Recovery from Power Sums

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Set up

Idea: there are n unknown complex numbers, $z_1, z_2, \ldots, z_n$, and we are given a set $\mathcal{A} = \{a_1, a_2, \ldots, a_m\}$ of m distinct positive integers.

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$$\sum_{i=1}^{n} z_i^{a_j} = c_j, \quad j = 1, \ldots, m.$$

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$$\sum_{i=1}^{n} z_i^{a_j} = c_j, \quad j = 1, \ldots, m.$$

Question: can we recover $z_1, \ldots, z_n$ from $c_1, \ldots, c_m$?

Example

$$z_1 = 6,\ z_2 = 8,\ z_3 = 13, \quad \mathcal{A} = \{2, 5, 7, 8\}$$

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We take 4 measurements:

$$\sum_{i=1}^{3} z_i^2 = 269,\ \sum_{i=1}^{3} z_i^5 = 411837,\ \sum_{i=1}^{3} z_i^7 = 65125605,\ \sum_{i=1}^{3} z_i^8 = 834187553.$$

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We compute the Gröbner basis and obtain

$$\{x_1+x_2+x_3-27, x_2^2+x_2x_3+x_3^2-27(x_2+x_3)+230, (x_3-6)(x_3-8)(x_3-13)\}.$$

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So $\{z_1, z_2, z_3\}$ recovered uniquely!

Model of te problem

For each pair $(n, \mathcal{A})$ we consider the map

$$\phi_{\mathcal{A}} \colon \mathbb{C}^n \longrightarrow \mathbb{C}^m,$$

where $\phi_j = x_1^{a_j} + \cdots + x_n^{a_j}$ for $j = 1, \ldots, m$, and we study its fibers.

Equivalently: we study the polynomial system

$$x_1^{a_j} + x_2^{a_j} + \cdots + x_n^{a_j} \;=\; c_j \qquad \text{for } j = 1, 2, \ldots, m.$$

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Three cases:

- $m > n \rightsquigarrow$ overconstrained: possibly no solutions.
- $m < n \rightsquigarrow$ underconstrained: positive dimensional variety.
- $m = n \rightsquigarrow$ square: finitely many points if $\mathcal{A}, c_1, \ldots, c_m$ 'general'.

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Recall: when we take all 4 measurements, we have unique recovery.

If we take $m = 3$ measurements with $\mathcal{A} = \{2, 5, 7\}$, solve

$$\sum_{i=1}^{3} x_i^2 = 269, \ \sum_{i=1}^{3} x_i^5 = 411837, \ \sum_{i=1}^{3} x_i^7 = 65125605.$$

We get 66 complex solutions (four less than the Bézout bound $70 = 2 \cdot 5 \cdot 7$).

Example (continued)

$z_1 = 6,\ z_2 = 8,\ z_3 = 13, \quad \mathcal{A} = \{2, 5, 7, 8\}$

Recall: when we take all 4 measurements, we have unique recovery.

If we take $m = 2$ measurements with $\mathcal{A} = \{2, 5\}$, solve

$$\sum_{i=1}^{3} x_i^2 = 269, \quad \sum_{i=1}^{3} x_i^5 = 411837.$$

We get an algebraic curve of degree $10 = 2 \cdot 5$.

Questions

$$\phi_{\mathcal{A}} = \phi_{\mathcal{A},\mathbb{C}} \colon \mathbb{C}^n \longrightarrow \mathbb{C}^m, \text{ where } \phi_j = x_1^{a_j} + \cdots + x_n^{a_j} \text{ for } j = 1, \ldots, m.$$

Define maps $\phi_{\mathcal{A},\mathbf{F}}$ similarly for $\mathbf{F} = \mathbb{R}, \mathbb{R}_{\geq 0}$.

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Questions:

- What is the expected dimension of fibers of $\phi_{\mathcal{A}}$?
- How many complex solution can we expect in the zero-dimensional case?
- When is the recovery from power sums given by a set $\mathcal{A}$ unique (over $\mathbb{C}/\mathbb{R}/\mathbb{R}_{\geq 0}$)?

Today

1. Fibers
2. Square case
3. Recovery from norms

Fibers

Fibers of $\phi_{\mathcal{A}}$

$$\phi_{\mathcal{A}} \colon \mathbb{C}^n \longrightarrow \mathbb{C}^m, \text{ where } \phi_j = x_1^{a_j} + \cdots + x_n^{a_j} \text{ for } j = 1, \ldots, m.$$

For $c = (c_1, \ldots, c_m) \in \mathbb{C}^m$, we have $\phi_{\mathcal{A}}^{-1}(c) = V(I_c)$ with

$$I_c \;=\; \langle\, \phi_1(x) - c_1, \ldots, \phi_m(x) - c_m \,\rangle \;\subseteq\; \mathbb{C}[x_1, \ldots, x_n].$$

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Proposition (Melánová, Sturmfels, W., '22)

Assume $m \leq n$. Then the following hold:

(i) *The map ϕ is dominant, i.e., the image of $\phi_{\mathcal{A}}$ is dense in $\mathbb{C}^m$.*

(ii) *For generic $c \in \mathbb{C}^m$, the ideal I_c is radical, and its variety $V(I_c)$ has dimension $n - m$.*

Unique recovery over $\mathbb{C}$

Recall previous **Example**: $\phi_{\mathcal{A}} : \mathbb{C}^n \to \mathbb{C}^m$ and $c = \phi_{\mathcal{A}}(6, 8, 13)$

- $n = 3, m = 3, \mathcal{A} = \{2, 5, 7\}, \quad |\phi_{\mathcal{A}}^{-1}(c)| = 66.$

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- $n = 3, m = 4, \mathcal{A} = \{2, 5, 7, 8\}, |\phi_{\mathcal{A}}^{-1}(c)| = 3!\ ((6, 8, 13), (6, 13, 8), \ldots)$

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Conjecture (Melánová, Sturmfels, W., '22)

The recovery of a set of n complex numbers from $n + 1$ power sums with coprime powers is unique. In other words, for $m = n + 1$, for generic points $z \in \mathbb{C}^n$, the fiber $\phi_{\mathcal{A}}^{-1}(\phi_{\mathcal{A}}(z))$ coincides with the set of $n!$ coordinate permutations of z.

We verified this using Gröbner bases in `magma` for a range of small cases ($n = 4$, $\sum_{a \in \mathcal{A}} \leq 52$, $\quad n = 5$, $\sum_{a \in \mathcal{A}} \leq 49$).

Square case

Square case: $m = n$

Set $\mathcal{A} = \{a_1, a_2, \ldots, a_n\}$ and consider the system

$$x_1^{a_j} + x_2^{a_j} + \cdots + x_n^{a_j} = c_j \qquad \text{for } j = 1, 2, \ldots, n,$$

for some $c = (c_1, \ldots, c_n) \in \mathbb{C}^n$.

Square case: $m = n$

Set $\mathcal{A} = \{a_1, a_2, \ldots, a_n\}$ and consider the homogenized system HS

$$x_1^{a_j} + x_2^{a_j} + \cdots + x_n^{a_j} = c_j x_0^{a_j} \qquad \text{for } j = 1, 2, \ldots, n,$$

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This gives an intersection of n hypersurfaces in $\mathbb{P}^n$.

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$$\begin{aligned} |\phi_{\mathcal{A}}^{-1}(c)| &= |\{\text{sol'ns to } HS\}| - |\{\text{ sol'ns to } HS \text{ with } x_0 = 0\}| \\ &= a_1 \cdots a_n - |\{\text{points at infinity}\}|. \end{aligned}$$

Example (continued)

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Bézout bound: 70
Number of solutions: 66

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We homogenize and find 2 points at infinity:

$$HS \cap \{w = 0\} = \{(1 : \zeta : \zeta^2 : 0), (1 : \zeta^2 : \zeta : 0)\}.$$

Each of these points has multiplicity 2 on HS.

The square case for $n = 2$

Consider $\mathcal{A} = \{a, b\}$ with $a < b$, and the corresponding system.

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How many solutions of S? $\Leftrightarrow$ How many solutions of HS with $z = 0$?

Two cases:

(i) $\gcd(a, b) = 1$,

(ii) $\gcd(a, b) > 1$.

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$(* : * : 0)$ solution to HS in $\mathbb{P}^2 \Leftrightarrow \gcd(x^a + y^a, x^b + y^b) \neq 1$

$\Leftrightarrow$ a, b are both odd $\Leftrightarrow \gcd(x^a + y^a, x^b + y^b) = x + y$.

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$$\Leftrightarrow \ a, b \text{ are both odd } \Leftrightarrow \gcd(x^a + y^a, x^b + y^b) = x + y.$$

We conclude:

$$HS \cap \{z = 0\} \neq \emptyset \Leftrightarrow a, b \text{ both odd},$$

and if this is the case, then

$$HS \cap \{z = 0\} = \{x + y = 0\} = \{(1 : -1 : 0)\}.$$

The square case for $n = 2$

(i) $\mathbf{gcd}(\mathrm{a}, \mathrm{b}) = 1, \quad a, b \text{ odd}, \quad P = (1 : -1 : 0) \in HS.$

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$$HS \cap \mathbb{A}^2 = \begin{cases} 1 + y^a &= c_a z^a \\ 1 + y^b &= c_b z^b \end{cases}$$

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$\mathcal{O}_{P,\mathbb{A}^2}$ local ring of $P = (-1, 0) \in \mathbb{A}^2$, ideal in $\mathcal{O}_{P,\mathbb{A}^2}$ given by

$$I = \left\langle 1 + y^a - c_a z^a, 1 + y^b - c_b z^b \right\rangle.$$

The square case for $n = 2$

(i) $\mathbf{gcd}(\mathrm{a}, \mathrm{b}) = 1$, a, b odd, $P = (1 : -1 : 0) \in HS$. $\mathbb{A}^2 \colon \{x \neq 0\}$.

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The multiplicity of P in HS is equal to $\dim_{\mathbb{C}}(\mathcal{O}_{P,\mathbb{A}^2}/I) = a$.

The square case for $n = 2$

Consider $\mathcal{A} = \{a, b\}$ with $a < b$, and the corresponding system.

$$HS = \begin{cases} x^a + y^a &= c_a z^a \\ x^b + y^b &= c_b z^b \end{cases}$$

(ii) $\mathbf{gcd}(\mathrm{a}, \mathrm{b}) = \mathrm{g} > 1$

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Rewrite the system

$$\left(x^g\right)^{\frac{a}{g}} + \left(y^g\right)^{\frac{a}{g}} = c_a z^a, \quad \left(x^g\right)^{\frac{b}{g}} + \left(y^g\right)^{\frac{b}{g}} = c_b z^b$$

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Conclude: $HS \cap \{z = 0\} \neq \emptyset \Leftrightarrow \frac{a}{g}, \frac{b}{g}$ both odd, in which case

$$HS \cap \{z = 0\} = \{x^g + y^g = 0\} = \{(\zeta^i : 1 : 0) \,:\, i = 1, \ldots, g\},$$

where ζ is a primitive g-th root of -1. Each of these points has multiplicity a in HS.

The square case for $n = 2$

Proposition (Melánová, Sturmfels, W., '22)

Let a, b be integers with $a < b$, and set $g = gcd(a, b)$. For generic constants $c_a, c_b \in \mathbb{C}$, the number of common solutions in $\mathbb{C}^2$ to the equations $x^a + y^a = c_a$ and $x^b + y^b = c_b$ equals

- $a(b - g)$, *if both a/g and b/g are odd,*
- *the Bézout bound ab, otherwise.*

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- *the Bézout bound ab, otherwise.*

Generalization to $n > 2$?

Examples square case for n=3

Recall: we want to count the number of complex solutions to the system

$$S = \begin{cases} x^{a_1} + y^{a_1} + z^{a_1} &= c_1, \\ x^{a_2} + y^{a_2} + z^{a_2} &= c_2, \\ x^{a_3} + y^{a_3} + z^{a_3} &= c_3, \end{cases} \quad \text{for} \quad \begin{array}{l} \mathcal{A} = \{a_1, a_2, a_3\}, \\ c = (c_1, c_2, c_3) \in \mathbb{C}^3. \end{array}$$

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Example

$\mathcal{A} = \{5, 7, 9\}$
$c = (5, 7, 9)$ Bézout: $5 \cdot 7 \cdot 9 = 315$ #sol'ns: 210

$\mathcal{A} = \{5, 7, 13\}$
$c = (5, 7, 13)$ Bézout: $5 \cdot 7 \cdot 13 = 455$ #sol'ns: 336

$\mathcal{A} = \{5, 8, 9\}$
$c = (5, 8, 9)$ Bézout: $5 \cdot 8 \cdot 9 = 360$ #sol'ns: 360

Examples square case for n=3

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$\mathcal{A} = \{5, 7, 9\}$

$$HS = \begin{cases} x^5 + y^5 + z^5 &= 5w^5 \\ x^7 + y^7 + z^7 &= 7w^7 \\ x^9 + y^9 + z^9 &= 9w^9 \end{cases}$$

Bézout bound: 315
Number of solutions: 210

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Bézout bound: 315
Number of solutions: 210

$$HS \cap \{w = 0\} = \{(1 : -1 : 0 : 0), (1 : 0 : -1 : 0), (0 : 1 : -1 : 0)\}.$$

Each of these points has multiplicity 35 on HS.

$$3 \cdot 35 = 105 = 315 - 210$$

Examples square case for n=3

Example

$\mathcal{A} = \{5, 7, 13\}$

$$HS = \begin{cases} x^5 + y^5 + z^5 = 5w^5 \\ x^7 + y^7 + z^7 = 7w^7 \\ x^{13} + y^{13} + z^{13} = 13w^{13} \end{cases}$$

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$$HS = \begin{cases} x^5 + y^5 + z^5 &= 5w^5 \\ x^7 + y^7 + z^7 &= 7w^7 \\ x^{13} + y^{13} + z^{13} &= 13w^{13} \end{cases}$$

Bézout bound: 455
Number of solutions: 336

$$HS \cap \{w = 0\} = \{(1 : -1 : 0 : 0), (1 : 0 : -1 : 0), (0 : 1 : -1 : 0), (1 : \zeta : \zeta^2 : 0), (1 : \zeta^2 : \zeta : 0)\}.$$

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Each of these points has multiplicity 35 or 7 on HS.

$$3 \cdot 35 + 2 \cdot 7 = 119 = 455 - 336$$

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Consider $\mathcal{A} = \{a, b, c\}$ with $\gcd(a, b, c) = 1$, and the corresponding system

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Conjecture (Melánová, Sturmfels, W., '22)

We have

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where ζ is a primitive cubic root of unity.

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Conjecture (Conca, Krattenthaler, Watanabe, '09)

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So our conjecture **implies** the CKW conjecture....

Points at infinity: results

Theorem (Conca, Krattenthaler, Watanabe, '09)

The CKW conjecture holds if either

- $\mathcal{A}$ *contains* 1 *and* n *with* $2 \leq n \leq 7$,
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Theorem (Melánová, Sturmfels, W., '22)

Our conjecture holds for all $\mathcal{A} = \{a, b, c\}$, $a < b < c$ *with* $a + b + c \leq 300$.

The square case for $n = 3$: conjecture

$\mathcal{A} = \{a_1, a_2, a_3\}$, $\mathcal{A}_p = \{a_1 \bmod p, a_2 \bmod p, a_3 \bmod p\}$

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$x = (x_1, \ldots, x_n) \in \mathbb{R}^n_{\geq 0}$, $\quad p$ positive integer, then we have

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Theorem (Melánová, Sturmfels, W., '22)

For $m = n$, recovery from p-norms is always unique. Given any set $\mathcal{A}$ of n positive integers, the map $\phi_{\mathcal{A},\mathbb{R}_{\geq 0}} : \mathbb{R}^n_{\geq 0} \to \mathbb{R}^n_{\geq 0}$ is injective up to permuting coordinates.

Describing the image of $\phi_{\mathcal{A},\mathbb{R}_{\geq 0}}$

Tarski-Seidenberg: the image of $\phi_{\mathcal{A},\mathbb{R}_{\geq 0}}$ is a semi-algebraic set.

Proposition

1. *$im(\phi_{\mathcal{A},\mathbb{R}_{\geq 0}})$ is closed in $\mathbb{R}^m_{\geq 0}$.*
2. *$m = 2, n \geq m$, then*

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Generalization to $m > 2$?

Generalizations to $m \geq 3$

$\Delta_m = \{u \in \mathbb{R}^{m+1}_{\geq 0} \colon u_1 + \cdots + u_{m+1} = 1\}$ probability symplex

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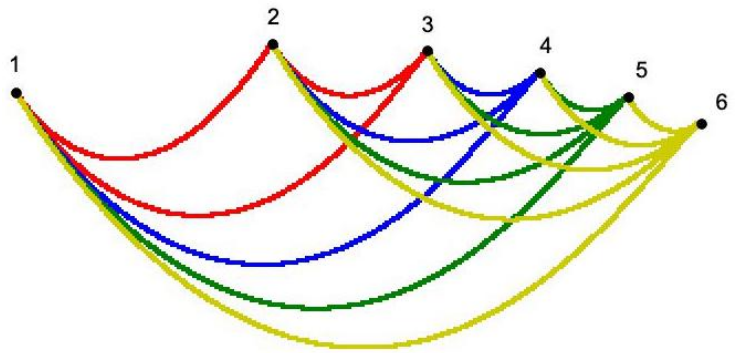
Example

$m = 3$ The image of $\psi_{\mathcal{A}}$ is an n-gon in $\{u_1 > u_2 > u_3\} \subset \Delta_2$.



If you want to read more ...

... everything I talked about can be found in our paper

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Recovery from Power Sums

Hana Melánová[a], Bernd Sturmfels[b,c], and Rosa Winter[d]

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ABSTRACT
We study the problem of recovering a collection of n numbers from the evaluation of m power sums. This yields a system of polynomial equations, which can be underconstrained ($m < n$), square ($m = n$), or overconstrained ($m > n$). Fibers and images of power sum maps are explored in all three regimes, and in settings that range from complex and projective to real and positive. This involves surprising deviations from the Bézout bound, and the recovery of vectors from length measurements by p-norms.

KEYWORDS
p-norms; solving polynomial systems; images and fibers; power sums

Thank you for your attention!