Euler-Kronecker constants: from Ramanujan to Ihara

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Values of the Euler phi-function not divisible by a given odd prime, and the distribution of Euler-Kronecker constants for cyclotomic fields, arXiv:1108.3805.
Definition

$K$, number field.

\[\zeta_K(s) = \sum_{\alpha} \frac{1}{(N\alpha)^s}, \quad \text{Re}(s) > 1.\]
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Laurent series:

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\zeta_K(s) = \frac{c_{-1}}{s - 1} + c_0 + O(s - 1).
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Euler-Kronecker constant of \( K \):

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\mathcal{E}K_K := \frac{c_0}{c_{-1}}.
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\lim_{s \to 1} \left( \frac{\zeta_K'(s)}{\zeta_K(s)} + \frac{1}{s - 1} \right) = \mathcal{E}_K
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\( \mathcal{E}_K \) is constant in logarithmic derivative of \( \zeta_K(s) \) at \( s = 1 \).
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\( \mathcal{EK}_K \) is constant in logarithmic derivative of \( \zeta_K(s) \) at \( s = 1 \).

Example. \( \zeta(s) = \sum n^{-s} = 1/(s - 1) + \gamma + O(s - 1). \)
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$\mathcal{EK}_K$ is constant in logarithmic derivative of $\zeta_K(s)$ at $s = 1$.

Example. $\zeta(s) = \sum n^{-s} = 1/(s - 1) + \gamma + O(s - 1)$.

$$\mathcal{EK}_\mathbb{Q} = \gamma/1 = \gamma = 0.577 \ldots$$

Euler-Mascheroni constant
Historical background

Sums of two squares

Landau (1908)

\[ B(x) = \sum_{n \leq x, \; n = a^2 + b^2} 1 \sim K \frac{x}{\sqrt{\log x}}. \]

Ramanujan (1913)

\[ B(x) = K \int_{x}^{\infty} \frac{dt}{\sqrt{\log t}} + O(x \log x), \]

where \( R \) is arbitrary.

\[ K = 0.764223653... \]: Landau-Ramanujan constant.

Shanks (1964): Ramanujan's claim is false for every \( R > 3/2 \).
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Non-divisibility of Ramanujan’s $\tau$

$$\Delta := q \prod_{m=1}^{\infty} (1 - q^m)^{24} = \sum_{n=1}^{\infty} \tau(n)q^n.$$ 

After setting $q = e^{2\pi iz}$, the function $\Delta(z)$ is the unique normalized cusp form of weight 12 for the full modular group $SL_2(\mathbb{Z})$. 

Fix a prime $q \in \{3, 5, 7, 23, 691\}$. For these primes $\tau(n)$ satisfies an easy congruence, e.g.,:

$$\tau(n) \equiv \sum_{d|n} d^{11} \pmod{691}.$$ 

Put $t_n = 1$ if $q \nmid \tau(n)$ and $t_n = 0$ otherwise.
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A further claim of Ramanujan

Ramanujan in last letter to Hardy (1920):

\[ \sum_{n} k \sim C q n \log \delta q n ; \tag{1} \]

and

\[ \sum_{n} k \sim C q \int_{n^2} dx \log \delta q x + O (n \log r n) , \tag{2} \]

where \( r \) is any positive number.'
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M. (2004): All estimates (2) are false for \( r > 1 + \delta_q \).
Non-divisibility of Euler’s $\varphi$-function

(Spearman-Williams, 2006). Put

$$E_q(x) = \sum_{n \leq x, \ q \nmid \varphi(n)} 1.$$  

**Question**

$$E_q(x) \sim c_q \frac{x}{\log^{1/(q-1)} x} \quad \text{or} \quad E_q(x) \sim c_q \int_{2}^{x} \frac{dt}{\log^{1/(q-1)} t}?$$

That is, is the Landau approximation or Ramanujan approximation better?
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Assume $(q, n) = 1$. We have $q \mid \varphi(n)$ iff $n$ does not have a prime divisor $p$ that splits completely in $\mathbb{Q}(\zeta_q)$. 
Euler-Kronecker constants of multiplicative sets

We say that $S$ is multiplicative if $m$ and $n$ are coprime integers then $mn$ is in $S$ iff both $m$ and $n$ are in $S$. 
We say that \( S \) is **multiplicative** if \( m \) and \( n \) are coprime integers then \( mn \) is in \( S \) iff both \( m \) and \( n \) are in \( S \).

Common example is where \( S \) is a multiplicative semigroup generated by \( q_i, i = 1, 2, \ldots \), with every \( q_i \) a prime power and \((q_i, q_j) = 1\).
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**Example I.** $n = X^2 + Y^2$.

**Example II.** If $q$ is a prime and $f$ a multiplicative function, then

$$\{n : q \nmid f(n)\}$$

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If $(m, n) = 1$, then

$$q \nmid f(mn) \iff q \nmid f(m)f(n) \iff q \nmid f(n) \text{ and } q \nmid f(m)$$
Euler-Kronecker constant of a multiplicative set

**Assumption.** There are some fixed $\delta, \rho > 0$ such that asymptotically

$$\pi_S(x) = \delta \pi(x) + O\left(\frac{x}{\log^{2+\rho} x}\right).$$
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We put

\[
L_S(s) := \sum_{n=1, n \in S} \infty \sum n^{-s}.
\]

Can show that, **Euler-Kronecker constant**

\[
\gamma_S := \lim_{s \to 1+0} \left( \frac{L'_S(s)}{L_S(s)} + \frac{\delta}{s-1} \right)
\]

exists.
The second order term and $\gamma_S$

We have

$$S(x) = C_0(S) x \log^{\delta-1} x \left( 1 + (1 + o(1)) \frac{C_1(S)}{\log x} \right), \quad \text{as} \quad x \to \infty,$$

where $C_1(S) = (1 - \delta)(1 - \gamma_S)$. 

*Theorem.* Suppose that $\delta < 1$. If $\gamma_S < 1/2$, the Ramanujan approximation is asymptotically better than the Landau one. If $\gamma_S > 1/2$, it is the other way around. 

Follows on noting that by partial integration we have

$$\int x^2 \log^{\delta-1} \log x \, dt = x \log^{\delta-1} x \left( 1 + (1 + o(1)) \frac{C_1(S)}{\log x} \right).$$

A Ramanujan type claim, if true, implies $\gamma_S = 0$. 

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Landau versus Ramanujan for $q \nmid \varphi$

**Theorem.** (M., 2006, unpublished). Assume ERH. For $q \leq 67$ we have $\gamma_{\varphi; q} < 1/2$ and Ramanujan’s approximation is better.
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\[ \gamma_{\varphi; q} = \gamma + O\left( \frac{\log^2 q}{\sqrt{q}} \right), \] effective constant.
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- $\gamma_{\varphi;q} = \gamma + O\left(\frac{\log q (\log \log q)}{q}\right)$, on ERH for $L$-functions mod $q$. 
Table: Overview of Euler-Kronecker constants discussed

<table>
<thead>
<tr>
<th>set</th>
<th>$\gamma_{\text{set}}$</th>
<th>winner</th>
<th>author</th>
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<tr>
<td>$\mathbb{Z}_{\geq 1}$</td>
<td>+0.5772...</td>
<td></td>
<td>Euler</td>
</tr>
<tr>
<td>$n = a^2 + b^2$</td>
<td>-0.1638...</td>
<td>Ramanujan</td>
<td>Shanks</td>
</tr>
<tr>
<td>$3 \nmid \tau$</td>
<td>+0.5349...</td>
<td>Landau</td>
<td>M.</td>
</tr>
<tr>
<td>$5 \nmid \tau$</td>
<td>+0.3995...</td>
<td>Ramanujan</td>
<td>M.</td>
</tr>
<tr>
<td>$7 \nmid \tau$</td>
<td>+0.2316...</td>
<td>Ramanujan</td>
<td>M.</td>
</tr>
<tr>
<td>$23 \nmid \tau$</td>
<td>+0.2166...</td>
<td>Ramanujan</td>
<td>M.</td>
</tr>
<tr>
<td>$691 \nmid \tau$</td>
<td>+0.5717...</td>
<td>Landau</td>
<td>M.</td>
</tr>
<tr>
<td>$q \nmid \varphi, q \leq 67$</td>
<td>&lt; 0.4977</td>
<td>Ramanujan</td>
<td>FLM</td>
</tr>
<tr>
<td>$q \nmid \varphi, q \geq 71$</td>
<td>&gt; 0.5023</td>
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</tbody>
</table>
Connection with $E\mathcal{K}_\mathbb{Q}(\zeta_q)$

Put $f_p = |\langle p \pmod{q} \rangle|$. 

$$S(q) := \sum_{p \neq q, f_p \geq 2} \frac{\log p}{p^{f_p} - 1},$$

We have

$$\gamma_{\varphi; q} = \gamma - \frac{(3 - q) \log q}{(q - 1)^2(q + 1)} - S(q) - \frac{E\mathcal{K}_\mathbb{Q}(\zeta_q)}{q - 1}$$

We have $S(q) \leq (\log q + 1)/2q$ (fairly easy).
Connection with $\mathcal{EK}_\mathbb{Q}(\zeta_q)$

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Given $\epsilon > 0$ we have $S(q) < \epsilon/q$ for a subset of primes of natural density 1.

Proof uses linear forms in logarithms in 3 variables (Matveev’s estimate).
\[ \mathcal{E} K_K = \lim_{x \to \infty} \left( \log x - \sum_{N_p \leq x} \frac{\log N_p}{N_p - 1} \right) \]

\[ \tilde{\zeta}_K(s) = \tilde{\zeta}_K(1 - s) \]

\[ \tilde{\zeta}_K(s) = \tilde{\zeta}_K(0)e^{\beta_K s} \prod_{\rho} \left(1 - \frac{s}{\rho}\right)e^{s/\rho} \]

\[ -\beta_K = \sum_{\rho} \frac{1}{\rho} \]

\[ -\beta_K = \mathcal{E} K_K - (r_1 + r_2) \log 2 + \frac{\log |D_K|}{2} - \frac{[K : \mathbb{Q}]}{2} (\gamma + \log \pi) + 1 \]

**Theorem.** (Ihara, 2006). *Under GRH we have*

\[ -c_1 \log |D_K| \leq \mathcal{E} K_K \leq c_2 \log \log |D_K| \]
\[ \mathcal{E}K_{\mathbb{Q}(\zeta_q)} \]
$\mathcal{EK}_Q(\zeta_q)$

Since $\zeta_Q(\zeta_q)(s) = \zeta(s)\prod_{\chi \neq \chi_0} L(s,\chi)$, we have

$\gamma_q = \gamma_q + \sum_{\chi \neq \chi_0} L'(1,\chi) L(1,\chi)$

Ihara's result implies, on GRH,

$-c_1 q \log q \leq \gamma_q \leq c_2 \log (q \log q)$

Badzyan (2010). On GRH, we have

$|\gamma_q| = O\left(\log q \log \log q\right)$

Ihara (2009).

(i) $\gamma_q > 0$ ('very likely')

(ii) Conjectures that

$1/2 - \epsilon \leq \gamma_q \log q \leq 3/2 + \epsilon$

for $q$ sufficiently large
Since \( \zeta_{\mathbb{Q}(\zeta q)}(s) = \zeta(s) \prod_{\chi \neq \chi_0} L(s, \chi) \), we have

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\gamma_q = \gamma + \sum_{\chi \neq \chi_0} \frac{L'(1, \chi)}{L(1, \chi)}
\]

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for \( q \) sufficiently large
\[ \frac{\gamma_q}{\log q} \text{ for } q \leq 50000 \]
Our results on $\gamma_q$

We have $\gamma_{964477901} = -0.1823\ldots$
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**Theorem.** On a quantitative version of the prime $k$-tuple conjecture we have

$$\lim_{q \to \infty} \inf \frac{\gamma_q}{\log q} = -\infty$$

**Conjecture.** For density 1 sequence of primes we have

$$1 - \epsilon < \frac{\gamma_q}{\log q} < 1 + \epsilon$$

(That is, $\gamma_q$ has normal order $\log q$)
Our results on \( \gamma_q \)

We have \( \gamma_{964477901} = -0.1823 \ldots \)

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\lim_{q \to \infty} \inf \frac{\gamma_q}{\log q} = -\infty
\]

**Conjecture.** For density 1 sequence of primes we have

\[
1 - \epsilon < \frac{\gamma_q}{\log q} < 1 + \epsilon
\]

(That is, \( \gamma_q \) has normal order \( \log q \))

We have

\[
\lim_{q \to \infty} \sup \frac{\gamma_q}{\log q} = 1
\]
Sketch of proof of theorem

On ERH we have (Ihara)

\[ \gamma_q = 2 \log q - q \sum_{\substack{p \leq q^2 \\ p \equiv 1 \pmod{q}}} \frac{\log p}{p - 1} + O(\log \log q) \]

Construct infinite sequence \( b_i, i = 1, 2, \ldots \) such that \( n, 1 + 2b_1 n, 1 + 2b_2 n, \ldots \) satisfies conditions of prime \( k \)-tuple conjecture AND

\[ \sum_{i=1}^{s} \frac{1}{b_i} \rightarrow \infty \]

Take \( s \) so large that sum is > 4.

By prime \( k \)-tuplet conjecture \( q, 1 + 2b_1 q, 1 + 2b_2 q, \ldots, 1 + 2b_s q \) are infinitely often ALL prime with \( 1 + 2b_s q \leq q^2 \). Then
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\[ q \sum_{\substack{p \leq q^2 \\ p \equiv 1 (\text{mod } q)}} \frac{\log p}{p - 1} > q \log q \sum_{i=1}^{s} \frac{1}{2b_i q} > (2 + \epsilon_0) \log q \]
Analogy with Kummer’s Conjecture

Kummer conjectured that

\[ h_1(p) = \frac{h(p)}{h_2(p)} \sim G(p) := 2p \left( \frac{p}{4\pi^2} \right)^{\frac{p-1}{4}} \]

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Granville: The quantities \( h_1(p)/G(p) \) and \( \gamma_q/\log q \) are (analytically) very similar.

Some of our lemmas can be already found in Granville, Inventiones, 1990.

In particular, he proved there that \( \sum_i \frac{1}{b_i} \) diverges.
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This solved a conjecture of Erdős from 1988.
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HAPPY RETIREMENT, HERMAN!
THANK YOU!