What Stochastic Dynamics can do for Space Weather

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ABSTRACT

Heliospheric and near-Earth plasmas, the evolution and interaction of which determine what we refer to as Space Weather, are open dynamical systems undergoing the action of irregular forcers, well representable as stochastic terms in equations.

The presence of stochastic terms in equations of space plasmas may be due to two reasons: on the one hand, any space plasma system interacts with spatially external forcers of unknown precise configuration (for instance, near-Earth plasma are forced by the solar wind fluctuations, triggered in turn by impulsive, unpredictable events on the Sun, well described probabilistically); on the other hand, any level of description of the plasma, e.g. MHD, multi-fluid, kinetic or other, use dynamical variables interacting with "modes" evolving on smaller time and space scales, that can be encoded in noise terms.

In this contribution, some examples are given of space plasma dynamics, relevant to Space Weather, cast into the form of Langevin equations, i.e. ODEs or PDEs, with stochastic terms of known statistics, making dynamical variables evolve probabilistically. Once Langevin equations are written for a plasma system, the statistical dynamics of the latter can be formulated through a functional formalism, based on path integrals, in which it is possible to calculate the probability of a particular evolution of the system.

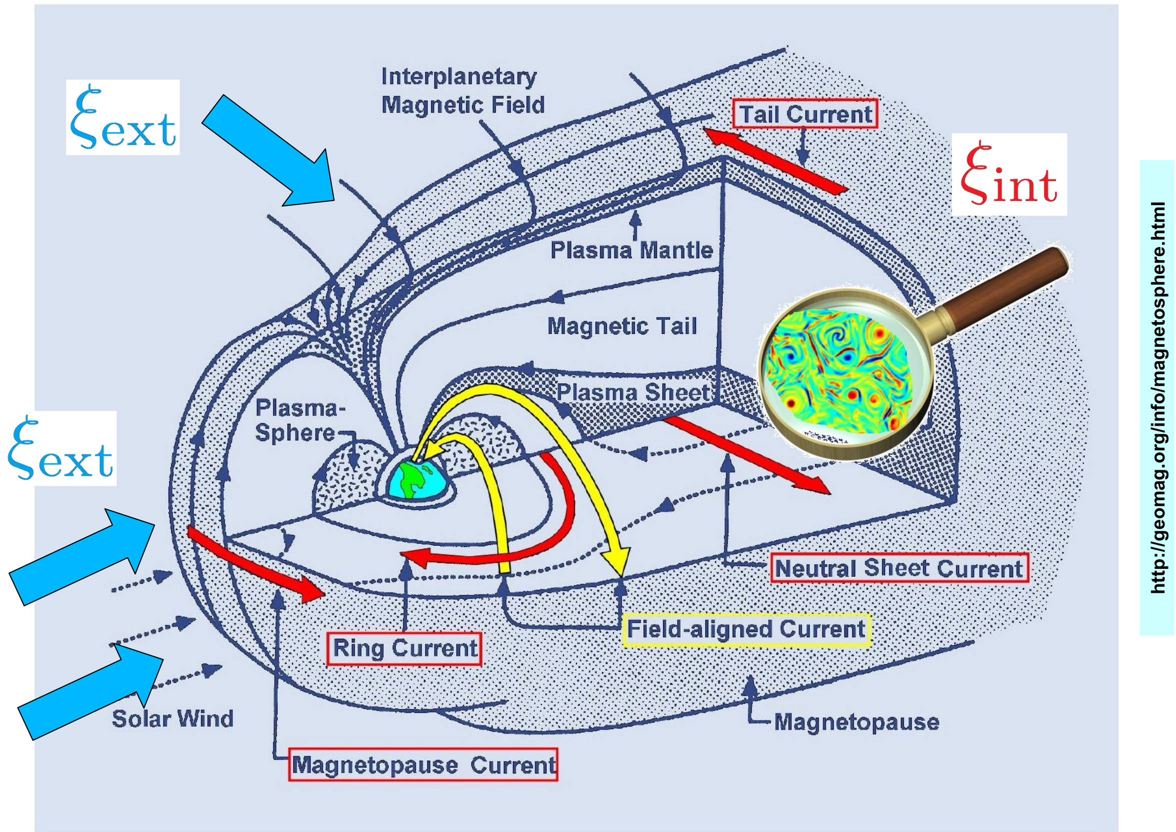
Finally, a roadmap, for the extensive use of these tools for Space Weather applications, is traced.

1. SPACE WEATHER AND NOISE

Any dynamical representation of the physics of near-Earth space plasma (NESP) must deal with two different kinds of noise:

1)- Sun's forcing on the system, in terms of Solar Wind and radiation, highly erratic in time and space: external noise ξ_{ext} ;

2)- At any given time or space scale ℓ erratic fluctuations from processes taking place at smaller scales emerge: internal noise ξ_{int} .



Effects of perturbations in solar-terrestrial relationship (e.g., CME, flares, irregularity in solar cycle) are represented via ξ_{ext} , while ξ_{int} is more suitable to represent the effects of turbulence and matter granularity.

2. FUNCTIONAL STOCHASTIC THEORIES

Noises may be included systematically in the dynamics of NESP systems resorting to the functional formalism for statistical dynamics (Phythian, 1977). The starting point is a set of equations of motion in which the system variables ψ evolve under the action of both deterministic terms $\Lambda(\psi)$, additive noises f and multiplicative noises g:

$$\dot{\psi} = \Lambda(\psi) + f + g \cdot \Gamma(\psi)$$

As a probability measure is given on the sample space of noise evolutions ($f(t)$,$g(t)$), a stochastic effective action $S[\psi]$ is calculated, encoding the whole statistical dynamics as a consequence of the statistical properties of noises. This $S[\psi]$ determines the probability measure $A[\psi]$ on the sample space of physical variables trajectories $\psi(t)$, taking place probabilistically:

$$A[\psi; t_0, t) = N_0 \exp\left(-iS[\psi; t_0, t)\right)$$

This $A[\psi]$ is the kernel of the stochastic path integral encoding all possible behaviours permitted to the system by the match between necessity (terms in Λ) and chance (noises f and g). Using this kernel, or its version $D[\psi,\chi]$ depending more explicitly on noise characteristic functional, it is possible to construct multi-point/time correlation functions and transition probabilities for physical variables, or the response functions with respect to noises:

$$A[\psi; t_0, t) = \int [d\chi]\, D[\psi, \chi; t_0, t)\,,$$

$$D[\chi, \psi; t_0, t) = N_0(t_0, t) \left\langle e^{-iS_{f,g}[f,g,\psi,\chi;t_0,t)} \right\rangle_{f,g} e^{-iS_0[\psi,\chi;t_0,t)}\,,$$

$$G_{\psi}^{I_1 \dots I_n}(\vec{x}_1, t_1, \dots \vec{x}_n, t_n) = \int [d\psi] \prod_{i=1}^{n} \psi^{I_i}(\vec{x}_i, t_i)\, A[\psi; t_0, t) \qquad \textit{corr. function}$$

$$K^{I}{}_{J}(\vec{x}_1, t_1, \vec{x}_2, t_2) = \int [d\psi] \frac{\delta \psi^I(\vec{x}_1, t_1)}{\delta f^J(\vec{x}_2, t_2)} A[\psi; t_0, t) \qquad \textit{response function}\,,$$

$$\mathcal{P}_{\psi_{\mathrm{i}} \to \psi_{\mathrm{f}}}(t_0, t) = \int_{\substack{\psi(t_0)=\psi_{\mathrm{i}} \\ \psi(t)=\psi_{\mathrm{f}}}} [d\psi]\, A[\psi; t_0, t) \qquad \textit{transition probability}$$

3. STOCHASTIC DYNAMICS APPLIED TO NESP

Some examples may be quoted in which the agenda in § 2 is applied to NESP systems. Other ones are under study.

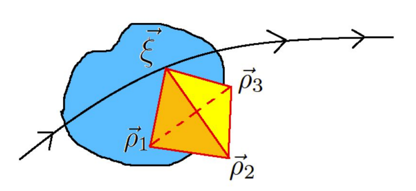
1)- Stochastic resistive MHD in local variables (Materassi & Consolini, 2008). MHD turbulent irregularities in density ρ, pressure p and current J are interpreted as additive and multiplicative internal noises Ξ , Θ and Δ:

$$\begin{cases} \partial_t B^i = B^j \partial_j V^i - B^i \partial_j V^j - V^j \partial_j B^i + \Xi^i, \\ \partial_t V^i = V^i \partial_j V^j - V^j \partial_j V^i + \Delta_j B_k \epsilon^{jki} + \Theta^i \\ \Xi^i \stackrel{\mathrm{def}}{=} -\epsilon^{ijk} \partial_j \left(\zeta_{kh} J^h\right), \ \Theta^i \stackrel{\mathrm{def}}{=} -\dfrac{\partial^i p}{\rho}, \ \Delta^i \stackrel{\mathrm{def}}{=} \dfrac{J^i}{\rho} \end{cases}$$

$$\begin{aligned} S_{f,g} = -\int_{t_0}^{t} dt \int d^3x \, [&\Xi^i(\vec{x},\tau)\,\Omega_i(\vec{x},\tau) + \\ &+\Theta^i(\vec{x},\tau)\,\Pi_i(\vec{x},\tau) + \epsilon^{ijk}\Delta_i(\vec{x},\tau)\,\Pi_j(\vec{x},\tau)\,B_k(\vec{x},\tau)] \end{aligned}$$

$$\begin{aligned} S_0 = \int_{t_0}^{t} dt \int d^3x \, \Big(&\Omega_i \dot{B}^i + \Pi_i \dot{V}^i + \\ &+ \left(B^i \partial_j V^j + V^j \partial_j B^i - B^j \partial_j V^i\right) \Omega_i + \\ &+ \left(V^j \partial_j V^i - V^i \partial_j V^j\right) \Pi_i - \frac{5}{2} i \partial_i V^i \Big) \end{aligned}$$

2)- Stochastic tetrad MHD in material variables (Materassi & Consolini, 2015). The evolution of a finite size parcel of plasma is represented along its material motion via the tetrad formalism (Chertkov et al., 1999). One has Langevin equations for coarse grained kinetic gradient M, magnetic gradient W and tetrad shape matrix ρ. Noises (U,h,θ) represent internal turbulence and chaotic interaction with nearby parcels of the plasma (internal noise):



$$\begin{cases} \dot{\varrho} = \varrho^{\dagger} \cdot M + U, \quad \Xi = W - W^{\dagger}, \\ \dot{M} = (\alpha_V - 1) \left\{ M^2 - \left[\mathrm{Tr}\left(M^2\right) - \dfrac{\mathrm{Tr}\left(\Xi^2\right)}{2\rho} \right] \Pi - \dfrac{\Xi \cdot W}{\rho} \right\} + h, \\ \dot{W} = (\alpha_B - 1)\,[W, M] + \theta \end{cases}$$

$$A[\varrho, M, W; t_0, t) = N_0(t_0, t)\, e^{-iS[\varrho, M, W; t_0, t)},$$

$$\begin{aligned} S = -\frac{i}{2} \int_{t_0}^{t} d\tau \Bigg\{ &\frac{\mathrm{Tr}\left[\left(\dot{\varrho} - \varrho^{\dagger} \cdot M\right)^{\dagger} \cdot \left(C_{//}^{-1} r + C_{\perp}(I - r)\right) \cdot \left(\dot{\varrho} - \varrho^{\dagger} \cdot M\right)\right]}{2\rho^2 \sqrt{\boldsymbol{M}^2}} + \\ &+ \frac{\boldsymbol{\rho}^2}{2C_h} \left\| \dot{M} - (\alpha_v - 1) \left[M^2 - \left(\boldsymbol{M}^2 - \frac{\boldsymbol{\Xi}^2}{2\rho} \right) \Pi(\boldsymbol{\rho}) - \frac{\Xi \cdot W}{\rho} \right] \right\|^2 + \\ &+ \frac{\boldsymbol{\rho}^2}{2C_\theta} \left\| \dot{W} - (\alpha_B - 1)\,[W, M] \right\|^2 - 2(\alpha_v - 1)\,\mathrm{Tr}\left(\Pi(\boldsymbol{\rho}) \cdot M\right) \Bigg\} \end{aligned}$$

3)- Many fluid representation of the Earth's ionosphere. In the evolution PDEs of the I-th chemical species forming the ionosphere, terms may be indicated the erratic aspect of which depends both on smaller scale irregularities (small scale fluctuations, internal noise) and on external forcing of solar, or space, origin (external noise):

$$\partial_t \vec{V}_I + \left(\vec{V}_I \cdot \vec{\partial}\right) \vec{V}_I + \frac{\Gamma_I}{N_I} \vec{V}_I = \vec{g}_{eff} + \frac{q_I}{m_I} \vec{E} + \frac{q_I}{m_I} \vec{V}_I \times \vec{B} +$$

$$-\frac{1}{m_I N_I} \vec{\partial} p_I - \sum_{J \neq I} \nu_{IJ} \left(\vec{V}_I - \vec{V}_J\right)$$

a)- Γ/N and N are influenced by local noise (turbulence) and Sun's action (flares, particle precipitation); b)- E and B may be both locally turbulent as influenced by external shocks, penetrating E-fields etc.; c)- pressure gradients and collision frequencies may get irregular due to turbulence (internal noise). *Full theory still to be done…*

4. WHAT CAN STOCHASTIC APPROACH ADD/TEACH TO SPACE WEATHER SCIENCE?

NESP systems are open systems characterized by high dimensional chaos, due to highly non-linear couplings and the overwhelming number of degrees of freedom involved. As a result, every space-weather proxy will look (weakly or strongly) erratic.

The stochastic formalism is an attempt to put under quantitative control the erratic fluctuations, letting them enter the equations explicitly. Doing so, one hopes to include "high order effects" in a strongly consistent way, e.g. performing expansions with respect to coupling constants instead of using perturbations in the solutions.

In Langevin equations, the chance vs necessity interplay (i.e. f and g vs Λ and Γ) becomes very transparent, and the relevance of interactions to the importance of fluctuations is put under control.

When functional formalism is used, transition probabilities and response functions become (in principle) calculable. In principle, if the equations are well written to a satisfactory extent, one should be able to predict theoretically the observed statistics of NESP quantities.

NEED OF WIDE DATABASES, AND CLEVER TOOLS TO INTERROGATE THEM…

5. A ROADMAP FOR SYNERGETIC SPACE WEATHER

In Haken (1983) the study of non-linear deterministic dynamics undergoing stochastic forcing is designed as "Synergetics": hence, we will refer to the present approach as Synergetic Space Weather (SSW). In order to obtain from SSW the profits described in § 4 above, we suggest the following roadmap.

A. CONSTRUCT STOCHASTIC DYNAMICAL MODELS. Dynamical models relevant to Space Weather should be re-formulated as noisy dynamical systems, to which the formalism in § 2 may be applied.

B. FEED MODELS WITH REAL NOISE DATA. Models need noise statistics in order to be usable: those will be provided by direct or indirect measurements of the fluctuations ξ_{int} and ξ_{ext} involved. Such information will be converted into the best fitting mathematical shapes of the distributions of (f,g) in § 2.

C. VALIDATE MODELS WITH PAST DATA, MAKE PREDICTIONS. Theoretical predictions of SSW models will be contrasted with historical series of NESP quantities, so to *validate and refine* their mathematics and physics. The ultimate goal is to be able to make predictions for reasonable scenarios of Space Weather events using SSW models.

Point A requires a great theoretical effort, both in analytical and numerical terms.

More than "new data", points B and C require tools for intelligent inspection of already existing large data collections. Nowadays, some decades of radio sounding, radar and in situ measurements throughout near-Earth and interplanetary plasmas exist: their most convenient use for SSW just needs our capacity to ask the right physical questions and dig with suitably clever algorithms into big data.

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