Recovery from Power Sums

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Set up

**Idea:** there are $n$ unknown complex numbers, $z_1, z_2, \ldots, z_n$, and we are given a set $A = \{a_1, a_2, \ldots, a_m\}$ of $m$ distinct positive integers.
Set up

**Idea:** there are $n$ unknown complex numbers, $z_1, z_2, \ldots, z_n$, and we are given a set $\mathcal{A} = \{a_1, a_2, \ldots, a_m\}$ of $m$ distinct positive integers.

We take ‘measurements’

$$\sum_{i=1}^{n} z_i^{a_j} = c_j, \quad j = 1, \ldots, m.$$
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We take ‘measurements’

$$\sum_{i=1}^{n} z_i^{a_j} = c_j, \quad j = 1, \ldots, m.$$ 

**Question:** can we recover $z_1, \ldots, z_n$ from $c_1, \ldots, c_m$?
Example

\[ z_1 = 6, \quad z_2 = 8, \quad z_3 = 13, \quad A = \{2, 5, 7, 8\} \]
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\]

We take 4 measurements:

\[
\sum_{i=1}^{3} z_i^2 = 269, \quad \sum_{i=1}^{3} z_i^5 = 411837, \quad \sum_{i=1}^{3} z_i^7 = 65125605, \quad \sum_{i=1}^{3} z_i^8 = 834187553.
\]
Example

\[ z_1 =?, \ z_2 =?, \ z_3 =?, \ \mathcal{A} = \{2, 5, 7, 8\} \]

We take 4 measurements:

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\sum_{i=1}^{3} x_i^2 = 269, \quad \sum_{i=1}^{3} x_i^5 = 411837, \quad \sum_{i=1}^{3} x_i^7 = 65125605, \quad \sum_{i=1}^{3} x_i^8 = 834187553.
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**Recovery:** replace \( z_i \) by variables \( x_i \) and solve the system of equations.
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We take 4 measurements:

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\begin{align*}
\sum_{i=1}^{3} x_i^2 &= 269, \\
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\end{align*}
\]

Recovery: replace \( z_i \) by variables \( x_i \) and solve the system of equations.

We compute the Gröbner basis and obtain

\[
\{ x_1 + x_2 + x_3 - 27, \ x_2^2 + x_2x_3 + x_3^2 - 27(x_2 + x_3) + 230, \ (x_3 - 6)(x_3 - 8)(x_3 - 13) \}. \]
Example

\[ z_1 = 6, \quad z_2 = 8, \quad z_3 = 13, \quad \mathcal{A} = \{2, 5, 7, 8\} \]

We take 4 measurements:

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\sum_{i=1}^{3} x_i^2 = 269, \quad \sum_{i=1}^{3} x_i^5 = 411837, \quad \sum_{i=1}^{3} x_i^7 = 65125605, \quad \sum_{i=1}^{3} x_i^8 = 834187553.
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**Recovery:** replace \( z_i \) by variables \( x_i \) and solve the system of equations.

We compute the Gröbner basis and obtain

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\{ x_1 + x_2 + x_3 - 27, \quad x_2^2 + x_2 x_3 + x_3^2 - 27(x_2 + x_3) + 230, \quad (x_3 - 6)(x_3 - 8)(x_3 - 13) \}.
\]

So \( \{z_1, z_2, z_3\} \) recovered uniquely!
Model of the problem

For each pair \((n, A)\) we consider the map

\[
\phi_A : \mathbb{C}^n \to \mathbb{C}^m,
\]

where \(\phi_j = x_1^{a_j} + \cdots + x_n^{a_j}\) for \(j = 1, \ldots, m\), and we study its fibers.

Equivalently: we study the polynomial system

\[
x_1^{a_j} + x_2^{a_j} + \cdots + x_n^{a_j} = c_j \quad \text{for } j = 1, 2, \ldots, m.
\]
Model of the problem

For each pair \((n, \mathcal{A})\) we consider the map

\[ \phi_{\mathcal{A}} : \mathbb{C}^n \longrightarrow \mathbb{C}^m, \]

where \(\phi_j = x_1^{a_j} + \cdots + x_n^{a_j}\) for \(j = 1, \ldots, m\), and we study its fibers.

Equivalently: we study the polynomial system

\[ x_1^{a_j} + x_2^{a_j} + \cdots + x_n^{a_j} = c_j \]

for \(j = 1, 2, \ldots, m\).

Three cases:

- \(m > n \implies\) overconstrained: possibly no solutions.
- \(m < n \implies\) underconstrained: positive dimensional variety.
- \(m = n \implies\) square: finitely many points if \(\mathcal{A}, c_1, \ldots, c_m\) ‘general’.
Example (continued)

\[ z_1 = 6, \ z_2 = 8, \ z_3 = 13, \ A = \{2, 5, 7, 8\} \]
Example (continued)

\[ z_1 = 6, \; z_2 = 8, \; z_3 = 13, \; \mathcal{A} = \{2, 5, 7, 8\} \]

**Recall:** when we take all 4 measurements, we have unique recovery.
Example (continued)

\[ z_1 = 6, \ z_2 = 8, \ z_3 = 13, \ \mathcal{A} = \{2, 5, 7, 8\} \]

**Recall:** when we take all 4 measurements, we have unique recovery.

If we take \( m = 3 \) measurements with \( \mathcal{A} = \{2, 5, 7\} \), solve

\[
\begin{align*}
\sum_{i=1}^{3} x_i^2 &= 269, \\
\sum_{i=1}^{3} x_i^5 &= 411837, \\
\sum_{i=1}^{3} x_i^7 &= 65125605.
\end{align*}
\]

We get 66 complex solutions (four less than the Bézout bound \( 70 = 2 \cdot 5 \cdot 7 \)).
Example (continued)

$z_1 = 6, \ z_2 = 8, \ z_3 = 13, \ \mathcal{A} = \{2, 5, 7, 8\}$

**Recall:** when we take all 4 measurements, we have unique recovery.

If we take $m = 2$ measurements with $\mathcal{A} = \{2, 5\}$, solve

$$\sum_{i=1}^{3} x_i^2 = 269, \quad \sum_{i=1}^{3} x_i^5 = 411837.$$  

We get an algebraic curve of degree $10 = 2 \cdot 5$.  

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Questions

\[ \phi_A = \phi_{A,C} : \mathbb{C}^n \longrightarrow \mathbb{C}^m, \quad \text{where} \quad \phi_j = x_1^{a_j} + \cdots + x_n^{a_j} \quad \text{for} \quad j = 1, \ldots, m. \]

Define maps \( \phi_{A,F} \) similarly for \( F = \mathbb{R}, \mathbb{R}_{\geq 0} \).
Questions

\[ \phi_A = \phi_{A,C} : \mathbb{C}^n \rightarrow \mathbb{C}^m, \text{ where } \phi_j = x_1^{a_j} + \cdots + x_n^{a_j} \text{ for } j = 1, \ldots, m. \]

Define maps \( \phi_{A,F} \) similarly for \( F = \mathbb{R}, \mathbb{R}_{\geq 0} \).

Questions:

- What is the expected dimension of fibers of \( \phi_A \)?
- How many complex solution can we expect in the zero-dimensional case?
- When is the recovery from power sums given by a set \( A \) unique (over \( \mathbb{C}/\mathbb{R}/\mathbb{R}_{\geq 0} \))?
Today

1. Fibers

2. Square case

3. Recovery from norms
Fibers
Fibers of $\phi_A$

$\phi_A : \mathbb{C}^n \rightarrow \mathbb{C}^m$, where $\phi_j = x_1^{a_j} + \cdots + x_n^{a_j}$ for $j = 1, \ldots, m$.

For $c = (c_1, \ldots, c_m) \in \mathbb{C}^m$, we have $\phi_A^{-1}(c) = V(I_c)$ with

$$I_c = \langle \phi_1(x) - c_1, \ldots, \phi_m(x) - c_m \rangle \subseteq \mathbb{C}[x_1, \ldots, x_n].$$
Fibers of $\phi_A$

\[ \phi_A : \mathbb{C}^n \longrightarrow \mathbb{C}^m, \text{ where } \phi_j = x_1^{a_j} + \cdots + x_n^{a_j} \text{ for } j = 1, \ldots, m. \]

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**Proposition (Melánová, Sturmfels, W., '22)**

Assume \( m \leq n \). Then the following hold:

(i) The map \( \phi \) is dominant, i.e., the image of \( \phi_A \) is dense in \( \mathbb{C}^m \).

(ii) For generic \( c \in \mathbb{C}^m \), the ideal \( I_c \) is radical, and its variety \( V(I_c) \) has dimension \( n - m \).
Unique recovery over $\mathbb{C}$

Recall previous Example: $\phi_A : \mathbb{C}^n \to \mathbb{C}^m$ and $c = \phi_A(6, 8, 13)$

- $n = 3, m = 3, A = \{2, 5, 7\}, \quad |\phi_A^{-1}(c)| = 66.$
Unique recovery over $\mathbb{C}$

Recall previous Example: $\phi_A : \mathbb{C}^n \to \mathbb{C}^m$ and $c = \phi_A(6, 8, 13)$

- $n = 3$, $m = 3$, $A = \{2, 5, 7\}$, $|\phi_A^{-1}(c)| = 66$.
- $n = 3$, $m = 4$, $A = \{2, 5, 7, 8\}$, $|\phi_A^{-1}(c)| = 3! ((6, 8, 13), (6, 13, 8), \ldots)$
Unique recovery over $\mathbb{C}$

Recall previous Example: $\phi_A : \mathbb{C}^n \to \mathbb{C}^m$ and $c = \phi_A(6, 8, 13)$

- $n = 3, m = 3, A = \{2, 5, 7\}, \ |\phi_A^{-1}(c)| = 66.$
- $n = 3, m = 4, A = \{2, 5, 7, 8\}, |\phi_A^{-1}(c)| = 3! \ ((6, 8, 13), (6, 13, 8), \ldots)$

Conjecture (Melánová, Sturmfels, W., ’22)

*The recovery of a set of $n$ complex numbers from $n + 1$ power sums with coprime powers is unique. In other words, for $m = n + 1$, for generic points $z \in \mathbb{C}^n$, the fiber $\phi_A^{-1}(\phi_A(z))$ coincides with the set of $n!$ coordinate permutations of $z$."

We verified this using Gröbner bases in magma for a range of small cases ($n = 4, \sum_{a \in A} \leq 52, \ n = 5, \sum_{a \in A} \leq 49$).
Square case
Square case: $m = n$

Set $\mathcal{A} = \{a_1, a_2, \ldots, a_n\}$ and consider the system

$$x_1^{a_j} + x_2^{a_j} + \cdots + x_n^{a_j} = c_j \quad \text{for } j = 1, 2, \ldots, n,$$

for some $c = (c_1, \ldots, c_n) \in \mathbb{C}^n$. 
Square case: \( m = n \)

Set \( A = \{a_1, a_2, \ldots, a_n\} \) and consider the homogenized system \( HS \)

\[
x_1^{a_j} + x_2^{a_j} + \cdots + x_n^{a_j} = c_jx_0^{a_j} \quad \text{for } j = 1, 2, \ldots, n,
\]

for some \( c = (c_1, \ldots, c_n) \in \mathbb{C}^n \).

This gives an intersection of \( n \) hypersurfaces in \( \mathbb{P}^n \).
Square case: \( m = n \)

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This gives an intersection of \( n \) hypersurfaces in \( \mathbb{P}^n \).

**Bézout**: the number of solutions to this system is \( a_1 \cdot a_2 \cdots a_n \), counted with multiplicities.
Square case: $m = n$

Set $A = \{a_1, a_2, \ldots, a_n\}$ and consider the homogenized system $HS$

$$x_1^{a_j} + x_2^{a_j} + \cdots + x_n^{a_j} = c_j x_0^{a_j}$$

for $j = 1, 2, \ldots, n$,

for some $c = (c_1, \ldots, c_n) \in \mathbb{C}^n$.

This gives an intersection of $n$ hypersurfaces in $\mathbb{P}^n$.

**Bézout**: the number of solutions to this system is $a_1 \cdot a_2 \cdots a_n$, counted with multiplicities.

**Question**: what can we say about the drop from this Bézout bound in the original (affine) system?
Square case: \( m = n \)

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\textbf{Bézout}: the number of solutions to this system is \( a_1 \cdot a_2 \cdots a_n \), counted with multiplicities.

\textbf{Question}: what can we say about the drop from this \textit{Bézout bound} in the original (affine) system?

\[
| \phi_A^{-1}(c) | = | \{ \text{sol'ns to } HS \} | - | \{ \text{sol'ns to } HS \text{ with } x_0 = 0 \} | = a_1 \cdots a_n - | \{ \text{points at infinity} \} |.
\]
Example (continued)

Example

\[ A = \{2, 5, 7\} \]

\[ S = \begin{cases} 
  x^2 + y^2 + z^2 = 269 \\
  x^5 + y^5 + z^5 = 411837 \\
  x^7 + y^7 + z^7 = 65125605 
\end{cases} \]

Bézout bound: 70
Number of solutions: 66
Example (continued)

Example

\( A = \{2, 5, 7\} \)

\[
HS = \begin{cases} 
    x^2 + y^2 + z^2 = 269w^2 \\
    x^5 + y^5 + z^5 = 411837w^5 \\
    x^7 + y^7 + z^7 = 65125605w^7
\end{cases}
\]

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Number of solutions: 66

We homogenize and find 2 points at infinity:

\[
HS \cap \{w = 0\} = \{(1 : \zeta : \zeta^2 : 0), (1 : \zeta^2 : \zeta : 0)\}.
\]

Each of these points has multiplicity 2 on \( HS \).
The square case for $n = 2$

Consider $\mathcal{A} = \{a, b\}$ with $a < b$, and the corresponding system.

$$S = \begin{cases} x^a + y^a = c_a \\ x^b + y^b = c_b \end{cases}$$
The square case for $n = 2$

Consider $\mathcal{A} = \{a, b\}$ with $a < b$, and the corresponding system.

$$HS = \begin{cases} x^a + y^a &= c_az^a \\ x^b + y^b &= c_bz^b \end{cases}$$

How many solutions of $S$? $\Leftrightarrow$ How many solutions of $HS$ with $z = 0$?

Two cases:

(i) $\gcd(a, b) = 1$,

(ii) $\gcd(a, b) > 1$. 

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The square case for $n = 2$

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$$HS = \begin{cases} x^a + y^a = c_a z^a \\ x^b + y^b = c_b z^b \end{cases}$$

(i) $\gcd(a, b) = 1$
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(i) $\gcd(a, b) = 1$

\((*:*:0)\) solution to $HS$ in $\mathbb{P}^2 \iff \gcd(x^a + y^a, x^b + y^b) \neq 1$
The square case for $n = 2$

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(i) $\gcd(a, b) = 1$

$(* : * : 0)$ solution to $HS$ in $\mathbb{P}^2 \iff \gcd(x^a + y^a, x^b + y^b) \neq 1$

$\iff a, b$ are both odd $\iff \gcd(x^a + y^a, x^b + y^b) = x + y$. 
The square case for $n = 2$

Consider $\mathcal{A} = \{a, b\}$ with $a < b$, and the corresponding system.

$$HS = \begin{cases} 
x^a + y^a = c_az^a \\
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\end{cases}$$

(i) $\gcd(a, b) = 1$

$(\ast : \ast : 0)$ solution to $HS$ in $\mathbb{P}^2 \iff \gcd(x^a + y^a, x^b + y^b) \neq 1$

$\iff a, b$ are both odd $\iff \gcd(x^a + y^a, x^b + y^b) = x + y$.

We conclude:

$$HS \cap \{z = 0\} \neq \emptyset \iff a, b$ both odd,$$
and if this is the case, then

$$HS \cap \{z = 0\} = \{x + y = 0\} = \{(1 : -1 : 0)\}.$
The square case for $n = 2$

(i) $\gcd(a, b) = 1$, $a, b$ odd, $P = (1 : -1 : 0) \in HS$.

$$HS = \begin{cases} 
  x^a + y^a &= c_aza^a \\
  x^b + y^b &= c_bzb^b 
\end{cases}$$
The square case for $n = 2$

(i) $\gcd(a, b) = 1, \ a, b$ odd, \ $P = (1 : -1 : 0) \in HS$. \ \(\mathbb{A}^2: \{x \neq 0\}\).

$$HS \cap \mathbb{A}^2 = \begin{cases} 1 + y^a = c_za^a \\ 1 + y^b = c_bz^b \end{cases}$$
The square case for $n = 2$

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$$HS \cap \mathbb{A}^2 = \begin{cases} 1 + y^a = c_az^a \\ 1 + y^b = c_bz^b \end{cases}$$

$\mathcal{O}_{P, \mathbb{A}^2}$ local ring of $P = (-1, 0) \in \mathbb{A}^2$, ideal in $\mathcal{O}_{P, \mathbb{A}^2}$ given by

$$I = \langle 1 + y^a - c_az^a, 1 + y^b - c_bz^b \rangle.$$
The square case for $n = 2$

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$$HS \cap \mathbb{A}^2 = \begin{cases} 
1 + y^a = c_a z^a \\
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\end{cases}$$

$\mathcal{O}_{P, \mathbb{A}^2}$ local ring of $P = (-1, 0) \in \mathbb{A}^2$, ideal in $\mathcal{O}_{P, \mathbb{A}^2}$ given by

$$I = \langle 1 + y^a - c_a z^a, 1 + y^b - c_b z^b \rangle.$$

The multiplicity of $P$ in $HS$ is equal to $\dim_{\mathbb{C}}(\mathcal{O}_{P, \mathbb{A}^2}/I) = a$. 
The square case for $n = 2$

Consider $\mathcal{A} = \{a, b\}$ with $a < b$, and the corresponding system.

$$HS = \begin{cases} x^a + y^a = c_az^a \\ x^b + y^b = c_bz^b \end{cases}$$

(ii) $\gcd(a, b) = g > 1$
The square case for $n = 2$

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(ii) $\gcd(a, b) = g > 1$

Rewrite the system

$$(x^g)^{\frac{a}{g}} + (y^g)^{\frac{a}{g}} = c_az^a, \quad (x^g)^{\frac{b}{g}} + (y^g)^{\frac{b}{g}} = c_bz^b$$
The square case for $n = 2$

Consider $\mathcal{A} = \{a, b\}$ with $a < b$, and the corresponding system.

$$HS = \begin{cases} 
  x^a + y^a = c_az^a \\
  x^b + y^b = c_bz^b 
\end{cases}$$

(ii) $\gcd(a, b) = g > 1$

Rewrite the system

$$\left( x^g \right)^{\frac{a}{g}} + \left( y^g \right)^{\frac{a}{g}} = c_az^a, \quad \left( x^g \right)^{\frac{b}{g}} + \left( y^g \right)^{\frac{b}{g}} = c_bz^b$$

Conclude: $HS \cap \{z = 0\} \neq \emptyset \iff \frac{a}{g}, \frac{b}{g}$ both odd, in which case

$$HS \cap \{z = 0\} = \{ x^g + y^g = 0 \} = \{(\zeta^i : 1 : 0) : i = 1, \ldots, g\},$$

where $\zeta$ is a primitive $g$-th root of $-1$. Each of these points has multiplicity $a$ in $HS$. 
The square case for $n = 2$

Proposition (Melánová, Sturmfels, W., '22)

Let $a, b$ be integers with $a < b$, and set $g = \gcd(a, b)$. For generic constants $c_a, c_b \in \mathbb{C}$, the number of common solutions in $\mathbb{C}^2$ to the equations $x^a + y^a = c_a$ and $x^b + y^b = c_b$ equals

- $a(b - g)$, if both $a/g$ and $b/g$ are odd,

- the Bézout bound $ab$, otherwise.
The square case for $n = 2$

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Let $a, b$ be integers with $a < b$, and set $g = \gcd(a, b)$. For generic constants $c_a, c_b \in \mathbb{C}$, the number of common solutions in $\mathbb{C}^2$ to the equations $x^a + y^a = c_a$ and $x^b + y^b = c_b$ equals

- $a(b - g)$, if both $a/g$ and $b/g$ are odd,
- the Bézout bound $ab$, otherwise.

Generalization to $n > 2$?
Examples square case for $n=3$

Recall: we want to count the number of complex solutions to the system

$$
S = \begin{cases}
  x^{a_1} + y^{a_1} + z^{a_1} = c_1, \\
  x^{a_2} + y^{a_2} + z^{a_2} = c_2, \\
  x^{a_3} + y^{a_3} + z^{a_3} = c_3,
\end{cases}
$$

for $A = \{a_1, a_2, a_3\}$, $c = (c_1, c_2, c_3) \in \mathbb{C}^3$. 

Example

$A = \{5, 7, 9\}$  
$c = (5, 7, 9)$  
Bézout: $5 \cdot 7 \cdot 9 = 315$  
#sol'ns: 210

$A = \{5, 7, 13\}$  
$c = (5, 7, 13)$  
Bézout: $5 \cdot 7 \cdot 13 = 455$  
#sol'ns: 336

$A = \{5, 8, 9\}$  
$c = (5, 8, 9)$  
Bézout: $5 \cdot 8 \cdot 9 = 360$  
#sol'ns: 360
Examples square case for \( n=3 \)

**Recall:** we want to count the number of complex solutions to the system

\[
S = \begin{cases} 
    x^{a_1} + y^{a_1} + z^{a_1} = c_1, \\
    x^{a_2} + y^{a_2} + z^{a_2} = c_2, \\
    x^{a_3} + y^{a_3} + z^{a_3} = c_3,
\end{cases}
\]

for \( A = \{a_1, a_2, a_3\} \), \( c = (c_1, c_2, c_3) \in \mathbb{C}^3 \).

**Example**

\[
A = \{5, 7, 9\} \quad \text{Bézout: } 5 \cdot 7 \cdot 9 = 315 \quad \#\text{sol’ns: } 210
\]

\[
A = \{5, 7, 13\} \quad \text{Bézout: } 5 \cdot 7 \cdot 13 = 455 \quad \#\text{sol’ns: } 336
\]

\[
A = \{5, 8, 9\} \quad \text{Bézout: } 5 \cdot 8 \cdot 9 = 360 \quad \#\text{sol’ns: } 360
\]
Examples square case for $n=3$

Example

$A = \{5, 7, 9\}$

\[
HS = \begin{cases} 
 x^5 + y^5 + z^5 = 5w^5 \\
 x^7 + y^7 + z^7 = 7w^7 \\
 x^9 + y^9 + z^9 = 9w^9 
\end{cases}
\]

Bézout bound: 315
Number of solutions: 210
Examples square case for $n=3$

Example

$A = \{5, 7, 9\}$

\[
HS = \begin{cases}
   x^5 + y^5 + z^5 = 5w^5 \\
   x^7 + y^7 + z^7 = 7w^7 \\
   x^9 + y^9 + z^9 = 9w^9
\end{cases}
\]

Bézout bound: 315
Number of solutions: 210

$HS \cap \{w = 0\} = \{(1 : -1 : 0 : 0), (1 : 0 : -1 : 0), (0 : 1 : -1 : 0)\}$.

Each of these points has multiplicity 35 on $HS$.

$3 \cdot 35 = 105 = 315 - 210$
Examples square case for $n=3$

Example

$A = \{5, 7, 13\}$

$HS = \begin{cases} 
    x^5 + y^5 + z^5 = 5w^5 \\
    x^7 + y^7 + z^7 = 7w^7 \\
    x^{13} + y^{13} + z^{13} = 13w^{13}
\end{cases}$

Bézout bound: 455
Number of solutions: 336
Examples square case for n=3

Example

\[ A = \{5, 7, 13\} \]

\[
HS = \begin{cases} 
  x^5 + y^5 + z^5 = 5w^5 \\
  x^7 + y^7 + z^7 = 7w^7 \\
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\end{cases}
\]

Bézout bound: 455
Number of solutions: 336

\[
HS \cap \{ w = 0 \} = \{(1 : -1 : 0 : 0), (1 : 0 : -1 : 0), (0 : 1 : -1 : 0), (1 : \zeta : \zeta^2 : 0), (1 : \zeta^2 : \zeta : 0)\}.
\]
Examples square case for $n=3$

Example

$A = \{5, 7, 13\}$

$HS = \begin{cases} 
  x^5 + y^5 + z^5 = 5w^5 \\
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Bézout bound: 455
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$HS \cap \{w = 0\} = \{(1 : -1 : 0 : 0), (1 : 0 : -1 : 0), (0 : 1 : -1 : 0), 
(1 : \zeta : \zeta^2 : 0), (1 : \zeta^2 : \zeta : 0)\}$.

Each of these points has multiplicity 35 or 7 on $HS$.

$3 \cdot 35 + 2 \cdot 7 = 119 = 455 - 336$
Points at infinity

Consider $\mathcal{A} = \{a, b, c\}$ with $\gcd(a, b, c) = 1$, and the corresponding system

$$S = \begin{cases} 
  x^a + y^a + z^a = d_a \\
  x^b + y^b + z^b = d_b \\
  x^c + y^c + z^c = d_c 
\end{cases}$$
Consider $\mathcal{A} = \{a, b, c\}$ with $\gcd(a, b, c) = 1$, and the corresponding system

$$HS = \begin{cases} 
  x^a + y^a + z^a = d_aw^a \\
  x^b + y^b + z^b = d_bw^b \\
  x^c + y^c + z^c = d_cw^c 
\end{cases}$$
Points at infinity

Consider $\mathcal{A} = \{a, b, c\}$ with $\gcd(a, b, c) = 1$, and the corresponding system

\[
HS = \begin{cases}
    x^a + y^a + z^a = d_a w^a \\
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\end{cases}
\]

Note:

\[
\{(1 : -1 : 0 : 0), (1 : 0 : -1 : 0), (0 : 1 : -1 : 0)\} \subset HS \cap \{w = 0\} \iff abc \neq 0 \mod 2
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\]

\[
\{(1 : \zeta : \zeta^2 : 0), (1 : \zeta^2 : \zeta : 0) : \zeta \text{ primitive cubic root of 1}\} \subseteq HS \cap \{w = 0\} \iff abc \neq 0 \mod 3
\]
Points at infinity

Consider $\mathcal{A} = \{a, b, c\}$ with $\gcd(a, b, c) = 1$, and the corresponding system

$$HS = \begin{cases} 
  x^a + y^a + z^a = d_a w^a \\
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\end{cases}$$

Conjecture (Melánová, Sturmfels, W., '22)

We have

$$HS \cap \{w = 0\} \subseteq \{(1 : -1 : 0 : 0), (1 : 0 : -1 : 0), (0 : 1 : -1 : 0), (1 : \zeta : \zeta^2 : 0), (1 : \zeta^2 : \zeta : 0)\},$$

where $\zeta$ is a primitive cubic root of unity.
Conjecture (Conca, Krattenthaler, Watanabe, ’09)

\[ A = \{a, b, c\}, \text{ with } \gcd(a, b, c) = 1, \text{ then } \phi_1, \phi_2, \phi_3 \text{ define a regular sequence iff} \]

\[ abc \equiv 0 \pmod{6}. \]
CKW Conjecture

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Note:

**NO** points at infinity

\[ \iff \]

(0, 0, 0) is the only solution of \( S \)

\[ \iff \]

\[ x^a + y^a + z^a, x^b + y^b + z^b, x^c + y^c + z^c \]

form a regular sequence.
CKW Conjecture

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Note:

\textbf{NO} points at infinity

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\[ (0, 0, 0) \text{ is the only solution of } S \]

\[ \iff \]

\[ x^a + y^a + z^a, x^b + y^b + z^b, x^c + y^c + z^c \]

form a regular sequence.

So our conjecture \textbf{implies} the CKW conjecture....
Points at infinity: results

Theorem (Conca, Krattenthaler, Watanabe, ’09)
The CKW conjecture holds if either

- A contains 1 and \( n \) with \( 2 \leq n \leq 7 \),
- A contains 2 and 3.

Theorem (Melanová, Sturmfels, W., ’22)
Our conjecture holds for all \( A = \{a, b, c\} \), \( a < b < c \) with \( a + b + c \leq 300 \).
Points at infinity: results

Theorem (Conca, Krattenthaler, Watanabe, ’09)

The CKW conjecture holds if either

- $A$ contains 1 and $n$ with $2 \leq n \leq 7$,
- $A$ contains 2 and 3.

Theorem (Melánová, Sturmfels, W., ’22)

Our conjecture holds for all $A = \{a, b, c\}$, $a < b < c$ with $a + b + c \leq 300$. 
The square case for $n = 3$: conjecture

$\mathcal{A} = \{a_1, a_2, a_3\}$, $\mathcal{A}_p = \{a_1 \mod p, a_2 \mod p, a_3 \mod p\}$

Conjecture (Melánová, Sturmfels, W., '22)

For $n = 3$ and $\gcd(a_1, a_2, a_3) = 1$, the following holds.
The square case for $n = 3$: conjecture

$A = \{a_1, a_2, a_3\}, \ A_p = \{a_1 \mod p, a_2 \mod p, a_3 \mod p\}$

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For $n = 3$ and $\gcd(a_1, a_2, a_3) = 1$, the following holds.

If $0 \in A_3$, then we have

$$\#\text{Solutions} = \begin{cases} a_1a_2a_3 & \text{if } A_2 = \{1, 0\}; \\ a_1a_2a_3 - 3a_1a_2 & \text{if } A_2 = \{1\}. \end{cases}$$
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Points at infinity: \((1 : -1 : 0 : 0), (1 : 0 : -1 : 0), (0 : 1 : -1 : 0)\), each with multiplicity \(a_1a_2\).
The square case for $n = 3$: conjecture

$\mathcal{A} = \{a_1, a_2, a_3\}$, $\mathcal{A}_p = \{a_1 \mod p, a_2 \mod p, a_3 \mod p\}$

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For $n = 3$ and $\gcd(a_1, a_2, a_3) = 1$, the following holds.

If $\mathcal{A}_3 = \{1\}$ or $\{2\}$, then we have

\[
\#\text{Solutions} = \begin{cases} 
  a_1 a_2 a_3 - 4a_1 & \text{if } \mathcal{A}_2 = \{1, 0\}; \\
  a_1 a_2 a_3 - 4a_1 - 3a_1 a_2 & \text{if } \mathcal{A}_2 = \{1\}.
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Points at infinity:
$(1 : -1 : 0 : 0), (1 : 0 : -1 : 0), (0 : 1 : -1 : 0)$, each with multiplicity $a_1 a_2$.
$(1 : \zeta : \zeta^2 : 0), (1 : \zeta^2 : \zeta : 0)$, each with multiplicity $2a_1$. 

Rosa Winter (KCL)
The square case for $n = 3$: conjecture

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Where $i_A$ is always either $a_1, a_2$, or $2a_1$. 
The square case for $n = 3$: conjecture

$\mathcal{A} = \{a_1, a_2, a_3\}$, $\mathcal{A}_p = \{a_1 \mod p, a_2 \mod p, a_3 \mod p\}$

Conjecture (Melánová, Sturmfels, W., '22)

For $n = 3$ and $\gcd(a_1, a_2, a_3) = 1$, the following holds.

If $\mathcal{A}_3 = \{1, 2\}$, then we have

$$\#\text{Solutions} = \begin{cases} a_1a_2a_3 - 2i_\mathcal{A} & \text{if } \mathcal{A}_2 = \{1, 0\}, \\ a_1a_2a_3 - 2i_\mathcal{A} - 3a_1a_2 & \text{if } \mathcal{A}_2 = \{1\}, \end{cases}$$

Where $i_\mathcal{A}$ is always either $a_1, a_2$, or $2a_1$.

Points at infinity:

$(1 : -1 : 0 : 0), (1 : 0 : -1 : 0), (0 : 1 : -1 : 0)$, each with multiplicity $a_1a_2$.

$(1 : \zeta : \zeta^2 : 0), (1 : \zeta^2 : \zeta : 0)$, each with multiplicity $i_\mathcal{A}$. 
What about $n > 3$?

For $n = 4$:

- For which $\mathcal{A}$ are there solutions at infinity?

$\leadsto$ CKW: conjecture some conditions, show that they are necessary.
What about $n > 3$?

For $n = 4$:

- For which $A$ are there solutions at infinity?
  $\Rightarrow$ CKW: conjecture some conditions, show that they are necessary.

- If there are points at infinity, what shape do they have?
  $\Rightarrow$ not always roots of unity for coordinates!
What about $n > 3$?

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Example

$n = 4$, $\mathcal{A} = \{2, 4, 9, 10\}$. The original system has 576 solutions, Bézout bound is 720. There are 72 distinct points at infinity. The minimal polynomial of each of the coordinates of the points has degree 36.
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**For** $n = 4$:

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$n > 4 \ldots$?
Recovery from norms
Recovery from norms

$x = (x_1, \ldots, x_n) \in \mathbb{R}_{\geq 0}^n$, $p$ positive integer, then we have

$$\sum_{i=1}^{n} x_i^p = (\|x\|_p)^p.$$
Recovery from norms

\[ x = (x_1, \ldots, x_n) \in \mathbb{R}_{\geq 0}^n, \quad p \text{ positive integer}, \text{ then we have} \]

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**Question:** Can we recover a vector in \( \mathbb{R}_{\geq 0}^n \) by taking its \( p \)-norm for different \( p \)?
Recovery from norms

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**Question:** Can we recover a vector in \( \mathbb{R}^n_{\geq 0} \) by taking its \( p \)-norm for different \( p \)?

**Already seen:** At least \( n \) different norms are needed for a unique recovery. Conjecturally: \( n + 1 \) different norms provide a unique recovery.
Recovery from norms

\[ x = (x_1, \ldots, x_n) \in \mathbb{R}_{\geq 0}^n, \quad p \text{ positive integer}, \text{ then we have} \]

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**Question:** Can we recover a vector in \( \mathbb{R}_{\geq 0}^n \) by taking its \( p \)-norm for different \( p \)?

**Already seen:** At least \( n \) different norms are needed for a unique recovery. Conjecturally: \( n + 1 \) different norms provide a unique recovery.

**Theorem (Melánová, Sturmfels, W., ’22)**

For \( m = n \), recovery from \( p \)-norms is always unique. Given any set \( A \) of \( n \) positive integers, the map \( \phi_{A, \mathbb{R}_{\geq 0}} : \mathbb{R}_{\geq 0}^n \to \mathbb{R}_{\geq 0}^n \) is injective up to permuting coordinates.
Describing the image of $\phi_{A, \mathbb{R}_{\geq 0}}$

Tarski-Seidenberg: the image of $\phi_{A, \mathbb{R}_{\geq 0}}$ is a semi-algebraic set.

**Proposition**

1. $\text{im}(\phi_{A, \mathbb{R}_{\geq 0}})$ is closed in $\mathbb{R}^m_{\geq 0}$.
2. $m = 2$, $n \geq m$, then

   $$\text{im}(\phi_{A, \mathbb{R}_{\geq 0}}) = \{ (c_1, c_2) \in \mathbb{R}^2_{\geq 0} \mid c_2^{a_1} \leq c_1^{a_2} \leq n^{a_2-a_1} c_1^{a_1} \}.$$
Describing the image of $\phi_{A,R_{\geq 0}}$.

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Proof of 2. $x \in \mathbb{R}^n_{\geq 0}$, $e = (1, 1, \ldots, 1)$. 
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**Proof of 2.** $x \in \mathbb{R}_\geq^n, e = (1, 1, \ldots, 1)$. $\|x\|_{a_1} \geq \|x\|_{a_2}$, so we have

$$1 \leq \frac{\|x\|_{a_1}}{\|x\|_{a_2}} \leq \frac{\|e\|_{a_1}}{\|e\|_{a_2}} = \frac{1}{n^{a_2}}.$$
Describing the image of $\phi_{A,R_{\geq 0}}$

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Proposition

1. $\text{im}(\phi_{A,R_{\geq 0}})$ is closed in $\mathbb{R}^m_{\geq 0}$.
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Proof of 2. $x \in \mathbb{R}^n_{\geq 0}$, $e = (1, 1, \ldots, 1)$. $\|x\|_{a_1} \geq \|x\|_{a_2}$, so we have

$$1 \leq \frac{(\phi_1(x))^{\frac{1}{a_1}}}{(\phi_2(x))^{\frac{1}{a_2}}} \leq \frac{\|e\|_{a_1}}{\|e\|_{a_2}} = \frac{\frac{1}{n^{a_1}}}{\frac{1}{n^{a_2}}}.$$
Describing the image of $\phi_{A,\mathbb{R}_{\geq 0}}$

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**Proposition**

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**Generalization to** $m > 2$?
Generalizations to $m \geq 3$

$$\Delta_m = \{ u \in \mathbb{R}_{\geq 0}^{m+1} : u_1 + \cdots + u_{m+1} = 1 \} \text{ probability symplex}$$

$$\psi_A : \mathbb{R}_{\geq 0}^n \rightarrow \Delta_{m-1} : x \mapsto \frac{1}{\sum_{j=1}^m \|x\|_{a_j}} (\|x\|_{a_1}, \ldots, \|x\|_{a_m})$$
Generalizations to \( m \geq 3 \)

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\Delta_m = \{ u \in \mathbb{R}_{\geq 0}^{m+1} : u_1 + \cdots + u_{m+1} = 1 \} \text{ probability symplex}
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\]

Example

\( m = 3 \) The image of \( \psi_A \) is an \( n \)-gon in \( \{ u_1 > u_2 > u_3 \} \subset \Delta_2 \).
Recovery from Power Sums

Hana Melánová\textsuperscript{a}, Bernd Sturmfels\textsuperscript{b,c}, and Rosa Winter\textsuperscript{d}

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\textbf{ABSTRACT}
We study the problem of recovering a collection of $n$ numbers from the evaluation of $m$ power sums. This yields a system of polynomial equations, which can be underconstrained ($m < n$), square ($m = n$), or overconstrained ($m > n$). Fibers and images of power sum maps are explored in all three regimes, and in settings that range from complex and projective to real and positive. This involves surprising deviations from the Bézout bound, and the recovery of vectors from length measurements by $p$-norms.

\textbf{KEYWORDS}
$p$-norms; solving polynomial systems; images and fibers; power sums

Thank you for your attention!