Generalized Nash equilibrium seeking

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Outline

1. Motivation
2. Generalized Nash equilibrium problem
3. Partial first order information setting
4. Zeroth order information settings
5. Conclusion and Outlook
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1 Motivation

2 Generalized Nash equilibrium problem

3 Partial first order information setting

4 Zeroth order information settings

5 Conclusion and Outlook
Motivation: Sharing resources in Complex Systems

- Game theory models complex systems/infrastructures where (possibly selfish) decision makers ("agents") share resources.

Game-Theoretic Methods for the Smart Grid

An overview of microgrid systems, demand-side management, and smart grid communications.

Convex Optimization, Game Theory, and Variational Inequality Theory

Basic theoretical foundations and main techniques in multiuser communication systems.

Game-Theoretical Methods in Control of Engineering Systems

An introduction to the special issue.

Coalitional Control

Cooperative Game Theory and Control.

Transforming Energy Networks via Peer-to-Peer Energy Trading

The potential of game-theoretic approaches.

Game Theory-Based Control System Algorithms with Real-Time Reinforcement Learning

How to solve multiplayer games online.
Motivation: Sharing resources in Complex Systems (+)

Pavel, Hong, Başar, Grammatico, Shanbhag, ...

Distributed Games over Networks, IEEE CSM, 2022
Example 1: Demand Side Management

**Demand Side Management:**

*de-synchronize*/**flatten** net energy demand of prosumers

**Prosumer** $i$:

\[
\begin{align*}
\min_{u_i} \quad & \text{discomfort}_i (u_i) \\
+ \quad & \text{congestion cost}(u_i, \sum_j u_j) \\
\text{s.t.} \quad & u_i \in \{\text{limits}(\sum_j u_j)\}
\end{align*}
\]

Mohsenian-Rad et al., *Autonomous demand-side management based on game-theoretic energy consumption scheduling for the future smart grid*, IEEE TSG, 2010

Saad, Han, Poor, Başar, *Game-theoretic methods for the smart grid*, IEEE MSP, 2012
Example 2: Day ahead Electricity Markets

**Electricity Generators:**

\[
(\forall i) \quad P_i : \left\{ \begin{array}{l}
\max_{s_i, u_i} \quad p\left(\sum_j s_j\right)^\top s_i - c_i(u_i) - \text{diag}(\lambda)(s_i - u_i) \\
\text{nodal prices} \quad \text{generation cost} \quad \text{fees}
\end{array} \right.
\]

\[\begin{align*}
\text{s.t.} & \quad (s_i, u_i) \in \{\text{limits}\} \\
\text{• } s_i & = \text{sale, } u_i = \text{generation (at all nodes)} \\
\text{• } \lambda & = \text{transmission fees}
\end{align*}\]

**TSO/ISO:**

\[
P_{iso} : \left\{ \begin{array}{l}
\max_{\lambda} \quad \text{revenue}(\lambda, s, g) \\
\text{s.t.} & \quad (s, u) \in \{\text{transmission capacities}\}
\end{array} \right.
\]

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Hobbs, Pang, *Nash–Cournot equilibria in electric power markets with piecewise linear demand functions and joint constraints*, OR, 2007
Example 3: Road Traffic Management

Road Traffic Management: alleviate road traffic congestion via routing control (e.g. traffic light control, ramp metering, tolls)

Vehicle group $i$:

\[
\begin{align*}
\min_{u_i} & \quad \text{std\_travel\_time}_i(u_i) \\
& + \text{congestion\_time}(u_i, \sum_j u_j) \\
\text{s.t.} & \quad u_i \in \{\text{limits}\}
\end{align*}
\]

Example 4: Multi-vehicle Automated Driving

Multi-vehicle Automated Driving: optimize motion planning via intelligent actions (e.g. acceleration/deceleration, lane change, overtaking)

Vehicle $i$:

$$\begin{aligned}
\min_{u_i} & \quad \text{objective}_i (u_i, u_{-i}) \\
\text{s.t.} & \quad u_i \in \text{vehicle
dynamics}_i \\
& \quad u_i \in \text{safety}_i (u_i, \sum_j u_j) \\
& \quad \text{neighbors}
\end{aligned}$$

- Fabiani, Grammatico, *Multi-vehicle automated driving as a generalized mixed-integer potential game*, IEEE T-ITS, 2021
Motivating features of complex systems of systems

1. **autonomous sub-systems** (local decision authority, preferences)
2. **self-interested decision makers** (e.g., energy companies)
3. **large-scale** (hundreds/thousands of decision makers)
4. locality of information/privacy
5. **disruptive** if left uncontrolled

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**power outages**  
**traffic jam**
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Game modeling and equilibrium seeking

Operator theoretic problem

Fixed point dynamics

Game equilibrium problem

Game equilibrium solution

optimization

game theory

complex system

system theory
Game theoretic setting

- \( N \) autonomous agents (decision makers)
- each agent \( i \) has decision variable (strategy) \( x_i \)
  \[ x := \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \text{ ("all")}; \quad (\forall i) \quad x_{-i} := \begin{bmatrix} x_{i-1} \\ x_i \\ x_{i+1} \\ \vdots \end{bmatrix} \text{ ("others")} \]
- each agent \( i \) has its individual objective:
  \[
  \begin{cases}
  \min_{x_i} \text{cost}_i (x_i, x_{-i}) \\
  \text{s.t.} \quad x_i \in \text{constraints}_i (x_{-i})
  \end{cases}
  \]
  \[
  \Rightarrow \text{ "Game" } = \{ \text{ inter-dependent optimization problems } \} \]
Generalized Nash equilibrium problem

\[ P_i(\mathbf{x}_{-i}) : \begin{cases} \min_{x_i} J_i(x_i, x_{-i}) \\ \text{s.t.} \quad x_i \in X_i \\ g(x_i, x_{-i}) \leq 0 \end{cases} \]

\[ x^* = \begin{bmatrix} x^*_1 \\ \vdots \\ x^*_N \end{bmatrix} \in \arg\min_{P_1} \ldots \arg\min_{P_N} x^*_1 \ldots x^*_N \]

Facchinei, Kanzow, Generalized Nash equilibrium problems, 2007
Generalized Nash equilibrium problem

\[ \mathbb{P}_i(\mathbf{x}_{-i}) : \begin{cases} 
\min_{\mathbf{x}_i} & J_i(\mathbf{x}_i, \mathbf{x}_{-i}) \\
\text{s.t.} & \mathbf{x}_i \in X_i \\
& g(\mathbf{x}_i, \mathbf{x}_{-i}) \leq 0
\end{cases} \leftarrow \text{local} \]

\[ \mathbf{g}(\mathbf{x}_i, \mathbf{x}_{-i}) \leq 0 \leftarrow \text{coupling} \]

\textbf{(Generalized) Nash equilibrium (GNE)}

\[ \mathbf{x}^* = \begin{bmatrix} x_1^* \\ \vdots \\ x_N^* \end{bmatrix} \in \begin{bmatrix} \arg\min \mathbb{P}_1(\mathbf{x}_{-1}^*) \\ \vdots \\ \arg\min \mathbb{P}_N(\mathbf{x}_{-N}^*) \end{bmatrix} =: \mathcal{T}_{BR}(\mathbf{x}^*) \]

Lagrangian duality: Decouple coupling constraints

**Lagrangian functions**: \( \nu = \text{indicator function} \)

\[
L_i (\mathbf{x}, \lambda_i) := J_i (x_i, \mathbf{x}_-i) + \nu X_i (x_i) + \lambda_i^\top g(\mathbf{x})
\]
Lagrangian duality: Decouple coupling constraints

**Lagrangian functions:** \((\iota = \text{indicator function})\)

\[
L_i(x, \lambda_i) := J_i(x_i, x_{-i}) + \iota x_i(x_i) + \lambda_i^\top g(x)
\]

**KKT Theorem:** \((\text{under regularity/convexity})\) \(x_i\) solves \(P_i(x_{-i})\) iff

\[
\begin{cases}
0 \in \partial x_i L_i(x_i, x_{-i}, \lambda_i) & \text{← stationarity} \\
0 \geq g(x) & \text{← feasibility} \\
0 \leq \lambda_i \perp g(x) & \text{← complementarity}
\end{cases}
\]
Karush–Kuhn–Tucker inclusion

usual simplification:

- **variational** GNE \( \Longrightarrow \lambda_i = \lambda \) (fairness)
Karush–Kuhn–Tucker inclusion

usual simplification:

- variational GNE $\implies \lambda_i = \lambda$ (fairness)

KKT system $\equiv$ **KKT inclusion:** ($N = \text{normal cone}$)

$0 \in \mathcal{T}_{\text{KKT}}(\mathbf{x}, \lambda) := \left[ F(\mathbf{x}) + N_{\mathbf{x}}(\mathbf{x}) + \nabla g(\mathbf{x}) \lambda \right] \quad \leftarrow \text{stationarity}$

$\quad N_{\geq 0}(\lambda) - g(\mathbf{x}) \quad \leftarrow \text{feasibility & compl.}$

$F = \text{pseudo-subdifferential mapping:}$

$F(\mathbf{x}) := \begin{bmatrix} \partial_{x_1} J_1(x_1, x_{-1}) \\ \vdots \\ \partial_{x_N} J_N(x_N, x_{-N}) \end{bmatrix}$
Generalized Nash equilibrium seeking

**Generalized Nash equilibrium seeking problem:**

Design dynamical systems (e.g. in discrete time, $k \in \mathbb{N}$)

\[
\forall i : \begin{bmatrix} x_i(k+1) \\ \lambda_i(k+1) \end{bmatrix} = f_i \left( \begin{bmatrix} x_i(k) \\ \lambda_i(k) \end{bmatrix}, x_{-i}(k) \right)
\]

s.t. \( \begin{bmatrix} x(k) \\ \lambda(k) \end{bmatrix} \rightarrow \begin{bmatrix} x^* \\ \lambda^* \end{bmatrix} \in \{ \text{KKT solutions} \}, \quad x(k) \rightarrow x^* \in \{ \text{GNE} \} \)

\[\text{zer}(\mathcal{T}_{\text{KKT}})\]

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Pavel, *Dissipativity theory in game theory: On the role of dissipativity and passivity in Nash equilibrium seeking*, IEEE CSM, 2022
Game modeling and equilibrium seeking (+)

\[ \text{zer} \left( \mathcal{T}_{\text{KKT}} \right) \]

\[ z^+ = \mathcal{F}(z) \]

- Optimization
- Game theory

- Complex system

\[ \text{argmin} \ J_i \]

\[ \mathcal{x} \rightarrow \mathcal{x}^* \]
Technical assumptions

Assumptions:

- non-empty, compact, **convex** feasible sets
- \( g \) **convex**, continuously **differentiable**
- constraint qualification
- \( J_i \) continuous, \( J_i(\cdot, x_{-i}) \), **convex**
- \( F \) (Lipschitz) continuous, (strongly) **monotone** \((\mu > 0)\)

\[
\langle F(\xi) - F(\zeta), \xi - \zeta \rangle \geq \mu \|\xi - \zeta\|^2
\]
Generalized Nash equilibrium seeking Literature

Existence and uniqueness:

- Debreu, *A social equilibrium existence theorem*, PNAS, 1952 ← 1st “GNE”

Equilibrium seeking dynamics:

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**Partial information:**

agent $i$ does **not** know $x_{-i} \implies$ cannot evaluate $\nabla J_i(x_i, x_{-i})$

- agent $i$ estimates $x_{-i}$:

  $x_{i,j} = \text{estimate by agent } i \text{ of } x_j$

  $x_{i,i} = x_i \text{ (estimate by agent } i \text{ of } x_i)$
Recent challenge: Partial information setting

**Partial information:**

agent $i$ does **not** know $x_{-i} \implies$ cannot evaluate $\nabla J_i(x_i, x_{-i})$

- agent $i$ estimates $x_{-i}$:
  
  $x_{i,j} = \text{estimate by agent } i \text{ of } x_j$
  
  $x_{i,i} = x_i \text{ (estimate by agent } i \text{ of } x_i)$

**Partial information setup:**

Agents exchange estimates on undirected, connected **graph**

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Pavel, *Distributed GNE seeking under partial-decision information over networks via a doubly-augmented operator splitting approach*, TAC, 2020
Extended pseudo-gradient is **not** monotone

**Extended pseudo-gradient mapping:**

\[
F(\mathbf{x}) := \begin{bmatrix}
\nabla_{x_1} J_1(x_1, x_{1,-1}) \\
\vdots \\
\nabla_{x_N} J_N(x_N, x_{N,-N})
\end{bmatrix} \quad \leftarrow \text{estimated partial gradients}
\]

- **\(F\)** **not** monotone
- all **estimates** equal \(\implies F(1_N \otimes \mathbf{x}) = F(\mathbf{x})\)
Distributed preconditioned proximal point algorithm

Our proposed solution: **Prox + Consensus**

\[
\begin{bmatrix}
x(k + 1) \\
z(k + 1) \\
\lambda(k + 1)
\end{bmatrix} = J_{\Phi^{-1}} A \begin{bmatrix}
x(k) \\
z(k) \\
\lambda(k)
\end{bmatrix}
\]

- J = resolvent operator
- preconditioning matrix $\Phi > 0 \rightarrow$ distributed computation
Our proposed solution: Prox + Consensus

\[
\begin{bmatrix}
\mathbf{x}(k+1) \\
\mathbf{z}(k+1) \\
\lambda(k+1)
\end{bmatrix} = J_{\Phi^{-1}}A \begin{bmatrix}
\mathbf{x}(k) \\
\mathbf{z}(k) \\
\lambda(k)
\end{bmatrix}
\]

- \( J = \) resolvent operator
- Preconditioning matrix \( \Phi > 0 \) \( \rightarrow \) distributed computation
- \( A \) s.t. \( \text{col}(\mathbf{x}^*, \mathbf{z}^*, \lambda^*) \in \text{zer}(A) \implies \mathbf{x}^* = 1_N \otimes \mathbf{x}^*, \quad \mathbf{x}^* \nu\text{-GNE} \)

Bianchi, Belgioioso, Grammatico, *Fast generalized Nash equilibrium seeking under partial-decision information*, *Automatica*, 2022
Distributed preconditioned proximal point algorithm (+)

Prox + Consensus $\rightarrow$ fully distributed algorithm:

1. distributed information exchange and averaging
2. local estimate $(x_{i,-i})$ update (consensus)
3. local strategy $(x_i)$ update (prox)
4. local auxiliary $(z_i)$ and dual $(\lambda_i)$ variables update (∼ consensus)
Sketch of convergence analysis

1. small step sizes $\Rightarrow \mathcal{A}$ restricted monotone:
   \[
   \langle y - y^*, x - x^* \rangle \geq 0
   \]
   for all $(x, y), (x^*, y^*) \in \text{gph}(\mathcal{A}), x^* \in \text{zer}(\mathcal{A}) (y^* = 0)$

2. $\Phi^{-1} \mathcal{A}$ restricted monotone ($\Phi$—induced space)

3. $J_{\Phi^{-1} \mathcal{A}}$ firmly quasi-nonexpansive ($\Phi$—induced space)

4. build upon

   Bauschke, Combettes, *Convex analysis and monotone operator theory in Hilbert spaces*
Aggregative games:

\[ J_i(x_i, x_{-i}) = \varphi_i(x_i, \text{avg}(x)), \quad \text{avg}(x) := \frac{1}{N} \sum_{i=1}^{N} x_i \]
Distributed algorithm for aggregative games

**Aggregative games:**

\[ J_i(x_i, x_{-i}) = \varphi_i(x_i, \text{avg}(x)), \quad \text{avg}(x) := \frac{1}{N} \sum_{i=1}^{N} x_i \]

- Agent \( i \) estimates \( \text{avg}(x) \), **not** all strategies:

  \[ \sigma_i = \text{estimate by agent } i \text{ of } \text{avg}(x) \]

  \[ s_i := \sigma_i - x_i = \text{extra auxiliary variable} \]

- different \( \Phi \), similar dynamics outcome
(Accelerated) Proximal vs Forward-Backward dynamics

1. Preconditioned forward-backward (FB) algorithm

Pavel, *Distributed GNE seeking under partial-decision information over networks via a doubly-augmented operator splitting*..., TAC, 2020

2. Preconditioned proximal point (PPP) algorithm

\[ \omega^{k+1} = \mathcal{T}_{\text{PPP}}(\omega^k) \]

3. PPP with over-relaxation \((1 \leq \eta < 2)\)

\[ \omega^{k+1} = \eta \mathcal{T}_{\text{PPP}}(\omega^k) + (1 - \eta)\omega^k \]

4. PPP with inertia \((\rho \geq 0)\)

\[ \omega^{k+1} = \mathcal{T}_{\text{PPP}}(\omega^k + \rho(\omega^k - \omega^{k-1})) \]

5. PPP with alternated inertia
Numerical experiments

- PPP faster than projected pseudo-gradient (FB)
Numerical experiments (+)

- **PPP** with **inexact** prox: few projected-gradient steps are enough

![Graph showing the number of projected gradient steps over iterations](image)
Numerical experiments (++)

![Graph showing iterations for convergence with different values of γ and η.](image)
Alternative: Equilibrium seeking in continuous time

**Continuous-time** GNE seeking dynamics in **partial information**:

- **projected pseudo-gradient** \( \dot{x}_i = \text{proj}_{TC}(x_i, \nabla J_i(x) + \ldots) \)
- **adaptive step sizes** \( \dot{k}_i = \|\text{local disagreement}\|^2 \)
- **nonlinear** coupling constraints \( \sum_i g_i(x_i) \leq 0 \)
- **multi-integrator dynamical agents** (e.g. \( \ddot{x}_i = u_i, \ x \rightarrow \text{GNE} \))

De Persis, Grammatico, “*Distributed averaging integral Nash equilibrium seeking algorithms on networks*”, *Automatica*, 2019

Bianchi, Grammatico, “*Continuous-time fully distributed generalized Nash equilibrium seeking for multi-integrator agents*”, *Automatica*, 2022
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Zeroth order generalized Nash equilibrium seeking

**Zeroth order information setting:**

- agent $i$ has **no access** to $\nabla x_i J_i$ (first order information)
- instead agent $i$ measures $J_i$ (zeroth order information)

$\implies$ agent $i$ estimates $\nabla x_i J_i$ via filtering available measures

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Frihauf, Krstic, Başar, *Nash equilibrium seeking in noncooperative games*, IEEE TAC, 2011 $\leftarrow$ non-generalized NE seeking


Zou, Lygeros, *Semi-decentralized zeroth-order algorithms for stochastic generalized Nash equilibrium seeking*, IEEE TAC, 2022
Stochastic generalized Nash equilibrium seeking

**Stochastic information setting:**
- expected value cost functions $\mathbb{E}_\xi [J_i(x_i, x_{-i}, \xi)]$, $\xi \sim \mathbb{P}$
- samples available $\xi^{(1)}, \xi^{(2)}, \ldots, \sim \mathbb{P}$

**Stochastic approximation method:**

$F \sim \hat{F} = \text{empirical sum of sampled pseudo-gradients}$

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Stochastic pseudo-gradient dynamics

Relaxation: $\bar{x}_i(k) = (1 - \delta)x_i(k) + \delta\bar{x}_i(k - 1)$  (same for $z_i$'s, $\lambda_i$'s)

Projected Pseudo-Gradient + Consensus dynamics:

1. local strategy ($x_i$) update (proj of empirical pseudo-gradient)
2. local auxiliary ($z_i$) and dual ($\lambda_i$) variables update ($\sim$ consensus)

Franci, Grammatico, *Stochastic generalized Nash equilibrium seeking in merely monotone games*, IEEE TAC, 2022

Franci, Grammatico, *Stochastic generalized Nash equilibrium seeking under partial-decision information*, Automatica, 2022
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Conclusion

• In **Convex-Monotone Games**
  
equilibria are zeros of (restricted) monotone operators

• **Fixed-point and Monotone Operator Theory**
  
are key to establish convergence to equilibria

• **Preconditioned Proximal Point** algorithms converge fast!

• **Application areas:** smart power grids, energy/data markets, road traffic control, automated driving, network congestion control, ...
Available **partial-information** dynamics require complete estimation graph (inefficient information exchange)

**Open problem:**
How to design the **best information exchange graph**

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**The END: Estimation Network Design for efficient distributed equilibrium seeking**

Mattia Bianchi, *Member, IEEE*, Sergio Grammatico, *Senior Member, IEEE*

Bianchi, Grammatico, *The END: Estimation Network Design for efficient distributed equilibrium seeking, arXiv, 2022*
Outlook (+)

- Available dynamics converge towards some equilibrium (possibly inefficient from a system-level perspective)

**Fundamental open problem:**
How to design dynamics convergent towards the best equilibrium

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Optimal selection and tracking of generalized Nash equilibria in monotone games

Emilio Benenati, Wicak Ananduta, and Sergio Grammatico

Benenati, Ananduta, Grammatico, *Optimal selection and tracking of generalized Nash equilibria in monotone games*, IEEE TAC (2P), 2022
Outlook (++)

- Available theorems mostly limited to **convex-monotone games**

**Practical open problem:**
How to design convergent dynamics for **non-convex-mon** games

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**Clarke’s Local Generalized Nash Equilibria with Nonconvex Coupling Constraints**

Paolo Scarabaggio, *Graduate Student Member, IEEE*, Raffaele Carli, *Senior Member, IEEE*, Sergio Grammatico, *Senior Member, IEEE*, and Mariagrazia Dotoli, *Senior Member, IEEE*

Scarabaggio, Carli, Grammatico, Dotoli, *Clarke’s local generalized Nash equilibria with nonconvex coupling constraints*, TechRxiv, 2022
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