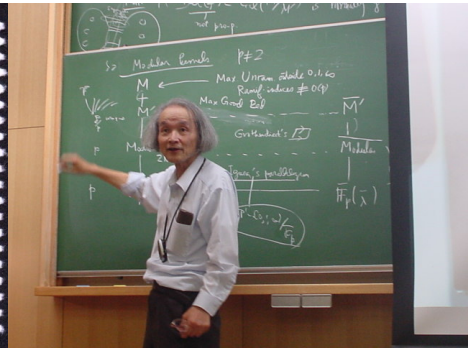
Euler-Kronecker constants: from Ramanujan to Ihara

Pieter Moree (MPIM, Bonn)



Amsterdam, CWI
December 2, 2011
Workshop Herman te Riele

(Partly) joint work with

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Kevin Ford (Urbana-Champaign, Illinois)

Values of the Euler phi-function not divisible by a given odd prime, and the distribution of Euler-Kronecker constants for cyclotomic fields, arXiv:1108.3805.

Definition

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$$\mathcal{EK}_{\mathbb{Q}} = \gamma/1 = \gamma = 0.577\ldots\ldots$$

Euler-Mascheroni constant

Historical background

Sums of two squares

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Shanks (1964): Ramanujan's claim is false for every $r > 3/2$.

Non-divisibility of Ramanujan's τ

$$\Delta := q \prod_{m=1}^{\infty} (1 - q^m)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n.$$

After setting $q = e^{2\pi i z}$, the function $\Delta(z)$ is the unique normalized cusp form of weight 12 for the full modular group $\mathrm{SL}_2(\mathbb{Z})$.

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For these primes $\tau(n)$ satisfies an easy congruence, e.g., :

$$\tau(n) \equiv \sum_{d|n} d^{11} \pmod{691}.$$

Put $t_n = 1$ if $q \nmid \tau(n)$ and $t_n = 0$ otherwise.

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It can be shown by transcendental methods that

$$\sum_{k=1}^{n} t_k \sim \frac{C_q n}{\log^{\delta_q} n}; \tag{1}$$

and

$$\sum_{k=1}^{n} t_k = C_q \int_2^n \frac{dx}{\log^{\delta_q} x} + O\Big(\frac{n}{\log^r n}\Big), \tag{2}$$

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M. (2004): All estimates (2) are false for $r > 1 + \delta_q$

Non-divisibility of Euler's φ-function

(Spearman-Williams, 2006). Put

$$\mathcal{E}_q(x) = \sum_{n \le x,\ q \nmid \varphi(n)} 1.$$

Question

$$\mathcal{E}_q(x) \sim c_q \frac{x}{\log^{1/(q-1)} x} \text{ or } \mathcal{E}_q(x) \sim c_q \int_2^x \frac{dt}{\log^{1/(q-1)} t}?$$

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Euler-Kronecker constants of multiplicative sets

We say that S is multiplicative if m and n are coprime integers then mn is in S iff both m and n are in S.

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If $(m, n) = 1$, then

$$q \nmid f(mn) \iff q \nmid f(m)f(n) \iff q \nmid f(n) \text{ and } q \nmid f(m)$$

Euler-Kronecker constant of a multiplicative set

Assumption. There are some fixed $\delta, \rho > 0$ such that asymptotically

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We put

$$L_S(s) := \sum_{n=1,\ n\in S}^{\infty} n^{-s}.$$

Can show that, Euler-Kronecker constant

$$\gamma_S := \lim_{s\to 1+0} \Big(\frac{L_S'(s)}{L_S(s)} + \frac{\delta}{s-1}\Big)$$

exists.

The second order term and γ_S

We have

$$S(x) = C_0(S)x\log^{\delta-1} x\Big(1+(1+o(1))\frac{C_1(S)}{\log x}\Big), \qquad as \quad x \to \infty,$$

where $C_1(S) = (1-\delta)(1-\gamma_S)$.

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Follows on noting that by partial integration we have

$$\int_2^x \log^{\delta-1} dt = x\log^{\delta-1} x\Big(1+\frac{1-\delta}{\log x}+O\Big(\frac{1}{\log^2 x}\Big)\Big).$$

A Ramanujan type claim, if true, implies $\gamma_S = 0$.

Landau versus Ramanujan for $q \nmid \varphi$

Theorem. (M., 2006, unpublished). *Assume ERH. For* $q \leq 67$ *we have* $\gamma_{\varphi;q} < 1/2$ *and Ramanujan's approximation is better.*

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Further, we have $\lim_{q\to\infty} \gamma_{\varphi;q} = \gamma$.

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- $\gamma_{\varphi;q} = \gamma + O(\frac{\log q(\log\log q)}{q})$, on ERH for L-functions mod q.

Table : Overview of Euler-Kronecker constants discussed

set	γ_{set}	winner	author
$\mathbb{Z}_{\geq 1}$	$+0.5772\ldots$		Euler
$n = a^2 + b^2$	$-0.1638\ldots$	Ramanujan	Shanks
$3 \nmid \tau$	$+0.5349\ldots$	Landau	M.
$5 \nmid \tau$	$+0.3995\ldots$	Ramanujan	M.
$7 \nmid \tau$	$+0.2316\ldots$	Ramanujan	M.
$23 \nmid \tau$	$+0.2166\ldots$	Ramanujan	M.
$691 \nmid \tau$	$+0.5717\ldots$	Landau	M.
$q \nmid \varphi$, $q \leq 67$	< 0.4977	Ramanujan	FLM
$q \nmid \varphi$, $q \geq 71$	> 0.5023	Landau	FLM

Connection with $\mathcal{EK}_{\mathbb{Q}(\zeta_q)}$

Put $f_p = |\langle p (\mathrm{mod}\ q)\rangle|$.

$$S(q) := \sum_{p \neq q,\ f_p \geq 2} \frac{\log p}{p^{f_p} - 1},$$

We have

$$\gamma_{\varphi;q} = \gamma - \frac{(3-q)\log q}{(q-1)^2(q+1)} - S(q) - \frac{\mathcal{EK}_{\mathbb{Q}(\zeta_q)}}{q-1}$$

We have $S(q) \leq (\log q + 1)/2q$ (fairly easy).

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Given $\epsilon > 0$ we have $S(q) < \epsilon/q$ for a subset of primes of natural density 1.

Proof uses linear forms in logarithms in 3 variables (Matveev's estimate).

$\mathcal{EK}_K$

$$\mathcal{EK}_K = \lim_{x\to\infty}\Big(\log x - \sum_{N\mathfrak{p}\le x}\frac{\log N\mathfrak{p}}{N\mathfrak{p}-1}\Big)$$

$$\tilde{\zeta}_K(s) = \tilde{\zeta}_K(1-s)$$

$$\tilde{\zeta}_K(s) = \tilde{\zeta}_K(0)e^{\beta_K s}\prod_{\rho}\Big(1-\frac{s}{\rho}\Big)e^{s/\rho}$$

$$-\beta_K = \sum_{\rho}\frac{1}{\rho}$$

$$-\beta_K = \mathcal{EK}_K - (r_1+r_2)\log 2 + \frac{\log|D_K|}{2} - \frac{[K:\mathbb{Q}]}{2}(\gamma+\log\pi)+1$$

Theorem. (Ihara, 2006). *Under GRH we have*

$$-c_1\log|D_K| \le \mathcal{EK}_K \le c_2\log\log|D_K|$$

$$\mathcal{EK}_{\mathbb{Q}(\zeta_q)}$$

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$\mathcal{EK}_{\mathbb{Q}(\zeta_q)} = \gamma_q$

Since $\zeta_{\mathbb{Q}(\zeta_q)}(s) = \zeta(s)\prod_{\chi\neq\chi_0} L(s,\chi)$, we have

$$\gamma_q = \gamma + \sum_{\chi\neq\chi_0} \frac{L'(1,\chi)}{L(1,\chi)}$$

Ihara's result implies, on GRH, $-c_1 q\log q \leq \gamma_q \leq c_2 \log(q\log q)$

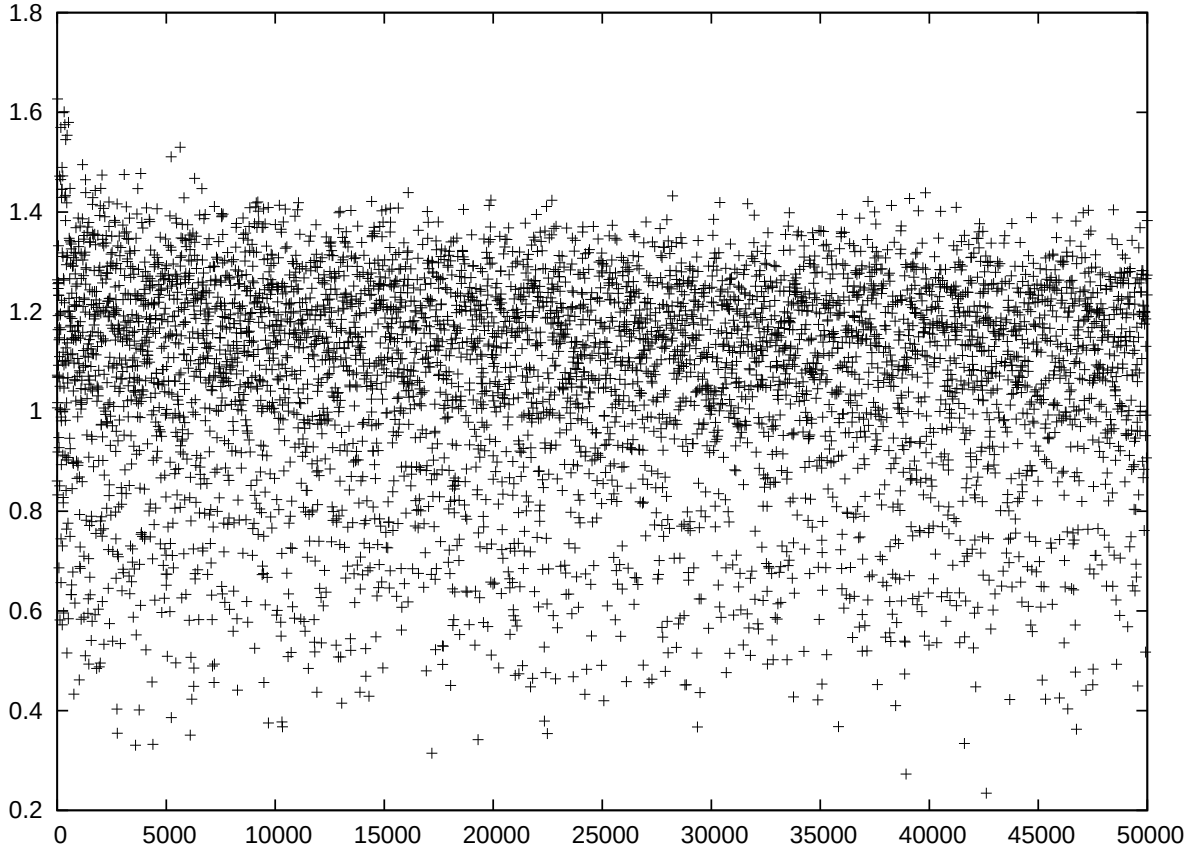
Badzyan (2010). On GRH, we have $|\gamma_q| = O(\log q \log\log q)$

Ihara (2009).
(i) $\gamma_q > 0$ ('very likely')
(ii) Conjectures that

$$\frac{1}{2} - \epsilon \leq \frac{\gamma_q}{\log q} \leq \frac{3}{2} + \epsilon$$

for q sufficiently large

$\frac{\gamma_q}{\log q}$ for $q \leq 50000$



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Theorem. On a quantitative version of the prime k-tuple conjecture we have

$$\liminf_{q\to\infty} \frac{\gamma_q}{\log q} = -\infty$$

Conjecture. For density 1 sequence of primes we have

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(That is, γ_q has normal order $\log q$)

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We have

$$\limsup_{q\to\infty} \frac{\gamma_q}{\log q} = 1$$

Sketch of proof of theorem

On ERH we have (Ihara)

$$\gamma_q = 2\log q - q \sum_{\substack{p \le q^2 \\ p \equiv 1 \pmod{q}}} \frac{\log p}{p-1} + O(\log\log q)$$

Construct infinite sequence b_i, $i = 1, 2, \ldots$ such that $n, 1 + 2b_1 n, 1 + 2b_2 n, \ldots$ satisfies conditions of prime k-tuple conjecture AND

$$\sum_{i=1}^{s} \frac{1}{b_i} \to \infty$$

Take s so large that sum is > 4.

By prime k-tuplet conjecture $q, 1 + 2b_1 q, 1 + 2b_2 q, \ldots, 1 + 2b_s q$ are infinitely often ALL prime with $1 + 2b_s q \le q^2$. Then

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$$q \sum_{\substack{p \le q^2 \\ p \equiv 1 \pmod{q}}} \frac{\log p}{p-1} > q \log q \sum_{i=1}^{s} \frac{1}{2b_i q} > (2 + \epsilon_0)\log q$$

Analogy with Kummer's Conjecture

Kummer conjectured that

$$h_1(p) = \frac{h(p)}{h_2(p)} \sim G(p) := 2p\left(\frac{p}{4\pi^2}\right)^{\frac{p-1}{4}}$$

(Ratio of the class number of $\mathbb{Q}(\zeta_p)$, respectively $\mathbb{Q}(\zeta_p + \zeta_p^{-1})$)

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This solved a conjecture of Erdős from 1988.

..finally...

Wikepedia: This is a Germanic name; the family name is te Riele, not Riele.

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HAPPY RETIREMENT, HERMAN!

THANK YOU!