Generalized Nash equilibrium seeking

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CWI Workshop on Polynomial Optimization and Applications in Control and Energy

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Outline

1. Motivation
2. Generalized Nash equilibrium problem
3. Partial first order information setting
4. Zeroth order information settings
5. Conclusion and Outlook

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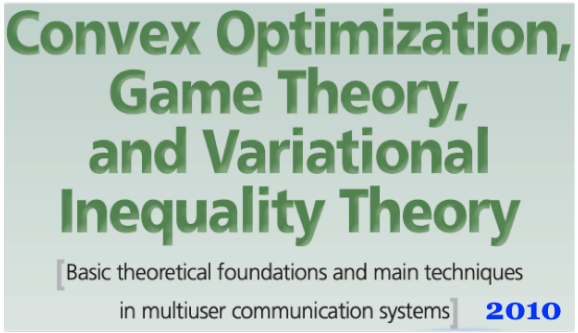
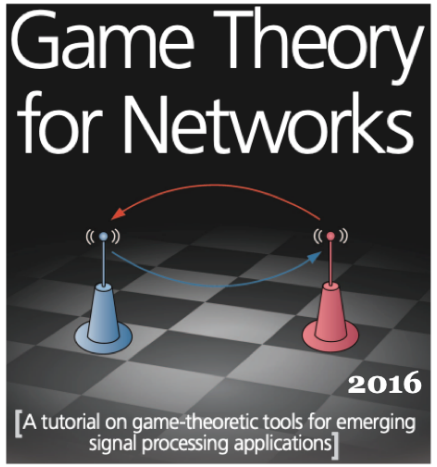
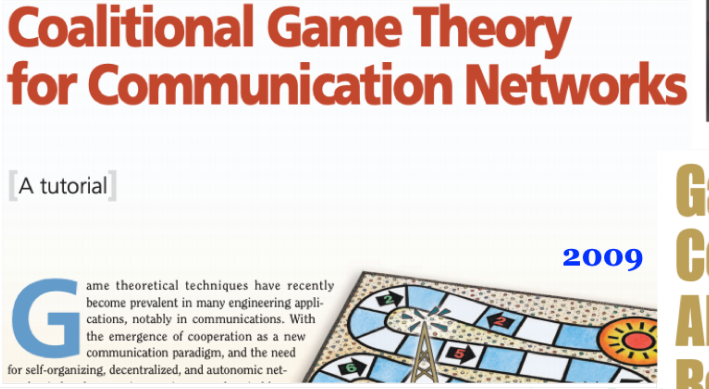
Motivation: Sharing resources in Complex Systems

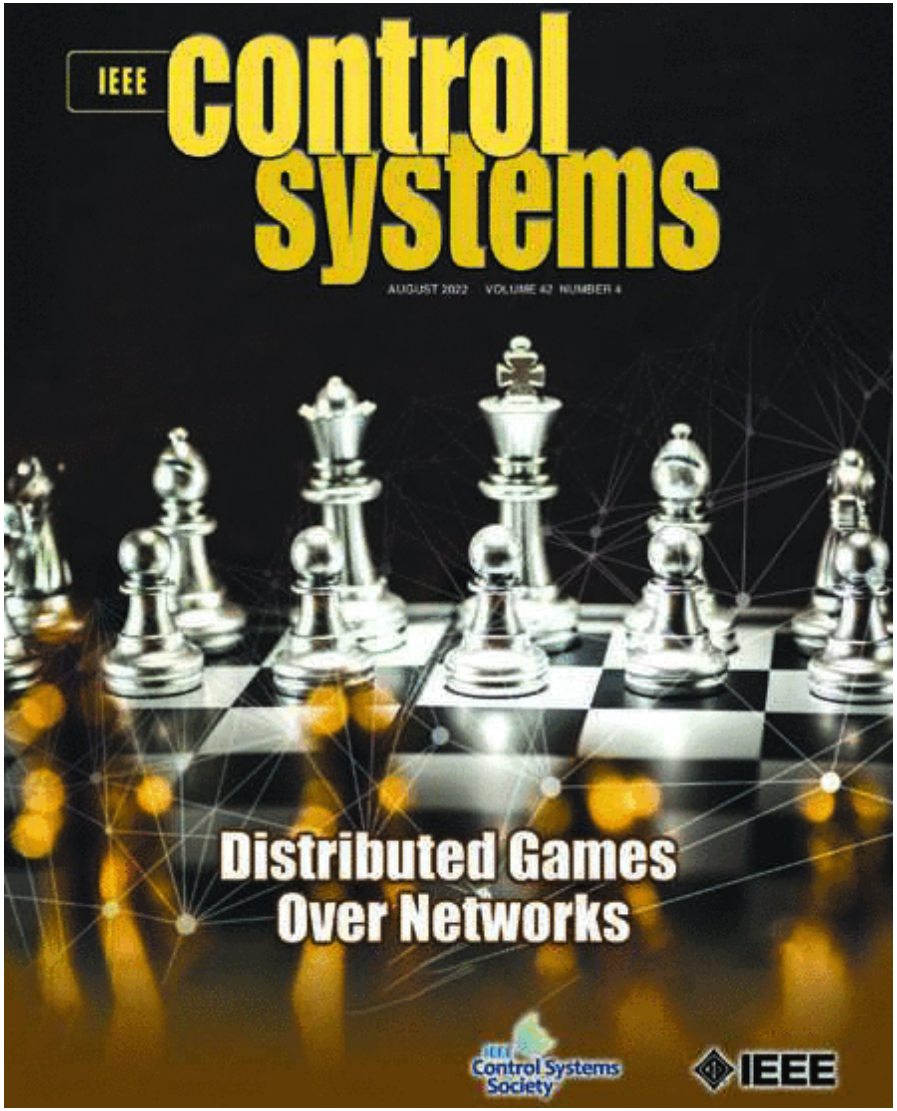
- **Game theory** models complex systems/infrastructures where (possibly selfish) decision makers ("agents") share resources







Motivation: Sharing resources in Complex Systems (+)



Pavel, Hong, Başar, Grammatico, Shanbhag, ...
Distributed Games over Networks, IEEE CSM, 2022

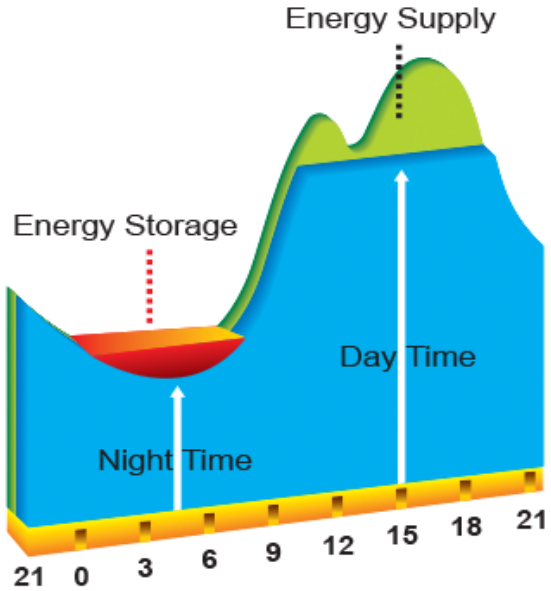
Example 1: Demand Side Management

Demand Side Management:

de-synchronize/flatten net energy demand of prosumers

Prosumer i:

$$\begin{cases} \min\limits_{u_i} & \text{discomfort}_i\left(u_i\right) \\ & \quad + \text{congestion\_cost}(u_i, \underbrace{\textstyle\sum_j u_j}_{\text{aggregation}}) \\ \text{s.t.} & u_i \in \{\text{limits}(\textstyle\sum_j u_j)\} \end{cases}$$



Mohsenian-Rad et al., *Autonomous demand-side management based on game-theoretic energy consumption scheduling for the future smart grid*, IEEE TSG, 2010

Saad, Han, Poor, Başar, *Game-theoretic methods for the smart grid*, IEEE MSP, 2012

Example 2: Day ahead Electricity Markets

Electricity Generators:

$$(\forall i) \quad \mathbb{P}_i : \begin{cases} \max\limits_{s_i,\, u_i} & \underbrace{p(\sum_j s_j)^\top s_i}_{\text{nodal prices}} - \underbrace{c_i(u_i)}_{\text{generation cost}} - \underbrace{\mathrm{diag}(\lambda)(s_i - u_i)}_{\text{fees}} \\ \text{s.t.} & (s_i, u_i) \in \{\text{limits}\} \end{cases}$$

- s_i = sale, u_i = generation (at all nodes)
- λ = transmission fees

TSO/ISO:

$$\mathbb{P}_{\text{iso}} : \begin{cases} \max\limits_{\lambda} & \text{revenue}(\lambda, \boldsymbol{s}, \boldsymbol{g}) \\ \text{s.t.} & (\boldsymbol{s}, \boldsymbol{u}) \in \{\text{transmission capacities}\} \end{cases}$$

Hobbs, Pang, *Nash–Cournot equilibria in electric power markets with piecewise linear demand functions and joint constraints*, OR, 2007

Example 3: Road Traffic Management

Road Traffic Management:

alleviate road traffic congestion via routing control
(e.g. traffic light control, ramp metering, tolls)

Vehicle group i:

$$\begin{cases} \min\limits_{u_i} & \text{std\_travel\_time}_i\left(u_i\right) \\ & \quad + \text{congestion\_time}(u_i, \underbrace{\textstyle\sum_j u_j}_{\text{all}}) \\ \\ \text{s.t.} & u_i \in \{\text{limits}\} \end{cases}$$

Groot, De Schutter, Hellendoorn, *Toward system-optimal routing in traffic networks: A reverse Stackelberg game approach*, IEEE T-ITS, 2012

Example 4: Multi-vehicle Automated Driving

Multi-vehicle Automated Driving:

optimize motion planning via intelligent actions
(e.g. acceleration/deceleration, lane change, overtaking)

Vehicle i:

$$\begin{cases} \min\limits_{u_i} & \text{objective}_i\left(u_i, \boldsymbol{u}_{-i}\right) \\ \text{s.t.} & u_i \in \text{vehicle\_dynamics}_i \\ & u_i \in \text{safety}_i\big(u_i, \underbrace{\textstyle\sum_j u_j}_{\text{neighbors}}\big) \end{cases}$$

Schwarting, Pierson, Alonso-Mora, Karaman, Rus, *Social behavior for autonomous vehicles*, PNAS, 2019

Fabiani, Grammatico, *Multi-vehicle automated driving as a generalized mixed-integer potential game*, IEEE T-ITS, 2021

Motivating features of complex systems of systems

1. autonomous sub-systems (local decision authority, preferences)
2. self-interested decision makers (e.g., energy companies)
3. **large-scale** (hundreds/thousands of decision makers)
4. locality of information/privacy
5. disruptive if left uncontrolled

power outages

traffic jam

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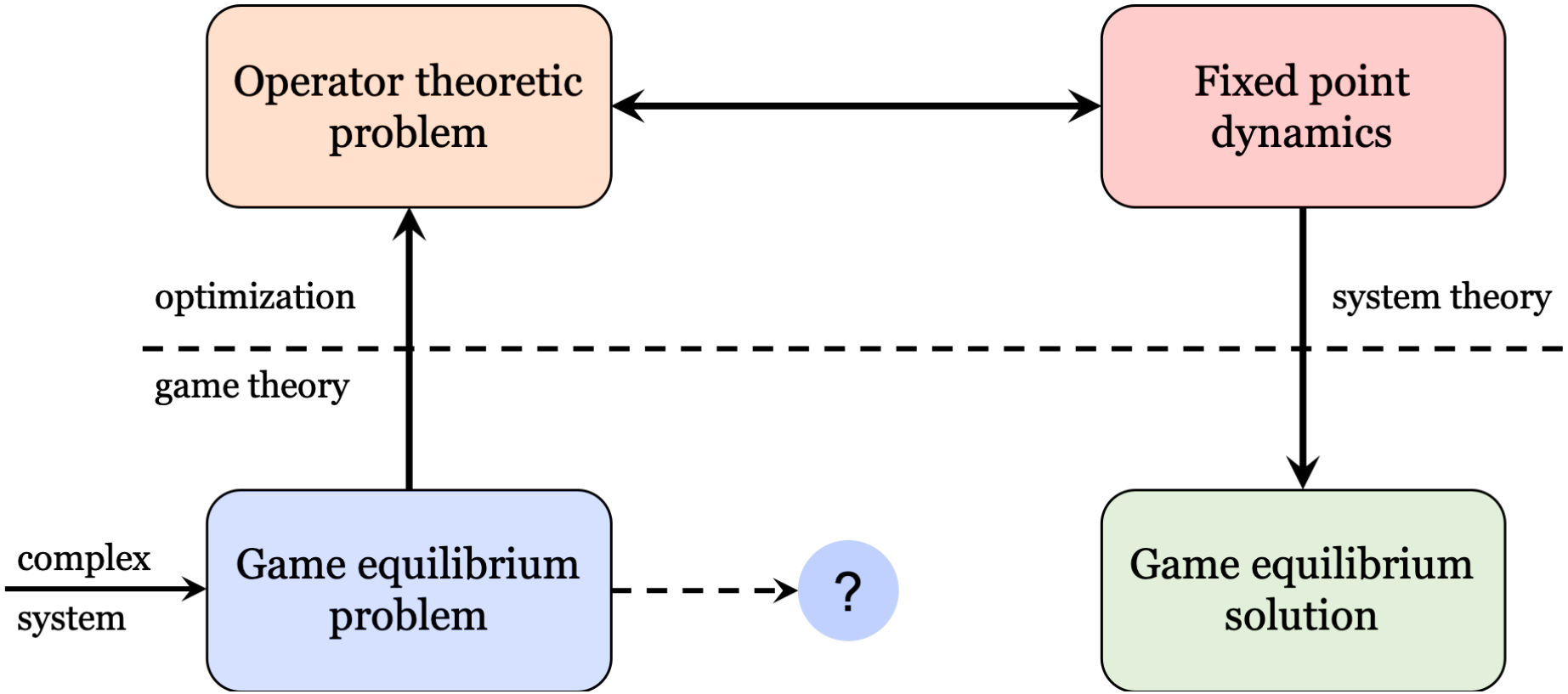
Game modeling and equilibrium seeking



Game theoretic setting

- N autonomous agents (decision makers)
- each agent i has decision variable (strategy) x_i
- $\boldsymbol{x} := \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}$ ("all"); $\quad (\forall i)\ \boldsymbol{x}_{-i} := \begin{bmatrix} \vdots \\ x_{i-1} \\ x_{i+1} \\ \vdots \end{bmatrix}$ ("others")
- each agent i has its individual objective:

$$\begin{cases} \min\limits_{x_i} & \text{cost}_i\left(x_i, \boldsymbol{x}_{-i}\right) \\ \text{s.t.} & x_i \in \text{constraints}_i\left(\boldsymbol{x}_{-i}\right) \end{cases}$$

$\Rightarrow$ "Game" $=$ { inter-dependent optimization problems }

Generalized Nash equilibrium problem

$$\mathbb{P}_i(\boldsymbol{x}_{-i}) : \begin{cases} \min\limits_{x_i} & J_i\left(x_i, \boldsymbol{x}_{-i}\right) \\ \text{s.t.} & x_i \in X_i \qquad \longleftarrow \text{ local} \\ & g\left(x_i, \boldsymbol{x}_{-i}\right) \leq 0 \qquad \longleftarrow \text{ coupling} \end{cases}$$

Generalized Nash equilibrium problem

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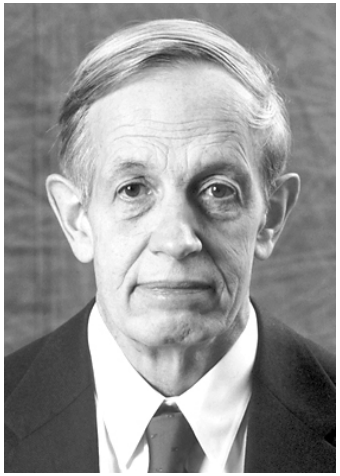
(Generalized) Nash equilibrium (GNE)

$$\boldsymbol{x}^* = \begin{bmatrix} x_1^* \\ \vdots \\ x_N^* \end{bmatrix} \in \begin{bmatrix} \operatorname{argmin} \mathbb{P}_1\left(\boldsymbol{x}_{-1}^*\right) \\ \vdots \\ \operatorname{argmin} \mathbb{P}_N\left(\boldsymbol{x}_{-N}^*\right) \end{bmatrix} =: \mathcal{T}_{\mathsf{BR}}(\boldsymbol{x}^*)$$

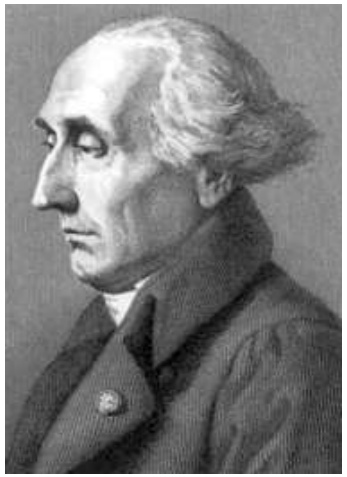


Facchinei, Kanzow, *Generalized Nash equilibrium problems*, 2007

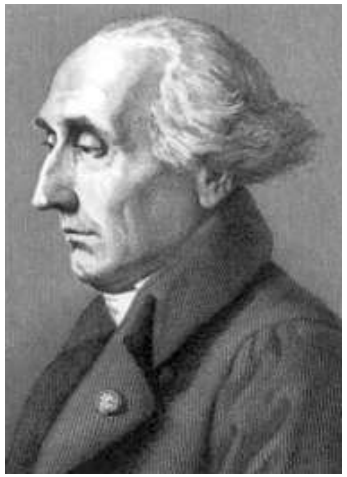
Lagrangian duality: Decouple coupling constraints

Lagrangian functions: (ι = indicator function)

$$L_i\left(\boldsymbol{x}, \lambda_i\right) := J_i\left(x_i, \boldsymbol{x}_{-i}\right) + \iota_{X_i}(x_i) + \lambda_i{}^\top g(\boldsymbol{x})$$



Lagrangian duality: Decouple coupling constraints



Lagrangian functions: (ι = indicator function)

$$L_i\left(\boldsymbol{x}, \lambda_i\right) := J_i\left(x_i,\, \boldsymbol{x}_{-i}\right) + \iota_{X_i}(x_i) + \lambda_i{}^{\top} g(\boldsymbol{x})$$

KKT Theorem: (under regularity/convexity) x_i solves $\mathbb{P}_i(\boldsymbol{x}_{-i})$ iff

$$\begin{cases} 0 \;\in\; \partial_{x_i} L_i(x_i, \boldsymbol{x}_{-i}, \lambda_i) & \longleftarrow \text{stationarity} \\ 0 \;\geq\; g(\boldsymbol{x}) & \longleftarrow \text{feasibility} \\ 0 \;\leq\; \lambda_i \perp g(\boldsymbol{x}) & \longleftarrow \textbf{complementarity} \end{cases}$$

Karush–Kuhn–Tucker inclusion

usual simplification:

- variational GNE $\implies \lambda_i = \lambda$ (fairness)

Karush–Kuhn–Tucker inclusion

usual simplification:

- variational GNE $\implies$ $\lambda_i = \lambda$ (fairness)

KKT system $\equiv$ **KKT inclusion**: (N = normal cone)

$$\mathbf{0} \in \mathcal{T}_{\mathsf{KKT}}(\boldsymbol{x}, \lambda) := \begin{bmatrix} F(\boldsymbol{x}) + \mathrm{N}_{\boldsymbol{X}}(\boldsymbol{x}) + \nabla g(\boldsymbol{x})\lambda \\ \mathrm{N}_{\geq 0}(\lambda) - g(\boldsymbol{x}) \end{bmatrix} \begin{array}{l} \longleftarrow \text{stationarity} \\ \longleftarrow \text{feasibility \& compl.} \end{array}$$

F = **pseudo-subdifferential mapping**:

$$F(\boldsymbol{x}) := \begin{bmatrix} \partial_{x_1} J_1(x_1, \boldsymbol{x}_{-1}) \\ \vdots \\ \partial_{x_N} J_N(x_N, \boldsymbol{x}_{-N}) \end{bmatrix}$$

Generalized Nash equilibrium seeking

Generalized Nash equilibrium seeking problem:

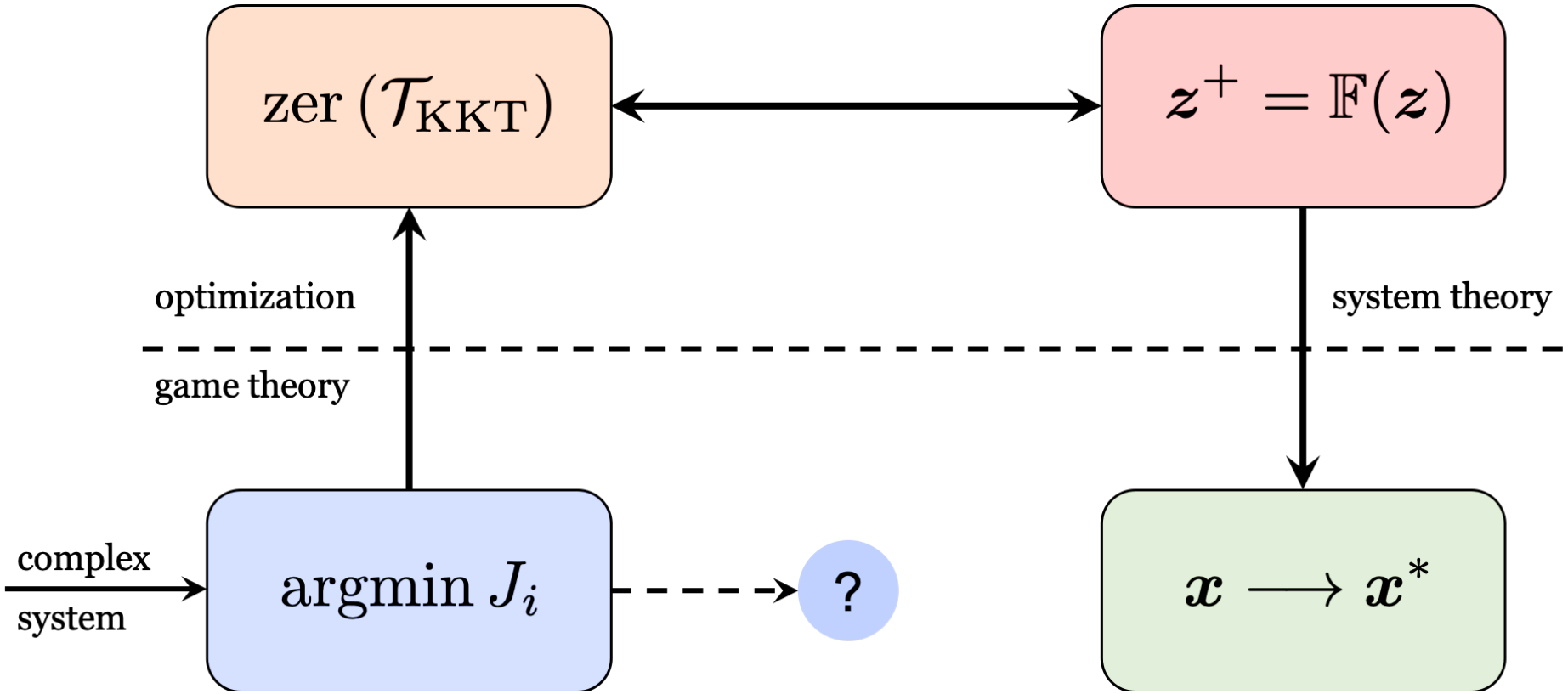
Design dynamical systems (e.g. in discrete time, $k \in \mathbb{N}$)

$$\forall i: \quad \begin{bmatrix} x_i(k+1) \\ \lambda_i(k+1) \end{bmatrix} = f_i\left(\begin{bmatrix} x_i(k) \\ \lambda_i(k) \end{bmatrix}, \boldsymbol{x}_{-i}(k) \right)$$

s.t. $\begin{bmatrix} \boldsymbol{x}(k) \\ \lambda(k) \end{bmatrix} \longrightarrow \begin{bmatrix} \boldsymbol{x}^* \\ \lambda^* \end{bmatrix} \in \underbrace{\{\textbf{KKT} \text{ solutions}\}}_{\mathrm{zer}(\mathcal{T}_{\mathsf{KKT}})}, \quad \boldsymbol{x}(k) \longrightarrow \boldsymbol{x}^* \in \{\textbf{GNE}\}$

Pavel, *Dissipativity theory in game theory: On the role of dissipativity and passivity in Nash equilibrium seeking*, IEEE CSM, 2022

Game modeling and equilibrium seeking (+)



Technical assumptions

Assumptions:

- non-empty, compact, **convex** feasible sets
- g **convex**, continuously **differentiable**
- constraint qualification
- J_i continuous, $J_i(\cdot, \boldsymbol{x}_{-i})$, **convex**
- F (Lipschitz) continuous, (strongly) **monotone** ($\mu > 0$)

$$\langle F(\xi) - F(\zeta), \xi - \zeta \rangle \;\geq\; \mu \left\| \xi - \zeta \right\|^2$$

Generalized Nash equilibrium seeking Literature

Existence and uniqueness:

- Debreu, *A social equilibrium existence theorem*, PNAS, 1952 ← 1st "GNE"
- Arrow, Debreu, *Existence of an equilibrium for a competitive economy*, ECONOMETRICA, 1954
- Rosen, *Existence and uniqueness of equilibrium points for concave N-person games*, ECONOMETRICA, 1965

Equilibrium seeking dynamics:

- Facchinei, Kanzow, *Generalized Nash equilibrium problems*, 2007
- Yi, Pavel, *An operator splitting approach for distributed generalized Nash equilibria computation*, AUTOMATICA, 2019
- Belgioioso, Grammatico, *Semi-decentralized generalized Nash equilibrium seeking in monotone aggregative games*, IEEE TAC, 2022

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Recent challenge: Partial information setting

Partial information:

agent i does **not** know $\boldsymbol{x}_{-i}$ $\implies$ **cannot** evaluate $\nabla J_i(x_i, \boldsymbol{x}_{-i})$

- agent i estimates $\boldsymbol{x}_{-i}$:

  $\boldsymbol{x}_{i,j}$ = estimate by agent i of x_j

  $\boldsymbol{x}_{i,i} = x_i$ (estimate by agent i of x_i)

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  $\boldsymbol{x}_{i,i} = x_i$ (estimate by agent i of x_i)

Partial information setup:

Agents exchange estimates on undirected, connected graph

Pavel, *Distributed GNE seeking under partial-decision information over networks via a doubly-augmented operator splitting approach*, TAC, 2020

Extended pseudo-gradient is **not** monotone

Extended pseudo-gradient mapping:

$$\boldsymbol{F}(\boldsymbol{x}) := \begin{bmatrix} \nabla_{x_1} J_1(x_1, \boldsymbol{x}_{1,-1}) \\ \vdots \\ \nabla_{x_N} J_N(x_N, \boldsymbol{x}_{N,-N}) \end{bmatrix} \longleftarrow \text{estimated partial gradients}$$

- $\boldsymbol{F}$ **not** monotone
- all estimates equal $\implies \boldsymbol{F}\underbrace{(\mathbf{1}_N \otimes \boldsymbol{x})}_{\text{consensus}} = F(\boldsymbol{x})$

Distributed preconditioned proximal point algorithm

Our proposed solution: **Prox + Consensus**

$$\begin{bmatrix} \boldsymbol{x}(k+1) \\ \boldsymbol{z}(k+1) \\ \boldsymbol{\lambda}(k+1) \end{bmatrix} = \mathrm{J}_{\Phi^{-1}\mathcal{A}} \left(\begin{bmatrix} \boldsymbol{x}(k) \\ \boldsymbol{z}(k) \\ \boldsymbol{\lambda}(k) \end{bmatrix} \right)$$

- J = resolvent operator
- preconditioning matrix $\Phi \succ 0 \longrightarrow$ distributed computation

Distributed preconditioned proximal point algorithm

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- J = resolvent operator
- preconditioning matrix $\Phi \succ 0 \longrightarrow$ distributed computation
- $\mathcal{A}$ s.t. $\mathrm{col}(\boldsymbol{x}^*, \boldsymbol{z}^*, \boldsymbol{\lambda}^*) \in \mathrm{zer}(\mathcal{A}) \Longrightarrow \boldsymbol{x}^* = \mathbf{1}_N \otimes \boldsymbol{x}^*$, $\boldsymbol{x}^*$ **v-GNE**

Bianchi, Belgioioso, Grammatico, *Fast generalized Nash equilibrium seeking under partial-decision information*, AUTOMATICA, 2022

Distributed preconditioned proximal point algorithm (+)

Prox + Consensus $\longrightarrow$ **fully distributed algorithm**:

1. distributed information exchange and averaging
2. local estimate ($x_{i,-i}$) update (**consensus**)
3. local strategy (x_i) update (**prox**)
4. local auxiliary (z_i) and dual (λ_i) variables update ($\sim$ **consensus**)

Sketch of convergence analysis

1. small step sizes $\implies$ $\mathcal{A}$ **restricted monotone**:

$$\langle y - y^* \,,\, x - x^* \rangle \geq 0$$

for all $(x, y), (x^*, y^*) \in \mathrm{gph}(\mathcal{A})$, $x^* \in \mathrm{zer}(\mathcal{A})$ $(y^* = 0)$

2. $\Phi^{-1}\mathcal{A}$ restricted monotone ($\Phi-$induced space)

3. $\mathrm{J}_{\Phi^{-1}\mathcal{A}}$ **firmly quasi-nonexpansive** ($\Phi-$induced space)

4. build upon

   Bauschke, Combettes, *Convex analysis and monotone operator theory in Hilbert spaces*

Distributed algorithm for aggregative games

Aggregative games:

$$J_i(x_i, \boldsymbol{x}_{-i}) = \varphi_i(x_i, \mathrm{avg}(\boldsymbol{x})), \qquad \mathrm{avg}(\boldsymbol{x}) := \frac{1}{N}\sum_{i=1}^{N} x_i$$

Distributed algorithm for aggregative games

Aggregative games:

$J_i(x_i, \boldsymbol{x}_{-i}) = \varphi_i(x_i, \mathrm{avg}(\boldsymbol{x})), \quad \mathrm{avg}(\boldsymbol{x}) := \frac{1}{N}\sum_{i=1}^{N} x_i$

- Agent i estimates $\mathrm{avg}(\boldsymbol{x})$, **not** all strategies:

  σ_i = estimate by agent i of $\mathrm{avg}(\boldsymbol{x})$

  $s_i := \sigma_i - x_i$ = extra auxiliary variable

- different Φ, similar dynamics outcome

(Accelerated) Proximal vs Forward-Backward dynamics

1. Preconditioned forward-backward (**FB**) algorithm

   Pavel, *Distributed GNE seeking under partial-decision information over networks via a doubly-augmented operator splitting...*, TAC, 2020

2. Preconditioned proximal point (**PPP**) algorithm
   $\boldsymbol{\omega}^{k+1} = \mathcal{T}_{\mathsf{PPP}}(\boldsymbol{\omega}^k)$

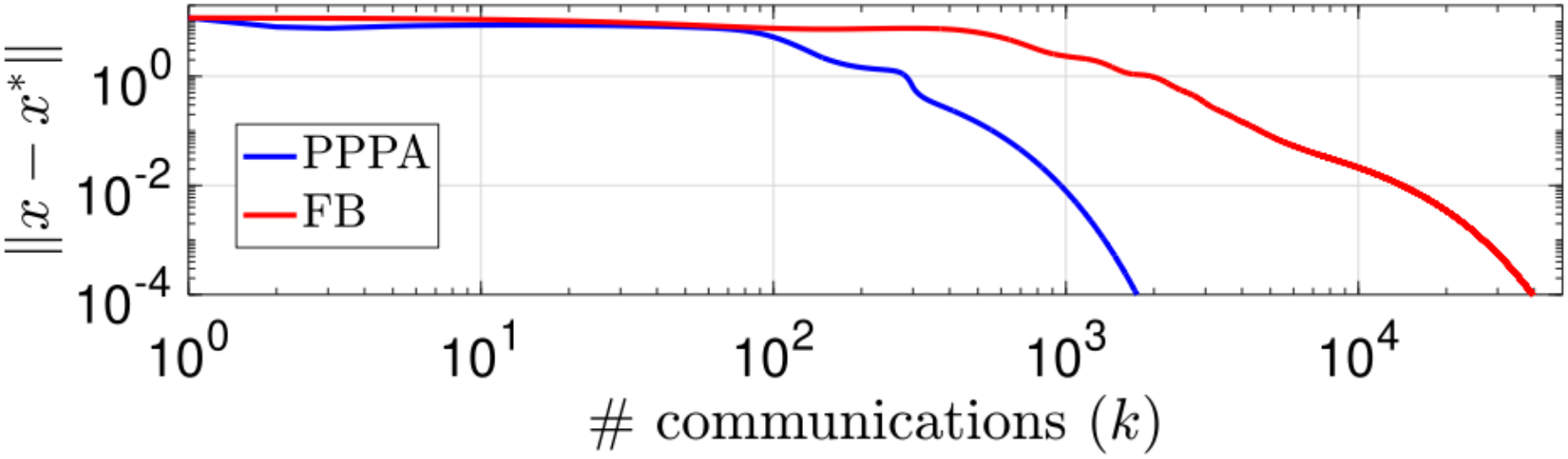
3. PPP with over-relaxation $(1 \leq \eta < 2)$
   $\boldsymbol{\omega}^{k+1} = \eta \mathcal{T}_{\mathsf{PPP}}(\boldsymbol{\omega}^k) + (1-\eta)\boldsymbol{\omega}^k$

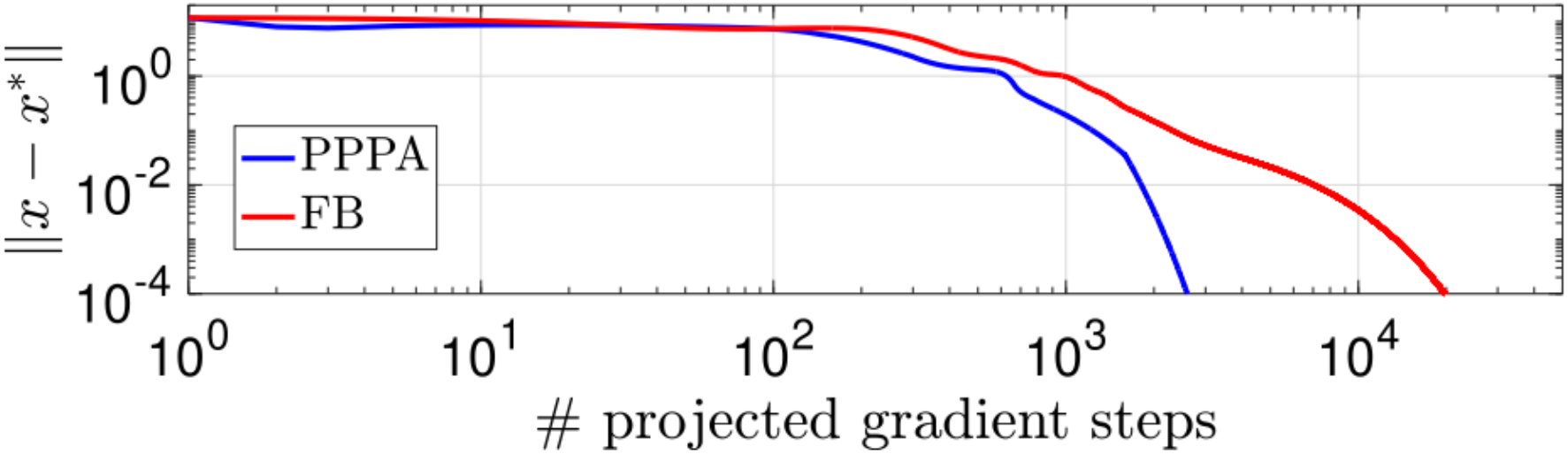
4. PPP with inertia $(\rho \geq 0)$
   $\boldsymbol{\omega}^{k+1} = \mathcal{T}_{\mathsf{PPP}}(\boldsymbol{\omega}^k + \rho(\boldsymbol{\omega}^k - \boldsymbol{\omega}^{k-1}))$

5. PPP with alternated inertia

Numerical experiments

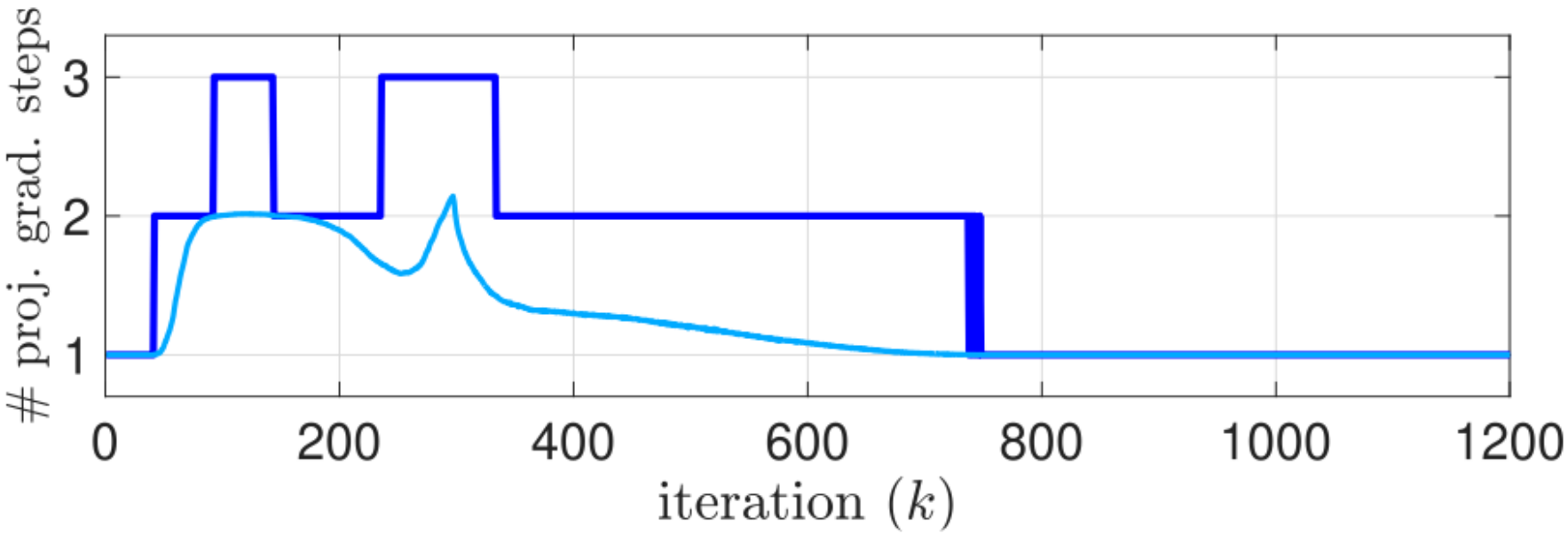
- PPP faster than projected pseudo-gradient (FB)



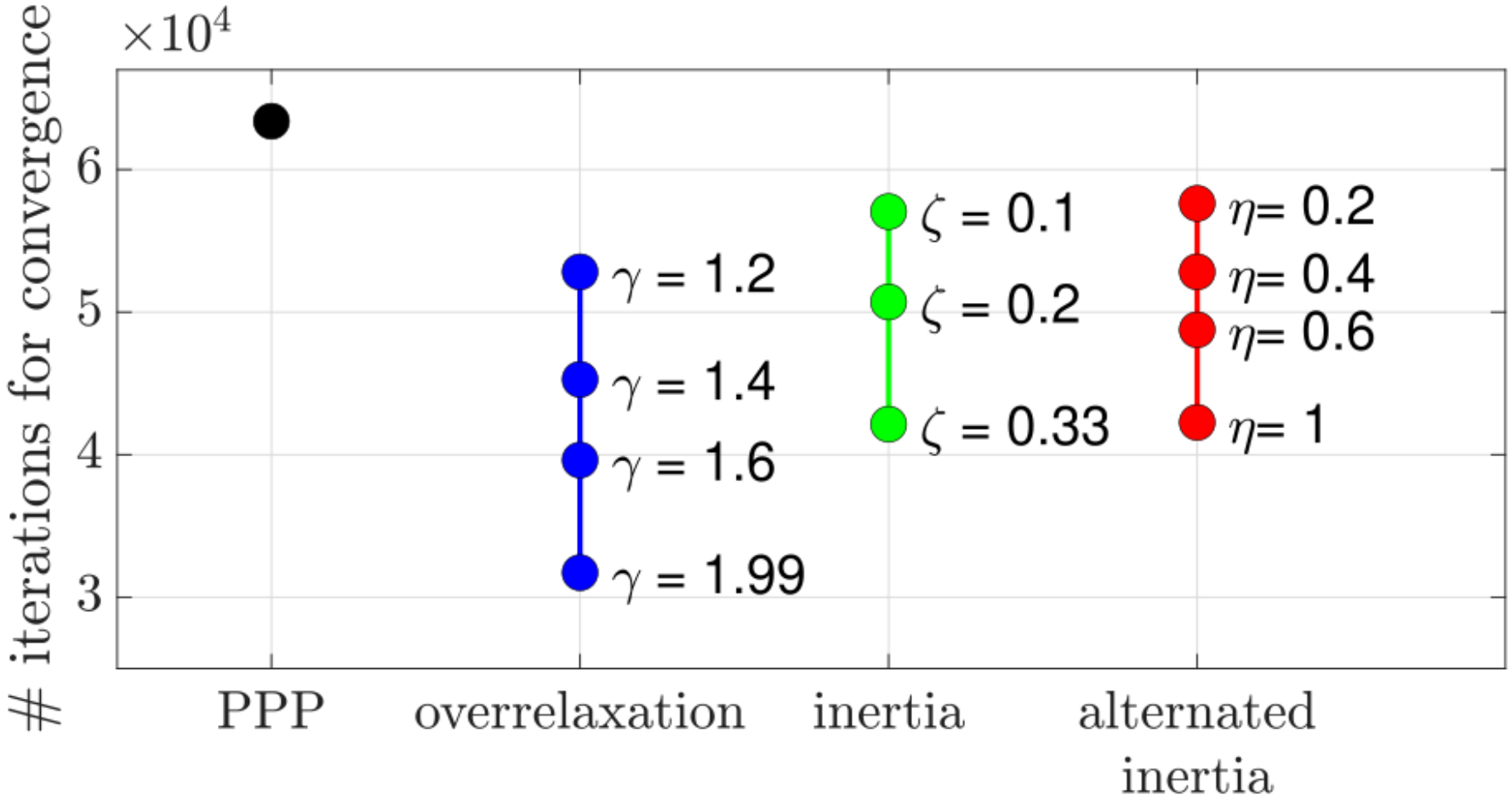


Numerical experiments (+)

- PPP with **inexact** prox: few projected-gradient steps are enough



Numerical experiments (++)



Alternative: Equilibrium seeking in continuous time

Continuous-time GNE seeking dynamics in **partial information**:

- projected pseudo-gradient ($\dot{x}_i = \mathrm{proj}_{\mathrm{TC}}\left(x_i, \nabla J_i(\boldsymbol{x}) + \ldots\right)$)
- **adaptive** step sizes ($\dot{k}_i = \|\text{local disagreement}\|^2$)
- **nonlinear** coupling constraints ($\sum_i g_i(x_i) \leq 0$)
- multi-integrator dynamical agents (e.g. $\dddot{x}_i = u_i, \ \ \boldsymbol{x} \to \mathrm{GNE}$)

De Persis, Grammatico, "*Distributed averaging integral Nash equilibrium seeking algorithms on networks*", AUTOMATICA, 2019

Bianchi, Grammatico, "*Continuous-time fully distributed generalized Nash equilibrium seeking for multi-integrator agents*", AUTOMATICA, 2022

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Zeroth order generalized Nash equilibrium seeking

Zeroth order information setting:

- agent i has **no access** to $\nabla_{x_i} J_i$ (first order information)
- instead agent i measures J_i (zeroth order information)

$\Longrightarrow$ agent i estimates $\nabla_{x_i} J_i$ via filtering available measures

Frihauf, Krstic, Başar, *Nash equilibrium seeking in noncooperative games*, IEEE TAC, 2011 $\leftarrow$ non-generalized NE seeking

Krilasevic, Grammatico, *Learning generalized Nash equilibria in multi-agent dynamical systems via extremum seeking control*, AUTOMATICA, 2021

Zou, Lygeros, *Semi-decentralized zeroth-order algorithms for stochastic generalized Nash equilibrium seeking*, IEEE TAC, 2022

Stochastic generalized Nash equilibrium seeking

Stochastic information setting:

- expected value cost functions $\mathbb{E}_\xi \left[J_i(x_i, \boldsymbol{x}_{-i}, \xi) \right], \ \ \xi \sim \mathbb{P}$
- samples available $\xi^{(1)}, \xi^{(2)}, \ldots, \sim \mathbb{P}$

Stochastic approximation method:

$F \simeq \hat{F} =$ empirical sum of sampled pseudo-gradients

Koshal, Nedic, Shanbhag, *Regularized iterative stochastic approximation methods for stochastic variational inequality problems*, IEEE TAC, 2013

Yu, van der Schaar, Sayed, *Distributed learning for stochastic generalized Nash equilibrium problems*, IEEE TSP, 2017

Stochastic pseudo-gradient dynamics

Relaxation: $\bar{x}_i(k) = (1-\delta)x_i(k) + \delta\bar{x}_i(k-1)$ (same for z_i's, λ_i's)

Projected Pseudo-Gradient + Consensus dynamics:

1. local strategy (x_i) update (**proj** of empirical pseudo-gradient)
2. local auxiliary (z_i) and dual (λ_i) variables update ($\sim$ **consensus**)

Franci, Grammatico, *Stochastic generalized Nash equilibrium seeking in merely monotone games*, IEEE TAC, 2022

Franci, Grammatico, *Stochastic generalized Nash equilibrium seeking under partial-decision information*, AUTOMATICA, 2022

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Conclusion

- In **Convex-Monotone Games**
  equilibria are zeros of (restricted) monotone operators

- Fixed-point and Monotone Operator Theory
  are key to establish convergence to equilibria

- **Preconditioned Proximal Point** algorithms converge fast!

- Application areas: smart power grids, energy/data markets, road traffic control, automated driving, network congestion control, ...

Outlook

- Available **partial-information** dynamics require **complete estimation graph** (inefficient information exchange)

Open problem:

How to design the **best information exchange graph**

The END: Estimation Network Design for efficient distributed equilibrium seeking

Mattia Bianchi, *Member, IEEE*, Sergio Grammatico, *Senior Member, IEEE*

Bianchi, Grammatico, *The END: Estimation Network Design for efficient distributed equilibrium seeking*, arXiv, 2022

Outlook (+)

- Available dynamics converge towards **some equilibrium** (possibly inefficient from a system-level perspective)

Fundamental open problem:

How to design dynamics convergent towards the **best equilibrium**

Optimal selection and tracking of generalized Nash equilibria in monotone games

Emilio Benenati, Wicak Ananduta, and Sergio Grammatico

Benenati, Ananduta, Grammatico, *Optimal selection and tracking of generalized Nash equilibria in monotone games*, IEEE TAC (2P), 2022

Outlook (++)

- Available theorems mostly limited to **convex-monotone games**

Practical open problem:

How to design convergent dynamics for **non-convex-mon** games

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Clarke's Local Generalized Nash Equilibria with Nonconvex Coupling Constraints

Paolo Scarabaggio, *Graduate Student Member, IEEE,* Raffaele Carli, *Senior Member, IEEE,* Sergio Grammatico, *Senior Member, IEEE,* and Mariagrazia Dotoli, *Senior Member, IEEE*

Scarabaggio, Carli, Grammatico, Dotoli, *Clarke's local generalized Nash equilibria with nonconvex coupling constraints*, TechRxiv, 2022

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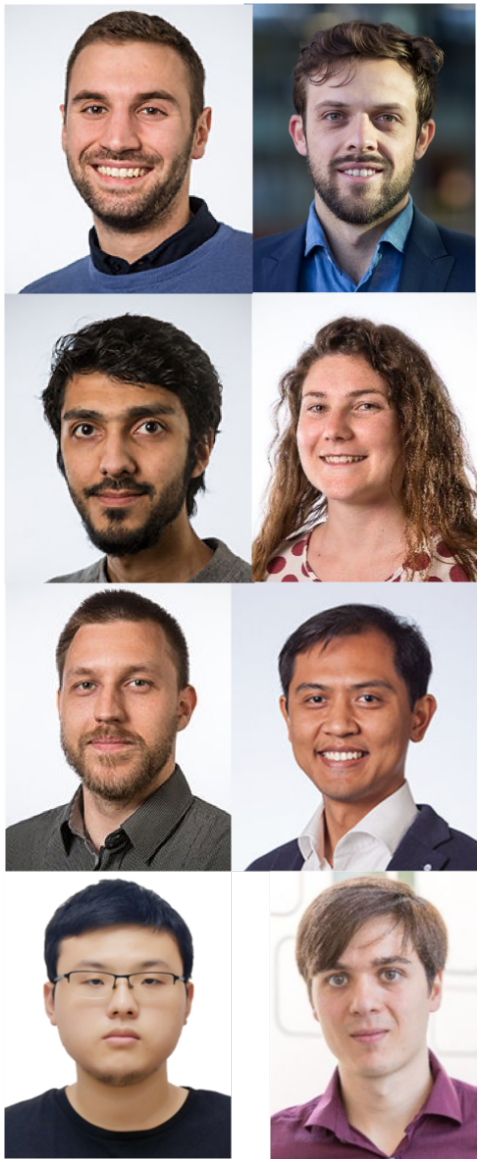
  - Aitazaz Raja ◦ Suad Krilasevic
  - (Barbara Franci) ◦ Wicak Ananduta
  - Emilio Benenati ◦ Luyao Zhang
  - Reza Baghbadorani ◦ Giorgios Pantakis



- **Funding agencies**

Thank you for your kind attention

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