Understanding Power Flow Solutions: History, Practice, Theory, Progress

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Nonlinear Systems

\[ \Sigma_1 \]

\[ v_1 \quad \Sigma_1 \quad y_1 \]

\[ y_2 \]

\[ \Sigma_2 \]

\[ v_2 \]

Feedback-Based Optimization

\[ \text{Inputs} \rightarrow \text{Process} \rightarrow \text{Measurements} \]

Optimization Alg.

\[ u_{k+1} = \text{Proj}_C (u_k - \alpha \nabla f(u_k, y_k)) \]

Network Dynamics & Control

3.3. Paths and connectivity in digraphs

(a) A periodic digraph with period 2

(b) An aperiodic digraph with cycles of length 1 and 2.

(c) An aperiodic digraph with cycles of length 2 and 3.

Figure 3.6: Example periodic and aperiodic digraphs.

3.3.3 Condensation digraphs

[Strongly connected components]

A subgraph \( H \) is a strongly connected component of \( G \) if \( H \) is strongly connected and any other subgraph of \( G \) strictly containing \( H \) is not strongly connected.

[Condensation digraph]

The condensation digraph of a digraph \( G \), denoted by \( C(G) \), is defined as follows: the nodes of \( C(G) \) are the strongly connected components of \( G \), and there exists a directed edge in \( C(G) \) from node \( H_1 \) to node \( H_2 \) if and only if there exists a directed edge in \( G \) from a node of \( H_1 \) to a node of \( H_2 \). The condensation digraph has no self-loops. This construction is illustrated in Figure 3.7.

(a) An example digraph \( G \)

(b) The strongly connected components of the digraph \( G \)

(c) The condensation digraph \( C(G) \)

Figure 3.7: An example digraph, its strongly connected components and its condensation digraph.

Lemma 3.2 (Properties of the condensation digraph).

For a digraph \( G \) and its condensation digraph \( C(G) \),

(i) \( C(G) \) is acyclic,

(ii) \( G \) is weakly connected if and only if \( C(G) \) is weakly connected, and

(iii) the following statements are equivalent:

a) \( G \) contains a globally reachable node,

b) \( C(G) \) contains a globally reachable node, and

c) \( C(G) \) contains a unique sink.
Prof. J. W. Simpson-Porco: energy systems

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Power Flow Analysis & Algorithms

Renewable Energy Integration

Microgrid Control & Optimization

Next-Generation Hierarchical Control
Problems in power system operations

Power Flow Analysis

Contingency Analysis [Hines et al.]

Optimal Power Flow [Molzahn et al.]

Transient Stability [Overbye et al.]
Modeling AC power flow

1. **Network Graph:** \((\mathcal{N}, \mathcal{E})\), complex weights \(y_{ij} = g_{ij} + jb_{ij}\)

2. **Nodal Variables:** voltage \(V_i e^{j\theta_i}\), power \(S_i = P_i + jQ_i\)

3. **Coupling Laws:** Kirchhoff & Ohm

4. **Admittance Matrix:** \(Y = G + jB = \text{Laplacian-like w/ weights } y_{ij}\)

5. **Lossless Lines:** \(G_{ij} = 0\)
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\[
\begin{align*}
P_i + j Q_i & \rightarrow Y_{ij} \rightarrow P_j + j Q_j \\
V_i e^{j\theta_i} & \quad y_{ij} \quad V_j e^{j\theta_j}
\end{align*}
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   \]

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- active power: $P_i = \sum_j V_i V_j B_{ij} \sin(\theta_i - \theta_j) + V_i V_j G_{ij} \cos(\theta_i - \theta_j)$
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   \[ V_i e^{j\theta_i} \quad y_{ij} \quad V_j e^{j\theta_j} \]

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![Diagram](image)

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6 Loads (●) and m Generators (□) \( \mathcal{N} = \mathcal{N}_L \cup \mathcal{N}_G \)

7 Load Model: PQ bus constant \( P_i \) constant \( Q_i \)

8 Generator Model: PV bus constant \( P_i \) constant \( V_i \),

\[ P_i = \sum_j V_i V_j B_{ij} \sin(\theta_i - \theta_j), \quad i \in \mathcal{N}_L \cup \mathcal{N}_G \]
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2n + m equations in variables \( \theta \in \mathbb{T}^{n+m} \) and \( V_L \in \mathbb{R}^n_{>0} \).
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Why study solvability of power flow problems?

1. Because it is interesting to do so

2. Numerical methods
   - understand convergence, divergence, and initialization issues

   - State vector: \( x = (\theta, V) \)
   
   - Newton iteration:
     \[
     x^{k+1} = x^k - J(\theta^k, V^k)^{-1} f(x^k)
     \]

3. Optimal power flow

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Motivation I: numerical methods for power flow

Power flow always solved with variant of Newton iteration

\[ x = (\theta \ V_L)^T, \quad x^{k+1} = x^k - J(x^k)^{-1} f(x^k). \]

- If **convergent**, may converge to “wrong” solution
- If **non-convergent**, several possibilities:
  1. No power flow solution exists
  2. Numerical instability (conditioning)
  3. \( x^0 \) not in any region of convergence
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To differentiate, need theory of power flow solvability
Motivation II: multimachine transient stability

Constrained Swing Dynamics

\[
\begin{align*}
\dot{\theta}_i &= \omega_i \\
M_i \dot{\omega}_i &= -D_i \omega_i + P_i - \sum_j V_i V_j B_{ij} \sin(\theta_i - \theta_j)
\end{align*}
\]

Challenge: Characterize equilibria, stability, basin of attraction

Approaches: Energy functions, nearest unstable eq. point, S.O.S., …
Motivation II: multimachine transient stability

Constrained Swing Dynamics

\begin{align*}
\text{Gen:} \quad & \dot{\theta}_i = \omega_i \\
& M_i \ddot{\omega}_i = -D_i \omega_i + P_i - \sum_j V_i V_j B_{ij} \sin(\theta_i - \theta_j) \\
\text{Load:} \quad & D_i \dot{\theta}_i = P_i - \sum_j V_i V_j B_{ij} \sin(\theta_i - \theta_j) \\
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\{Equilibria\} = \{Power Flow Solutions\}
Motivation III: optimal power flow

**Idea:** Optimally match supply and demand (with constraints)

\[
\text{minimize} \quad \theta, \ V, \ P_G, \ X_i \in \mathcal{N} \\
\text{subject to} \quad P_i = X_j V_i V_j B_{ij} \sin(\theta_i - \theta_j) \\
Q_i = -X_j V_i V_j B_{ij} \cos(\theta_i - \theta_j) \\
V_{\text{min}} \leq V_i \leq V_{\text{max}} \\
S_{\text{min}} \leq |P_i + jQ_i| \leq S_{\text{max}} \\
s_{\text{min}} \leq |p_{i \rightarrow j} + jq_{i \rightarrow j}| \leq s_{\text{max}} \quad (i, j) \in \mathcal{E}, \\
\text{non-convex, solved every 5-15 min. via linearization,} \quad (\quad )
\]

"Today, 50 years after the problem was formulated, we still do not have a fast, robust solution technique for the full ACOPF." — Richard P. O'Neill (Chief Economic Advisor, FERC, 2016)
Motivation III: optimal power flow

**Idea:** Optimally match supply and demand (with constraints)

$$\text{minimize} \quad \sum_{i \in \mathcal{N}_G} f_i(P_i)$$

subject to

$$P_i = \sum_j V_i V_j B_{ij} \sin(\theta_i - \theta_j) \quad i \in \mathcal{N}_L \cup \mathcal{N}_G,$$

$$Q_i = -\sum_j V_i V_j B_{ij} \cos(\theta_i - \theta_j) \quad i \in \mathcal{N}_L,$$

$$V_{i \min} \leq V_i \leq V_{i \max} \quad i \in \mathcal{N}_L,$$

$$S_{i \min} \leq |P_i + jQ_i| \leq S_{i \max} \quad i \in \mathcal{N}_G,$$

$$s_{ij \min} \leq |p_{i \to j} + jq_{i \to j}| \leq s_{ij \max} \quad (i, j) \in \mathcal{E},$$
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& \quad Q_i = -\sum_j V_i V_j B_{ij} \cos(\theta_i - \theta_j) \quad i \in \mathcal{N}_L, \\
& \quad V_i^{\min} \leq V_i \leq V_i^{\max} \quad i \in \mathcal{N}_L, \\
& \quad S_i^{\min} \leq |P_i + jQ_i| \leq S_i^{\max} \quad i \in \mathcal{N}_G, \\
& \quad s_{ij}^{\min} \leq |p_{i\to j} + jq_{i\to j}| \leq s_{ij}^{\max} \quad (i, j) \in \mathcal{E},
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- non-convex, solved every 5-15 min. via linearization, (**$$**)

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& \quad V_i^{\min} \leq V_i \leq V_i^{\max} \quad i \in \mathcal{N}_L, \\
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“Today, 50 years after the problem was formulated, we still do not have a fast, robust solution technique for the full ACOPF.”

— Richard P. O’Neill (Chief Economic Advisor, FERC, 2016)
Motivation III: optimal power flow

**Idea:** Optimally match supply and demand (with constraints)

\[
\text{minimize} \quad \sum_{i \in \mathcal{N}_G} f_i(P_i)
\]

subject to

\[
P_i = \sum_j V_i V_j B_{ij} \sin(\theta_i - \theta_j) \quad i \in \mathcal{N}_L \cup \mathcal{N}_G,
\]

\[
Q_i = -\sum_j V_i V_j B_{ij} \cos(\theta_i - \theta_j) \quad i \in \mathcal{N}_L,
\]

\[
V_i^{\text{min}} \leq V_i \leq V_i^{\text{max}} \quad i \in \mathcal{N}_L,
\]

\[
S_i^{\text{min}} \leq |P_i + jQ_i| \leq S_i^{\text{max}} \quad i \in \mathcal{N}_G,
\]

\[
s_{ij}^{\text{min}} \leq |p_{i \rightarrow j} + jq_{i \rightarrow j}| \leq s_{ij}^{\text{max}} \quad (i, j) \in \mathcal{E},
\]

- non-convex, solved every 5-15 min. via linearization, (\\$\\$\\$\))

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Intuition on power flow solutions

1. ‘Normally’, exists unique high-voltage soln:
   - voltage magnitude $V_i \approx 1$
   - phase diff $|\theta_i - \theta_j| \ll 1$
   - current flows from high $V$ to low $V$!

2. Lightly loaded systems: many low-voltage solutions

3. Heavily loaded systems: Few solutions or infeasible
   - saddle node bifurcations
   - maximum power transfer limit
   - non-convex feasible set in $(P, Q)$-space

[Josz et al.]
Intuition on power flow solutions

1. ‘Normally’, exists unique **high-voltage** soln:
   - voltage magnitude $V_i \simeq 1$
   - phase diff $|\theta_i - \theta_j| \ll 1$
   - current flows from high $V$ to low $V$!

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Literature on Power Flow Solvability

Given data: network topology, impedances, generation & loads

Q: ∃ “stable high-voltage” solution? unique? properties?

Many approaches over \(45+\) years of literature:

- [Weedy '67]: Jacobian singularity
- [Korsak '72]: Multiple “stable” solutions
- [Wu & Kumagai '77, '80, '82]: Fixed-point analysis of existence
- [Araposthatis, Sastry & Varaiya, '81]: Jacobian analysis
- [Baillieul and Byrnes '82]: Counting # of solutions, Bezout/Morse analysis
- [Illic '86, '92]: “no-gain” results, nonlinear resistive networks
- [Makarov, Hill & Hiskens '00]: Solution insights for general quadratic equations
- [Dörfler, Chertkov & Bullo '12]: Existence/uniqueness for lossless \(P/\theta\) problem
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Many approaches over 45+ years of literature:

Main insight: stiffness vs. loading

1. Stiff network + light loading $\Rightarrow$ feasible
2. Weak network + heavy loading $\Rightarrow$ infeasible

Q: How to quantify network stiffness vs. loading?
Literature on Power Flow Solvability

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**Main insight: stiffness vs. loading**

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Q: How to quantify network stiffness vs. loading?
Solution of two-bus system

\[ P_L = b V_G V_L \sin(-\eta) \]
\[ P_G = b V_G V_L \sin(\eta) \]
\[ Q_L = b V_L^2 - b V_L V_G \cos(\eta) \]

**Figure 2.6** Voltage as a function of load active and reactive powers
Solution of two-bus system

\[ p = bV_G V_L \sin(\eta) \]
\[ Q_L = bV_L^2 - bV_L V_G \cos(\eta) \]

1 Change Variables

\[ \nu := \frac{V_L}{V_G} \quad \Gamma := \frac{p}{bV_G^2} \quad \Delta := \frac{Q_L}{-\frac{1}{4} bV_G^2} \]

2 Square equations, add, and solve quadratic in \( \nu^2 \)

\[ \nu_{\pm} = \sqrt{\frac{1}{2} \left( 1 - \frac{\Delta}{2} \pm \sqrt{1 - (4\Gamma^2 + \Delta)} \right)} \]

3 Nec. & Suff. Condition

\[ 4\Gamma^2 + \Delta < 1 \]
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1. **High-voltage** solution
   \[ v_+ \in \left[ \frac{1}{2}, 1 \right) \]

2. **Low-voltage** solution
   \[ v_- \in \left[ 0, \frac{1}{\sqrt{2}} \right) \]

Angle: \( \sin(\eta_\pm) = \frac{\Gamma}{v_\pm} \)

1. **Small-angle** solution
   \[ \eta_- \in \left[ 0, \frac{\pi}{4} \right) \]

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   \[ \eta_+ \in \left[ 0, \frac{\pi}{2} \right) \]
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Solution of two-bus system

- Squaring and adding equations does not generalize to networks.
- Is there any hope then?

\[ \Gamma = v \sin(\eta) \]
\[ \Delta = -4v^2 + 4v \cos(\eta) \]

- Use \( \cos(\eta) = \sqrt{1 - \sin^2(\eta)} \) \[ \Rightarrow \quad \Delta = -4v^2 + 4v \sqrt{1 - \left(\frac{\Gamma}{v}\right)^2} \]
- Rearrange to get fixed-point equation

\[ v = f(v) := -\frac{1}{4} \frac{\Delta}{v} + \sqrt{1 - \left(\frac{\Gamma}{v}\right)^2} \]

This generalizes! Leverage intuition.
Solution of two-bus system

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Notation I: branches and bus types

\[ P_i = \sum_j V_i V_j B_{ij} \sin(\theta_i - \theta_j), \quad i \in \mathcal{N}_L \cup \mathcal{N}_G \]

\[ Q_i = -\sum_j V_i V_j B_{ij} \cos(\theta_i - \theta_j), \quad i \in \mathcal{N}_L \]

Bus partitioning \( \mathcal{N} = \mathcal{N}_L \cup \mathcal{N}_G \) induces **branch partitioning**

\[ \mathcal{E} = \mathcal{E}^{ll} \cup \mathcal{E}^{gl} \cup \mathcal{E}^{gg}, \quad A = \begin{pmatrix} A_L \\ A_G \end{pmatrix} = \begin{pmatrix} A_L^{ll} & A_L^{gl} & 0 \\ 0 & A_G^{gl} & A_G^{gg} \end{pmatrix} \]
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Notation II: open-circuit voltages

\[ P_i = \sum_j V_i V_j B_{ij} \sin(\theta_i - \theta_j), \quad i \in \mathcal{N}_L \cup \mathcal{N}_G \]

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- **Generators** \( \mathcal{N}_G \): \( V_i \) fixed
- **Loads** \( \mathcal{N}_L \): \( V_i \) free

Partitioned Variables

\[ V = \begin{pmatrix} V_L \\ V_G \end{pmatrix}, \quad B = \begin{pmatrix} B_{LL} & B_{LG} \\ B_{GL} & B_{GG} \end{pmatrix} \]
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**Open-circuit voltages**

\[ V_L^* \triangleq -B_{LL}^{-1} B_{LG} \cdot V_G \]

"Generators → Loads"
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**Open-circuit voltages**

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\[ v_i \triangleq \frac{V_i}{V_i^*} \]
Notation III: stiffness matrices

\[ V = \left( \begin{array}{c} V_L \\ V_G \end{array} \right), \quad B = \left( \begin{array}{c|c} B_{LL} & B_{LG} \\ \hline B_{GL} & B_{GG} \end{array} \right), \quad V_L^* = -B_{LL}^{-1} B_{LG} V_G \]

- Need to non-dimensionalize power flow equations
- Stiffness matrices quantify grid strength in units of power

1. **Nodal** stiffness matrix

   \[ S \triangleq \frac{1}{4} [V_L^*] \cdot B_{LL} \cdot [V_L^*] \]

2. **Branch** stiffness matrix

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Active power flow reformulation

**Notation:**

\[
h_e(v) = \begin{cases} 
  v_i v_j & \text{if } e = (i, j) \in \mathcal{E}^{\ell}\ell \\
  v_j & \text{if } e = (i, j) \in \mathcal{E}^g\ell \\
  1 & \text{if } e = (i, j) \in \mathcal{E}^{gg}
\end{cases}
\]

**Active Power:**

\[
P_i = \sum_j V_i V_j B_{ij} \sin(\theta_i - \theta_j)
\]

\[
P = \underbrace{A}_{\text{Incidence}} \underbrace{D}_{\text{Branch Stiff.}} \begin{bmatrix} h(v) \end{bmatrix} \begin{bmatrix} \sin(A^T \theta) \end{bmatrix}
\]

- Let columns of \( C \) be a basis for \( \ker(A) \), let \( p_c \in \mathbb{R}^c \)

**Semi-Explicit Solution**

\[
\sin(A^T \theta) = \psi(v, p_c) \triangleq \begin{bmatrix} h(v) \end{bmatrix}^{-1} \left( A^T L^+ P + D^{-1} C p_c \right)
\]

\[
\emptyset = C^T \text{arcsin}(\psi)
\]
Active power flow reformulation

Notation:

\[ h_e(v) = \begin{cases} 
  v_i v_j & \text{if } e = (i, j) \in \mathcal{E}^\ell\ell \\
  v_j & \text{if } e = (i, j) \in \mathcal{E}^g\ell \\
  1 & \text{if } e = (i, j) \in \mathcal{E}^{gg}
\end{cases} \]

Active Power:

\[ P_i = \sum_j V_i V_j B_{ij} \sin(\theta_i - \theta_j) \]

\[ P = \underbrace{A}_{\text{Incidence}} \underbrace{D}_{\text{Branch Stiff.}} \underbrace{[h(v)] \sin(A^T \theta)}_{\text{Voltages sin(\theta_i - \theta_j)}} \]

- Let columns of \( C \) be a basis for \( \ker(A) \), let \( p_c \in \mathbb{R}^c \)

Semi-Explicit Solution

\[ \sin(A^T \theta) = \psi(v, p_c) \triangleq [h(v)]^{-1} \left( A^T L^\dagger P + D^{-1} C p_c \right) \]

\[ \emptyset = C^T \arcsin(\psi) \]
Active power flow reformulation

Notation:
\[ h_e(v) = \begin{cases} 
  v_i v_j & \text{if } e = (i, j) \in E_{\ell \ell} \\
  v_j & \text{if } e = (i, j) \in E_{g \ell} \\
  1 & \text{if } e = (i, j) \in E_{gg} 
\end{cases} \]

Active Power:
\[ P_i = \sum_j V_i V_j B_{ij} \sin(\theta_i - \theta_j) \]

\[ P = \begin{bmatrix} A & D \end{bmatrix} \begin{bmatrix} [h(v)] \sin(A^T \theta) \end{bmatrix} \]

- Let columns of \( C \) be a basis for \( \ker(A) \), let \( p_c \in \mathbb{R}^c \)

Semi-Explicit Solution
\[ \sin(A^T \theta) = \psi(v, p_c) \triangleq [h(v)]^{-1} \left( A^T L^+ P + D^{-1} C p_c \right) \]
\[ \emptyset = C^T \arcsin(\psi) \]
Active power flow reformulation

Notation:

\[ h_e(v) = \begin{cases} 
v_i v_j & \text{if } e = (i, j) \in \mathcal{E}^{ll} \\
v_j & \text{if } e = (i, j) \in \mathcal{E}^{gl} \\
1 & \text{if } e = (i, j) \in \mathcal{E}^{gg}
\end{cases} \]

Active Power:

\[ P_i = \sum_j V_i V_j B_{ij} \sin(\theta_i - \theta_j) \]

\[
P = \underbrace{A}_{\text{Incidence}} \underbrace{D}_{\text{Branch Stiff.}} \begin{bmatrix} h(v) \end{bmatrix} \begin{bmatrix} \sin(A^T \theta) \end{bmatrix}
\]

- Let columns of \( C \) be a basis for \( \ker(A) \), let \( p_c \in \mathbb{R}^c \)

Semi-Explicit Solution

\[
\sin(A^T \theta) = \psi(v, p_c) \triangleq [h(v)]^{-1} \left( A^T L^\dagger P + D^{-1} C p_c \right)
\]

\[ \Theta = C^T \arcsin(\psi) \]
Active power flow reformulation

Notation:

\[ h_e(v) = \begin{cases} 
  v_i v_j & \text{if } e = (i, j) \in \mathcal{E}^{\ell \ell} \\
  v_j & \text{if } e = (i, j) \in \mathcal{E}^{g \ell} \\
  1 & \text{if } e = (i, j) \in \mathcal{E}^{gg}
\end{cases} \]

Active Power:

\[ P_i = \sum_j V_i V_j B_{ij} \sin(\theta_i - \theta_j) \]

\[ P = \underbrace{A}_{\text{Incidence}} \underbrace{D}_{\text{Branch Stiff.}} \left[ \begin{bmatrix} h(v) \end{bmatrix} \sin(A^T \theta) \right] \]

- Let columns of \( C \) be a basis for \( \ker(A) \), let \( p_c \in \mathbb{R}^c \)

Semi-Explicit Solution

\[ \sin(A^T \theta) = \psi(v, p_c) \triangleq [h(v)]^{-1} \left( A^T L^\dagger P + D^{-1} C p_c \right) \]

\[ \emptyset = C^T \arcsin(\psi) \]
Reactive power flow reformulation
Skipping some details . . .

\[ Q_L = -4[v]S(v - 1_n) + |A|_LD [h(v)] (1|\varepsilon| - \cos(A^T \theta)). \]

- Rearrange for \( v \)

\[
v = f(v, \theta) = 1_n - \frac{1}{4}S^{-1}[Q_L][v]^{-1}1_n + \frac{1}{4}S^{-1}[v]^{-1}|A|_LD [h(v)] (1|\varepsilon| - \cos(A^T \theta)),
\]

- Now plug in \( \cos(z) = \sqrt{1 - \sin^2(z)} \)!
Reactive power flow reformulation
Skipping some details . . .

\[ Q_L = -4[v]S (v - 1_n) + |A|_L D[h(v)](1_{|E|} - \cos(A^T \theta)). \]

- Rearrange for \( v \)

\[ v = f(v, \theta) = 1_n - \frac{1}{4} S^{-1}[Q_L][v]^{-1} 1_n \]
\[ + \frac{1}{4} S^{-1}[v]^{-1}|A|_L D[h(v)] (1_{|E|} - \cos(A^T \theta)), \]

- Now plug in \( \cos(z) = \sqrt{1 - \sin^2(z)} \)!
Reactive power flow reformulation
Skipping some details . . .

\[ Q_L = -4[v]S(v - \mathbb{1}_n) + |A|_L D [h(v)](\mathbb{1}_{|\mathcal{E}|} - \cos(A^T \theta)). \]

- Rearrange for \( v \)

\[ v = f(v, \theta) = \mathbb{1}_n - \frac{1}{4} S^{-1}[Q_L][v]^{-1}\mathbb{1}_n \]
\[ + \frac{1}{4} S^{-1}[v]^{-1}|A|_L D [h(v)](\mathbb{1}_{|\mathcal{E}|} - \cos(A^T \theta)), \]

- Now plug in \( \cos(z) = \sqrt{1 - \sin^2(z)} \)!
Reactive power flow reformulation

Skipping some details . . .

\[ Q_L = -4[v]S(v - \mathbb{1}_n) + |A|_L \begin{bmatrix} h(v) \end{bmatrix} (|\mathbb{1}|_E - \cos(A^T \theta)) \text{ Branch Stiff.} \]

- Rearrange for \( v \)

\[ v = f(v, \theta) = \mathbb{1}_n - \frac{1}{4} S^{-1}[Q_L][v]^{-1} \mathbb{1}_n \]

\[ + \frac{1}{4} S^{-1}[v]^{-1} |A|_L D [h(v)] (|\mathbb{1}|_E - \cos(A^T \theta)) , \]

- Now plug in \( \cos(z) = \sqrt{1 - \sin^2(z)} \)!
Reactive power flow reformulation

Skipping some details . . .

\[ Q_L = -4[v]S(v - 1_n) + |A|_L D[h(v)](1_{|E|} - \cos(A^T \theta)). \]

- Rearrange for \( v \)

\[
    v = f(v, \theta) = 1_n - \frac{1}{4}S^{-1}[Q_L][v]^{-1}1_n
    + \frac{1}{4}S^{-1}[v]^{-1}|A|_L D[h(v)](1_{|E|} - \cos(A^T \theta)),
\]

- Now plug in \( \cos(z) = \sqrt{1 - \sin^2(z)} \)!
Reactive power flow reformulation

Skipping some details . . .

\[ Q_L = -4[v]S(v - \mathbb{1}_n) + |A|_L D [h(v)](\mathbb{1}_n|e| - \cos(A^T \theta)). \]

- Rearrange for \( v \)

\[
v = f(v, \theta) = \mathbb{1}_n - \frac{1}{4}S^{-1}[Q_L][v]^{-1}\mathbb{1}_n
+ \frac{1}{4}S^{-1}[v]^{-1}|A|_L D [h(v)] (\mathbb{1}_n|e| - \cos(A^T \theta)),
\]

- Now plug in \( \cos(z) = \sqrt{1 - \sin^2(z)} \)!
Reactive power flow reformulation

Skipping some details . . .

\[ Q_L = -4[\nu]S(\nu - 1_n) + |A|_L D [h(\nu)](1|\varepsilon| - \cos(A^T \theta)). \]

- Rearrange for \( \nu \)

\[ \nu = f(\nu, \theta) = 1_n - \frac{1}{4}S^{-1}[Q_L][\nu]^{-1}1_n \]

\[ + \frac{1}{4}S^{-1}[\nu]^{-1}|A|_L D [h(\nu)] (1|\varepsilon| - \cos(A^T \theta)). \]

- Now plug in \( \cos(z) = \sqrt{1 - \sin^2(z)} \)!
Reactive power flow reformulation

Skipping some details . . .

\[ Q_L = -4[v]S(v - 1_n) + |A|L D [h(v)][1|\varepsilon| - \cos(A^T \theta)]. \]

- Rearrange for \( v \)

\[
v = f(v, \theta) = 1_n - \frac{1}{4} S^{-1}[Q_L][v]^{-1} 1_n
\]
\[
+ \frac{1}{4} S^{-1}[v]^{-1}|A|L D [h(v)] (1|\varepsilon| - \cos(A^T \theta)),
\]

- Now plug in \( \cos(z) = \sqrt{1 - \sin^2(z)} \)!
(θ, V_L) is a power flow solution iff (ν, p_c) solves the FPPF

\[ ν = f(ν, p_c) \triangleq 1_n - \frac{1}{4} S^{-1}[Q_L][ν]^{-1} 1_n \]
\[ + \frac{1}{4} S^{-1}[ν]^{-1}|A|_L D [h(ν)] u(ν, p_c), \]

0_c = C^T \text{arcsin}(ψ(ν, p_c)).

where

\[ u(ν, p_c) \triangleq 1 - \sqrt{1 - [ψ]ψ} \]
\[ ψ(ν, p_c) = [h(ν)]^{-1} \left( A^T L^+ P + D^{-1} C p_c \right), \]

with the phase angles \( A^T θ = \text{arcsin}(ψ) \).
(θ, VL) is a power flow solution iff (v, pc) solves the FPPF

\[ v = f(v, pc) \triangleq 1_n - \frac{1}{4} S^{-1}[Q_L][v]^{-1} 1_n \]
\[ + \frac{1}{4} S^{-1}[v]^{-1}|A|_L D [h(v)] u(v, pc), \]

0c = C^T \arcsin(ψ(v, pc)).

where

\[ u(v, pc) \triangleq 1 - \sqrt{1 - [ψ]ψ} \]
\[ ψ(v, pc) = [h(v)]^{-1} \left( A^T L^\dagger P + D^{-1} C pc \right), \]

with the phase angles \( A^T \theta = \arcsin(ψ) \).
$(\theta, V_L)$ is a power flow solution iff $(v, p_c)$ solves the FPPF

$$v = f(v, p_c) \triangleq 1_n - \frac{1}{4} S^{-1} [Q_L][v]^{-1} 1_n$$
$$+ \frac{1}{4} S^{-1} [v]^{-1} |A|_L D [h(v)] u(v, p_c),$$

$$0_c = C^T \arcsin(\psi(v, p_c)),$$

where

$$u(v, p_c) \triangleq 1 - \sqrt{1 - [\psi] \psi}$$

$$\psi(v, p_c) = [h(v)]^{-1} \left( A^T L^\dagger P + D^{-1} C p_c \right),$$

with the phase angles $A^T \theta = \arcsin(\psi)$. 
$(\theta, V_L)$ is a power flow solution iff $(v, p_c)$ solves the FPPF

\[
v = f(v, p_c) \triangleq 1_n - \frac{1}{4} S^{-1} [Q_L][v]^{-1} 1_n + \frac{1}{4} S^{-1} [v]^{-1} |A|_L D [h(v)] u(v, p_c),
\]

\[\theta_c = C^T \arcsin(\psi(v, p_c)).\]

where

\[
u(v, p_c) \triangleq 1 - \sqrt{1 - [\psi][\psi]}
\]

\[
\psi(v, p_c) = [h(v)]^{-1} \left( A^T L^\dagger P + D^{-1} C p_c \right),
\]

with the phase angles $A^T \theta = \arcsin(\psi)$. 
(θ, V_L) is a power flow solution iff (v, p_c) solves the FPPF

\[
v = f(v, p_c) \triangleq 1_n - \frac{1}{4} S^{-1} [Q_L][v]^{-1} 1_n
\]

\[+ \frac{1}{4} S^{-1} [v]^{-1} |A|_L D [h(v)] u(v, p_c),\]

\[0_c = C^T \arcsin(\psi(v, p_c)).\]

where

\[u(v, p_c) \triangleq 1 - \sqrt{1 - [\psi]\psi}\]

\[\psi(v, p_c) = [h(v)]^{-1} \left( A^T L^\dagger P + D^{-1} C p_c \right),\]

with the phase angles \( A^T \theta = \arcsin(\psi) \).
Fixed-Point Power Flow: Meshed Networks

\((\theta, V_L)\) is a power flow solution iff \((v, p_c)\) solves the FPPF

\[
v = f(v, p_c) \triangleq 1_n - \frac{1}{4} S^{-1}[Q_L][v]^{-1} 1_n
\]

\[
+ \frac{1}{4} S^{-1}[v]^{-1}|A|_L \text{D} [h(v)] u(v, p_c),
\]

\[\mathbf{0}_c = C^T \arcsin(\psi(v, p_c)).\]

where

\[
u(v, p_c) \triangleq 1 - \sqrt{1 - [\psi]_\psi}
\]

\[
\psi(v, p_c) = [h(v)]^{-1} \left( A^T L^\dagger P + D^{-1} C p_c \right),
\]

with the phase angles \(A^T \theta = \arcsin(\psi).\)
(θ, VL) is a power flow solution iff (v, pc) solves the FPPF

\[
v = f(v, pc) \triangleq 1_n - \frac{1}{4} S^{-1} [QL][v]^{-1} 1_n \\
+ \frac{1}{4} S^{-1} [v]^{-1} |A| L D [h(v)] u(v, pc),
\]

\[\theta_c = C^T \arcsin(\psi(v, pc)).\]

where

\[
u(v, pc) \triangleq 1 - \sqrt{1 - [\psi] \psi}
\]

\[
\psi(v, pc) = [h(v)]^{-1} \left( A^T L^\dagger P + D^{-1} C pc \right),
\]

with the phase angles \( A^T \theta = \arcsin(\psi) \).
(θ, V_L) is a power flow solution iff (v, p_c) solves the FPPF
\[
v = f(v, p_c) \triangleq 1_n - \frac{1}{4} S^{-1} [Q_L][v]^{-1} 1_n \]
\[
+ \frac{1}{4} S^{-1} [v]^{-1} |A_L| D [h(v)] u(v, p_c),
\]
\[
0_c = C^T \arcsin(\psi(v, p_c)).
\]

where
\[
u(v, p_c) \triangleq 1 - \sqrt{1 - [\psi]^2}
\]
\[
\psi(v, p_c) = [h(v)]^{-1} \left( A^T L^\dagger P + D^{-1} C_{p_c} \right),
\]
with the phase angles \( A^T \theta = \arcsin(\psi) \).
An approximate power flow solution

- The model says $v = f(v, p_c)$, and $\sin(A^T \theta) = \psi(v, p_c)$.
- By construction, when $P = Q_L = 0$, a solution is
  \[ v = 1_n, \quad p_c = 0_c, \quad A^T \theta = 0|_{\varepsilon}. \]

Taylor expand FPPF model around this solution

\[
A^T \theta_{\text{approx}} = A^T L^\dagger P \\
v_{\text{approx}} \approx 1_n - \frac{1}{4} S^{-1} Q_L + \frac{1}{8} S^{-1}|A|_L D [A^T L^\dagger P] A^T L^\dagger P \\
p_{c, \text{approx}} = 0
\]
An approximate power flow solution

- The model says $v = f(v, p_c)$, and $\sin(A^T \theta) = \psi(v, p_c)$.
- By construction, when $P = Q_L = 0$, a solution is

$$v = 1_n, \quad p_c = 0_c, \quad A^T \theta = 0_{|\mathcal{E}|}.$$ 

- **Taylor expand** FPPF model around this solution

$$A^T \theta_{\text{approx}} = A^T L^\dagger P$$

$$v_{\text{approx}} \approx 1_n - \frac{1}{4} S^{-1} Q_L + \frac{1}{8} S^{-1} |A|_L D [A^T L^\dagger P] A^T L^\dagger P$$

$$p_{c, \text{approx}} = 0$$
An approximate power flow solution

- The model says \( v = f(v, p_c) \), and \( \sin(A^T \theta) = \psi(v, p_c) \).
- By construction, when \( P = Q_L = 0 \), a solution is
  \[
  v = 1_n, \quad p_c = 0_c, \quad A^T \theta = 0_{|\mathcal{E}|}.
  \]

- **Taylor expand** FPPF model around this solution
  \[
  A^T \theta_{\text{approx}} = A^T L^\dagger P
  \]
  \[
  v_{\text{approx}} \approx 1_n - \frac{1}{4} S^{-1} Q_L + \frac{1}{8} S^{-1} |A|_L D [A^T L^\dagger P] A^T L^\dagger P
  \]
  \[
  p_{c,\text{approx}} = 0
  \]
An approximate power flow solution

- The model says \( v = f(v, p_c) \), and \( \sin(A^T \theta) = \psi(v, p_c) \).
- By construction, when \( P = Q_L = 0 \), a solution is

\[
\begin{align*}
  v &= 1_n, \\
  p_c &= 0_c, \\
  A^T \theta &= 0_{|E|}.
\end{align*}
\]

- **Taylor expand** FPPF model around this solution

\[
A^T \theta_{\text{approx}} = A^T L^\dagger P
\]

\[
\nu_{\text{approx}} \simeq 1_n - \frac{1}{4} S^{-1} Q_L + \frac{1}{8} S^{-1} |A|_L D [A^T L^\dagger P] A^T L^\dagger P
\]

\[
p_{c, \text{approx}} = 0
\]
An approximate power flow solution

\[ A^T \theta_{\text{approx}} = A^T L^\dagger P \]

\[ \nu_{\text{approx}} \simeq 1_n - \frac{1}{4} S^{-1} Q_L + \frac{1}{8} S^{-1} |A|_L D [A^T L^\dagger P] A^T L^\dagger P \]
An approximate power flow solution

\[ A^T \theta_{\text{approx}} = A^T L^\dagger P \]

\[ v_{\text{approx}} \approx 1_n - \frac{1}{4} S^{-1} Q_L + \frac{1}{8} S^{-1} |A|_L D [A^T L^\dagger P] A^T L^\dagger P \]

---

![Graph showing voltage magnitude vs. bus number](image-url)
Numerical results I

\[ \delta_{\text{max}} = \| \mathbf{v} - \mathbf{v}_{\text{approx}} \|_\infty, \quad \delta_{\text{avg}} = \frac{1}{n} \| \mathbf{v} - \mathbf{v}_{\text{approx}} \|_1 \]

<table>
<thead>
<tr>
<th>Test Case</th>
<th>Base Load</th>
<th>High Load</th>
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<td>FPPF</td>
<td>( \delta_{\text{max}} ) (p.u.)</td>
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</table>
Numerical results II – convergence rates

- IEEE 300 bus system under heavy loading

![Graph showing convergence rates](image)

Legend:
- NR
- FDLF (XB)
- FPPF
Numerical results III – sensitivity to initialization

- perturb voltage magnitude initialization randomly
- IEEE 118 bus system, base case

<table>
<thead>
<tr>
<th>IC Spread (α)</th>
<th>NR</th>
<th>FDLF</th>
<th>FPPF</th>
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<td>0.9</td>
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<td>0.99</td>
</tr>
</tbody>
</table>

- extreme insensitivity to initialization (contraction)
(θ, VL) is a power flow solution iff ν is a fixed point of

\[ f(ν) \triangleq 1_n - \frac{1}{4} S^{-1} [QL][ν]^{-1} 1_n + \frac{1}{4} S^{-1} [ν]^{-1} A|_L D [h(ν)] u(ν), \]

where

\[ u(ν) \triangleq 1 - \sqrt{1 - [ψ]} \]

\[ ψ(ν) = [h(ν)]^{-1} D^{-1} p \]

\[ p = (A^T A)^{-1} A^T P \]

with the phase angles \( A^T θ = \arcsin(ψ) \).
$(\theta, V_L)$ is a power flow solution iff $v$ is a fixed point of

$$f(v) \triangleq \mathbb{1}_n - \frac{1}{4} S^{-1}[Q_L][v]^{-1}\mathbb{1}_n + \frac{1}{4} S^{-1}[v]^{-1}|A|_L D [h(v)] u(v),$$

where

$$u(v) \triangleq 1 - \sqrt{1 - [\psi][\psi]}$$

$$\psi(v) = [h(v)]^{-1} D^{-1} p$$

$$p = (A^T A)^{-1} A^T P$$

with the phase angles $A^T \theta = \arcsin(\psi)$. 

On what invariant set is $f$ a contraction?
\((\theta, V_L)\) is a power flow solution iff \(v\) is a fixed point of

\[
f(v) \triangleq \mathbb{1}_n - \frac{1}{4} S^{-1} [Q_L][v]^{-1} \mathbb{1}_n + \frac{1}{4} S^{-1} [v]^{-1} [A]_L D [h(v)] u(v),
\]

where

\[
u(v) \triangleq 1 - \sqrt{1 - [\psi][\psi]}
\]

\[
\psi(v) = [h(v)]^{-1} D^{-1} p
\]

\[
p = (A^T A)^{-1} A^T P
\]

with the phase angles \(A^T \theta = \text{arcsin}(\psi)\).
Fixed-Point Power Flow: Radial Networks

\((\theta, V_L)\) is a power flow solution iff \(v\) is a fixed point of

\[
f(v) \triangleq 1_n - \frac{1}{4} S^{-1}[Q_L][v]^{-1} 1_n + \frac{1}{4} S^{-1}[v]^{-1} |A|_L D [h(v)] u(v),
\]

where

\[
u(v) \triangleq 1 - \sqrt{1 - [\psi] \psi}
\]

\[
\psi(v) = [h(v)]^{-1}D^{-1}p
\]

\[
p = (A^T A)^{-1} A^T P
\]

with the phase angles \(A^T \theta = \arcsin(\psi)\).
(θ, VL) is a power flow solution iff ν is a fixed point of

\[ f(\nu) \triangleq 1_n - \frac{1}{4} S^{-1}[Q_L][\nu]^{-1} 1_n + \frac{1}{4} S^{-1}[\nu]^{-1} |A|_L D [h(\nu)] u(\nu), \]

where

\[
    u(\nu) \triangleq 1 - \sqrt{1 - [\psi]\psi} \\
    \psi(\nu) = [h(\nu)]^{-1} D^{-1} p \\
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On what invariant set is \( f \) a contraction?
Solvability results for different tree topologies
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PQ buses have one PV bus neighbor

Sufficient + Necessary
Existence + Uniqueness
Solvability results for different tree topologies

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General interconnections

Sufficient
Existence
Partitioning of voltage space

\[ \max_{i \in \mathcal{N}_L} \Delta_i + 4\Gamma_i^2 < 1 \]

\[ \max_{(i,j) \in \mathcal{E}^{gg}} \Gamma_{ij} < 1, \]

\[ v_{i,\pm} \triangleq \sqrt{\frac{1}{2} \left( 1 - \frac{\Delta_i}{2} \right) \pm \sqrt{1 - (\Delta_i + 4\Gamma_i^2)}} \]

\[ v_{i,+}^2 - v_{i,-}^2 = \sqrt{1 - (\Delta_i + 4\Gamma_i^2)}. \]
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n_{i,\pm} &\triangleq \sqrt{\frac{1}{2} \left( 1 - \frac{\Delta_i}{2} \right) \pm \sqrt{1 - (\Delta_i + 4\Gamma_i^2)}} \\
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Summary
Framework for studying **Lossless Power Flow**:

1. Fixed-Point Power Flow
2. Approximate solution

New **conditions for power flow solvability**:

3. Contractive iteration
4. Existence/uniqueness
5. Generalizes known results

What’s unresolved?

1. **Lossless meshed** case
2. **Lossy meshed** case; algorithms in Chen & JWSP 2022, but theory is hard
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Final Thoughts

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Questions

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