for Polymatroids.

Compact Representations of Strategies

$$P(\mathcal{M}) := \left\{ x \in \mathbb{R}^{\mathcal{S}}_{\geq 0} \;\middle|\; \sum_{S \in \mathcal{S}_i} x_S = d_i \text{ for all } i \in N \right\}.$$

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Resource-based polytope $\tilde{P}(\mathcal{M}) \subseteq \mathbb{R}^{R}_{\geq 0}$

$$\tilde{P}(\mathcal{M}) := \left\{ \sum_{i \in N} \sum_{S \in \mathcal{S}_i} x_S \cdot \chi_S \;\middle|\; x \in P(\mathcal{M}) \right\},$$

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Theorem (Beckmann et al. '56)

x is a Wardrop equilibrium if and only if it solves

$$\min_{x \in \tilde{P}(\mathcal{M})} \left\{ \Phi(x) := \sum_{r \in R} \int_0^{x_r} c_r(t)\, dt \right\}. \tag{2}$$

We call Φ the Beckmann potential.

Polymatroids and Submodular Functions

Definition (submodular, monotone, normalized)

- $f : 2^R \to \mathbb{R}$ is submodular if $f(U) + f(V) \geq f(U \cup V) + f(U \cap V)$ for all $U, V \in 2^R$.
- f is monotone, if $U \subseteq V \Rightarrow f(U) \leq f(V)$
- f is normalized, if $f(\emptyset) = 0$.

$$P_h := \left\{ x \in \mathbb{R}_+^R \mid x(U) \leq h(U) \; \forall U \subseteq R, \; x(R) = h(R) \right\},$$

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Matroid Base Polytope

$$P_{d_i \cdot \mathrm{rk}_i} = \left\{ x_i \in \mathbb{R}_+^R \mid x_i(U) \leq d_i \cdot \mathrm{rk}_i(U) \forall U \subseteq R, x_i(R) = d_i \cdot \mathrm{rk}_i(R) \right\}$$

$$\tilde{P}(\mathcal{M}) := \sum_{i \in N} P_{d_i \cdot \mathrm{rk}_i} = P_{\sum_{i \in N} d_i \cdot \mathrm{rk}_i}$$

Sensitivity in Polymatroid Optimization

$$\min_{x \in P_h} \left\{ \Phi(x) := \sum_{r \in R} \int_0^{x_r} c_r(t)\, dt \right\}, \tag{3}$$

where P_h is a polymatroid base polytope with rank function h and for all $r \in R$, $c_r : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, are non-decreasing and continuous functions.

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Optimality Conditions

$x \in P_h$ is optimal if and only if

$$c_e(x_e) \leq c_f(x_f) \text{ for all } e, f \in R, x_e > 0 \text{ with}$$
$$x' := x + \epsilon(\chi_f - \chi_e) \in P_h \text{ for some } \epsilon > 0.$$

Matroids are Immune to Braess Paradox

To show $\bar{c}_e(\bar{x}_e) \leq c_e(x_e)$ for all $e \in R$.

Assume $\bar{c}_e(\bar{x}_e) > c_e(x_e)$ for some $e \in R$.

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Strong exchange property of polymatroids: $\exists f \in R - e$ with $\bar{x}_f < x_f$ und $\epsilon > 0$ such that

$$x - \epsilon\chi_f + \epsilon\chi_e \in P_h \quad \text{and} \quad \bar{x} - \epsilon\chi_e + \epsilon\chi_f \in P_h.$$

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Strong exchange property of polymatroids: $\exists f \in R - e$ with $\bar{x}_f < x_f$ und $\epsilon > 0$ such that

$$x - \epsilon\chi_f + \epsilon\chi_e \in P_h \quad \text{and} \quad \bar{x} - \epsilon\chi_e + \epsilon\chi_f \in P_h.$$

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We get the following contradiction

$$c_f(x_f) \leq c_e(x_e) < \bar{c}_e(\bar{x}_e) \leq \bar{c}_f(\bar{x}_f) \leq c_f(x_f).$$

$\square$

Summary

Part I matroids lead to fast convergence of best-response in congestion games

Part II polymatroids allow PNE in integral splittable congestion games

Part III matroids are immune to Braess Paradox

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- all results are tight (in some sense)!