Four-Dimensional Lie Algebras Revisited

Bernd Sturmfels
MPI Leipzig and UC Berkeley

with Laurent Manivel
and Svala Sverrisdóttir

Workshop on Solving Polynomial Equations and Applications
CWI Amsterdam, October 7, 2022

Lie Algebras

Fix the vector space $V = \mathbb{C}^4$ with standard basis $\{e_1, e_2, e_3, e_4\}$.

A *Lie algebra structure* is a bilinear operation $V \times V \to V$ that is skew-symmetric and satisfies the Jacobi identity:

$$\text{For all } u, v, w \in V, \text{ we have } \quad [v, u] = -[u, v] \quad \text{and}$$
$$[\,u, [v, w]\,] \;+\; [\,v, [w, u]\,] \;+\; [\,w, [u, v]\,] \;=\; 0.$$

Lie Algebras

Fix the vector space $V = \mathbb{C}^4$ with standard basis $\{e_1, e_2, e_3, e_4\}$.

A *Lie algebra structure* is a bilinear operation $V \times V \to V$ that is skew-symmetric and satisfies the Jacobi identity:

$$\text{For all } u, v, w \in V, \text{ we have } \quad [v,u] = -[u,v] \quad \text{and}$$
$$[\,u,[v,w]\,] \;+\; [\,v,[w,u]\,] \;+\; [\,w,[u,v]\,] \;=\; 0.$$

Define a Lie algebra on the basis vectors:

$$[e_i, e_j] \;=\; a_{ij1}e_1 + a_{ij2}e_2 + a_{ij3}e_3 + a_{ij4}e_4.$$

Since $[e_i, e_j] = -[e_j, e_i]$, we have $a_{iik} = 0$ and $a_{jik} = -a_{ijk}$.

Thus *a Lie algebra structure on V is encoded by the*

24 structure constants a_{ijk}

where $1 \le i < j \le 4$ *and* $k = 1, 2, 3, 4$.

Our Variety

Fix the projective space $\mathbb{P}^{23}$ with coordinates

$$A \quad = \quad \begin{bmatrix} a_{121} & a_{131} & a_{141} & a_{231} & a_{241} & a_{341} \\ a_{122} & a_{132} & a_{142} & a_{232} & a_{242} & a_{342} \\ a_{123} & a_{133} & a_{143} & a_{233} & a_{243} & a_{343} \\ a_{124} & a_{134} & a_{144} & a_{234} & a_{244} & a_{344} \end{bmatrix}.$$

Let Lie_4 be the subvariety whose points are the Lie algebra structures on $\mathbb{C}^4$. This is defined by 16 quadratic equations.

The 16 quadrics in the a_{ijk} are found by substituting

$$[e_i, e_j] \;=\; a_{ij1}e_1 + a_{ij2}e_2 + a_{ij3}e_3 + a_{ij4}e_4.$$

into the Jacobi identify

$$[\,e_i, [e_j, e_k]\,] \;+\; [\,e_j, [e_k, e_i]\,] \;+\; [\,e_k, [e_i, e_j]\,] \quad = \quad 0.$$

16 Quadrics

The variety $\mathrm{Lie}_4 \subset \mathbb{P}^{23}$ is defined by 16 quadrics in 24 unknowns.

Here are the equations we wish to solve:

```
I = ideal(
 -a121*a133-a122*a233+a123*a131+a123*a232+a124*a343-a134*a243+a143*a234,
 -a121*a144-a122*a244-a123*a344+a124*a141+a124*a242+a134*a243-a143*a234,
 a121*a242-a122*a241-a123*a341+a131*a243+a141*a244-a143*a231-a144*a241,
 -a121*a142+a122*a141-a123*a342+a132*a243+a142*a244-a143*a232-a144*a242,
 -a121*a132+a122*a131+a124*a342+a132*a233-a133*a232-a134*a242+a142*a234,
  a121*a232-a122*a231+a124*a341+a131*a233-a133*a231-a134*a241+a141*a234,
  a121*a342+a131*a343-a132*a241-a133*a341+a141*a344+a142*a231-a144*a341,
  -a122*a341+a132*a241-a142*a231+a232*a343-a233*a342+a242*a344-a244*a342,
  a123*a342-a131*a143-a132*a243+a133*a141+a142*a233+a143*a344-a144*a343,
  -a123*a341+a133*a241-a143*a231-a232*a243+a233*a242+a243*a344-a244*a343,
  a124*a342-a131*a144-a132*a244-a133*a344+a134*a141+a134*a343+a142*a234,
 -a124*a341+a134*a241-a144*a231-a232*a244-a233*a344+a234*a242+a234*a343,
 -a121*a143-a122*a243+a123*a141+a123*a242-a123*a343+a133*a243-a143*a233+a143*a244-a144*a243,
 -a121*a134-a122*a234+a124*a131+a124*a232+a124*a344-a133*a234+a134*a233-a134*a244+a144*a234,
 -a121*a341+a131*a241-a141*a231+a231*a242+a231*a343-a232*a241-a233*a341+a241*a344-a244*a341,
 a122*a342-a131*a142+a132*a141-a132*a242+a132*a343-a133*a342+a142*a232+a142*a344-a144*a342);
```

Each solution is one Lie algebra.

Theoretical Mathematics

Dimension **11**, *Degree* **1033**, *Four Components*

ON THE VARIETY OF FOUR DIMENSIONAL LIE ALGEBRAS

LAURENT MANIVEL

Abstract. Lie algebras of dimension n are defined by their structure constants, which can be seen as sets of $N = n^2(n-1)/2$ scalars (if we take into account the skew-symmetry condition) to which the Jacobi identity imposes certain quadratic conditions. Up to rescaling, we can consider such a set as a point in the projective space $\mathbf{P}^{N-1}$. Suppose n =4, hence $N = 24$. Take a random subspace of dimension 12 in $\mathbf{P}^{23}$, over the complex numbers. We prove that this subspace will contain exactly 1033 points giving the structure constants of some four dimensional Lie algebras. Among those, 660 will be isomorphic to $\mathbf{g}l_2$, 195 will be the sum of two copies of the Lie algebra of one dimensional affine transformations, 121 will have an abelian, three-dimensional derived algebra, and 57 will have for derived algebra the three dimensional Heisenberg algebra. This answers a question of Kirillov and Neretin.

Journal of Lie Theory **26** (2016) 1–10

Nonlinear Algebra People

Svala Sverrisdóttir

My name is Svala Sverrisdóttir and I will be starting as a first-year PhD student at UC Berkeley in the fall. My current mathematical interests lie in the realms of combinatorics and algebraic geometry and even though my PhD research topic is not yet decided I am hoping my work will lie at the interface of these two areas. This summer, I am visiting the nonlinear algebra group at the MPI and working on a project about nilpotent 7-dimensional Lie algebras.

Svala's First Experiment

```
I = ideal(
 -a121*a133-a122*a233+a123*a131+a123*a232+a124*a343-a134*a243+a143*a234,
 -a121*a144-a122*a244-a123*a344+a124*a141+a124*a242+a134*a243-a143*a234,
 a121*a242-a122*a241-a123*a341+a131*a243+a141*a244-a143*a231-a144*a241,
 -a121*a142+a122*a141-a123*a342+a132*a243+a142*a244-a143*a232-a144*a242,
 -a121*a132+a122*a131+a124*a342+a132*a233-a133*a232-a134*a242+a142*a234,
  a121*a232-a122*a231+a124*a341+a131*a233-a133*a231-a134*a241+a141*a234,
  a121*a342+a131*a343-a132*a241-a133*a341+a141*a344+a142*a231-a144*a341,
  -a122*a341+a132*a241-a142*a231+a232*a343-a233*a342+a242*a344-a244*a342,
  a123*a342-a131*a143-a132*a243+a133*a141+a142*a233+a143*a344-a144*a343,
  -a123*a341+a133*a241-a143*a231-a232*a243+a233*a242+a243*a344-a244*a343,
  a124*a342-a131*a144-a132*a244-a133*a344+a134*a141+a134*a343+a142*a234,
 -a124*a341+a134*a241-a144*a231-a232*a244-a233*a344+a234*a242+a234*a343,
 -a121*a143-a122*a243+a123*a141+a123*a242-a123*a343+a133*a243-a143*a233+a143*a244-a144*a243,
 -a121*a134-a122*a234+a124*a131+a124*a232+a124*a344-a133*a234+a134*a233-a134*a244+a144*a234,
 -a121*a341+a131*a241-a141*a231+a231*a242+a231*a343-a232*a241-a233*a341+a241*a344-a244*a341,
 a122*a342-a131*a142+a132*a141-a132*a242+a132*a343-a133*a342+a142*a232+a142*a344-a144*a342);
```

She ran `HomotopyContinuation.jl` on the 16 quadrics, augmented by some random linear equations, and she found

Dimension **11**, *Degree* **832**

Manivel had proved Degree **1033**.

How to explain this?

Why Do We Care?

A. Kirillov and Y. Neretin:
The variety A_n of structures of n-dimensional Lie algebras,
American Mathematical Society Translations **137** (1987).

A description of the variety of these structures even for small dimensions would be of great interest, since the Lie superalgebras are more and more frequently finding applications in current research, both in mathematics and in mathematical physics. Only the very first steps have been made in this direction.([4])

The problem remains unsolved of computing the degrees of the irreducible components $A_n^{(k)}$ for $n \geq 4$. For example, for $n = 4$ all four components $A_n^{(k)}$ have the same dimension. Therefore the answer to the question of what is the "typical" 4-dimensional Lie algebra depends on the degrees of these components.

Attempts to compute the Hilbert polynomial for these varieties by the methods of representation theory have so far not produced any result.

Our Results

Theorem

The variety Lie_4 *has four irreducible components* C_1, C_2, C_3, C_4 *in the ambient space* $\mathbb{P}^{23}$*. Each of these components has dimension* 11*. Their degrees are* $55, 361, 121, 295$*.*

[Manivel '16] had reported $660, 57, 121, 195$.

Our Results

Theorem

The variety Lie_4 *has four irreducible components* C_1, C_2, C_3, C_4 *in the ambient space* $\mathbb{P}^{23}$. *Each of these components has dimension* 11. *Their degrees are* $55, 361, 121, 295$.

[Manivel '16] had reported $660, 57, 121, 195$.

Theorem

The ideal I_{C_1} *is generated by* 4 *linear forms,* 10 *quadrics,* 20 *cubics. The ideal* I_{C_2} *is generated by* 16 *quadrics and* 44 *cubics,* I_{C_3} *is generated by* 26 *quadrics and* 40 *cubics, and* I_{C_4} *is generated by* 16 *quadrics,* 60 *cubics. The Hilbert polynomials of the varieties are*

$$\begin{aligned}
\mathrm{Hilb}_{C_1} &= \mathbf{55}\,[\mathbb{P}^{11}] - 120\,[\mathbb{P}^{10}] + 86\,[\mathbb{P}^{9}] - 20\,[\mathbb{P}^{8}],\\
\mathrm{Hilb}_{C_2} &= \mathbf{361}\,[\mathbb{P}^{11}] - 1184\,[\mathbb{P}^{10}] + 1526\,[\mathbb{P}^{9}] - 964\,[\mathbb{P}^{8}] + 298\,[\mathbb{P}^{7}] - 36\,[\mathbb{P}^{6}],\\
\mathrm{Hilb}_{C_3} &= \mathbf{121}\,[\mathbb{P}^{11}] - 284\,[\mathbb{P}^{10}] + 220\,[\mathbb{P}^{9}] - 56\,[\mathbb{P}^{8}],\\
\mathrm{Hilb}_{C_4} &= \mathbf{295}\,[\mathbb{P}^{11}] - 920\,[\mathbb{P}^{10}] + 1114\,[\mathbb{P}^{9}] - 652\,[\mathbb{P}^{8}] + 184\,[\mathbb{P}^{7}] - 20\,[\mathbb{P}^{6}].
\end{aligned}$$

Here $[\mathbb{P}^d]$ is the polynomial $n \mapsto \binom{n+d}{d}$.

FAIR Data

MATHREPO
MATHEMATICAL RESEARCH-DATA REPOSITORY

Four-Dimensional Lie Algebras Revisited

This page contains auxiliary files to the paper:
Laurent Manivel, Bernd Sturmfels and Svala Sverrisdóttir: Four-Dimensional Lie Algebras Revisited
ARXIV: http://arxiv.org/abs/2208.14631 CODE: https://mathrepo.mis.mpg.de/Lie4

ABSTRACT: The projective variety of Lie algebra structures on a 4-dimensional vector space has four irreducible components of dimension 11. We compute their prime ideals in the polynomial ring in 24 variables. By listing their degrees and Hilbert polynomials, we correct an earlier publication and we answer a 1987 question by Kirillov and Neretin.

Verifications of Theorems 1 and 2

We used Macaulay2 (version 1.20) to verify Theorems 1 and 2 from the paper.

The file Lie 4 Component includes the explicit generators of the irreducible components C_1, C_2, C_3, C_4 which are explained in Section 3 of our paper. We calculate the dimension, degree, Betti numbers and the Hilbert polynomial of each of these component. We also verify that our idels for C_1, C_3, C_4 are prime. To show that C_1 and C_3 are prime it is enough to run the *isPrime*

Changing Bases

Two Lie algebra structures on $\mathbb{C}^4$ are isomorphic if they are in the same orbit under the action of $G = \mathrm{GL}(4, \mathbb{C})$. We write the G-action by matrix multiplication:

$$G \times \mathbb{C}[A] \to \mathbb{C}[A], \ (g, A) \mapsto g^{-1} \cdot A \cdot \wedge_2(g).$$

The entries of the 6×6 matrix $\wedge_2(g)$ are the 2×2 minors of g.

The induced $\mathbb{Z}^4$-grading of the polynomial ring $\mathbb{C}[A]$ equals

$$\deg(a_{ijk}) \quad = \quad e_i + e_j - e_k.$$

Changing Bases

Two Lie algebra structures on $\mathbb{C}^4$ are isomorphic if they are in the same orbit under the action of $G = \mathrm{GL}(4, \mathbb{C})$.
We write the G-action by matrix multiplication:

$$G \times \mathbb{C}[A] \to \mathbb{C}[A], \; (g, A) \mapsto g^{-1} \cdot A \cdot \wedge_2(g).$$

The entries of the 6×6 matrix $\wedge_2(g)$ are the 2×2 minors of g.

The induced $\mathbb{Z}^4$-grading of the polynomial ring $\mathbb{C}[A]$ equals

$$\deg(a_{ijk}) \quad = \quad e_i + e_j - e_k.$$

Space of linear forms decomposes into two irreducible representations:

$$\mathbb{C}[A]_1 \;=\; S_{(1,0,0,0)}(\mathbb{C}^4) \;\oplus\; S_{(1,1,0,-1)}(\mathbb{C}^4) \;\simeq\; \mathbb{C}^4 \oplus \mathbb{C}^{20}.$$

The two G-invariant subspaces are

$$\mathbb{C}\{\underline{a_{122} + a_{133} + a_{144}}, -a_{121}+a_{233}+a_{244}, -a_{131}-a_{232}+a_{344}, -a_{141}-a_{242}-a_{343}\},$$

$$\mathbb{C}\{\underline{a_{124}}, a_{123}, a_{132}, a_{134}, a_{142}, a_{143}, a_{231}, a_{234}, a_{241}, a_{243}, a_{341}, a_{342}, \; a_{122}-a_{133}, \\ a_{133}-a_{144}, a_{121}+a_{233}, a_{233}-a_{244}, a_{131}-a_{232}, a_{232}+a_{344}, a_{141}-a_{242}, a_{242}-a_{343}\}.$$

Sixteen Quadrics

The 16 Jacobi quadrics form a 4×4 matrix $\Theta = (\theta_{ij})$. The columns are labeled e_4, e_3, e_2, e_1, and the rows are labeled $123, -124, 134, -234$. For instance, the second row of Θ is

$$-[e_{\mathbf{1}}, [e_{\mathbf{2}}, e_{\mathbf{4}}]] - [e_2, [e_4, e_1]] - [e_4, [e_1, e_2]] = \theta_{21} e_4 + \theta_{22} e_3 + \theta_{23} e_2 + \theta_{24} e_1$$

$$\theta_{21} = a_{121} a_{144} - a_{124} a_{141} + a_{143} a_{234} - a_{124} a_{242} - a_{134} a_{243} + a_{122} a_{244} + a_{123} a_{344}.$$

Sixteen Quadrics

The 16 Jacobi quadrics form a 4×4 matrix $\Theta = (\theta_{ij})$. The columns are labeled e_4, e_3, e_2, e_1, and the rows are labeled $123, -124, 134, -234$. For instance, the second row of Θ is

$$-[e_1,[e_2,e_4]] - [e_2,[e_4,e_1]] - [e_4,[e_1,e_2]] = \theta_{21}e_4+\theta_{22}e_3+\theta_{23}e_2+\theta_{24}e_1$$

$$\theta_{21} = a_{121}a_{144} - a_{124}a_{141} + a_{143}a_{234} - a_{124}a_{242} - a_{134}a_{243} + a_{122}a_{244} + a_{123}a_{344}.$$

Our matrix represents

$$\Theta : \wedge_3\mathbb{C}^4 \hookrightarrow \wedge_2\mathbb{C}^4 \otimes \mathbb{C}^4 \to \mathbb{C}^4 \otimes \mathbb{C}^4 \to \wedge_2\mathbb{C}^4 \to \mathbb{C}^4.$$

Second and fourth map are Lie algebra multiplication, given by our 4×6 matrix A. The Jacobi identity states that $\Theta = 0$.

Sixteen Quadrics

The 16 Jacobi quadrics form a 4×4 matrix $\Theta = (\theta_{ij})$. The columns are labeled e_4, e_3, e_2, e_1, and the rows are labeled $123, -124, 134, -234$. For instance, the second row of Θ is

$$-[e_1,[e_2,e_4]] - [e_2,[e_4,e_1]] - [e_4,[e_1,e_2]] = \theta_{21}e_4 + \theta_{22}e_3 + \theta_{23}e_2 + \theta_{24}e_1$$

$$\theta_{21} = a_{121}a_{144} - a_{124}a_{141} + a_{143}a_{234} - a_{124}a_{242} - a_{134}a_{243} + a_{122}a_{244} + a_{123}a_{344}.$$

Our matrix represents

$$\Theta : \wedge_3\mathbb{C}^4 \hookrightarrow \wedge_2\mathbb{C}^4 \otimes \mathbb{C}^4 \rightarrow \mathbb{C}^4 \otimes \mathbb{C}^4 \rightarrow \wedge_2\mathbb{C}^4 \rightarrow \mathbb{C}^4.$$

Second and fourth map are Lie algebra multiplication, given by our 4×6 matrix A. The Jacobi identity states that $\Theta = 0$.

The space $\mathbb{C}[A]_2 \simeq \mathbb{C}^{300}$ breaks into eight irreducibles, including

$$\begin{array}{rcccl} S_{(1,1,1,-1)}(\mathbb{C}^4) & = & \mathbb{C}\{\text{entries of } \Theta + \Theta^T\} & \simeq & \mathbb{C}^{10} \quad \text{and} \\ S_{(1,1,0,0)}(\mathbb{C}^4) & = & \mathbb{C}\{\text{entries of } \Theta - \Theta^T\} & \simeq & \mathbb{C}^6. \end{array}$$

Highest weight vectors are θ_{11} and $\theta_{12} - \theta_{21}$.
$G = \mathrm{GL}(4,\mathbb{C})$ acts by congruence: $\Theta \mapsto g^T \Theta g$.

Lie Theory

The four components of Lie_4*:*

C_1 : The Lie algebra $\mathfrak{gl}_2$ of the general linear group $\mathrm{GL}(2,\mathbb{C})$.
Its derived algebra is $\mathfrak{sl}_2$.

C_2 : Lie algebras whose derived algebra is the Heisenberg algebra $\mathfrak{he}_3$, of dimension 3.

C_3 : Lie algebras whose derived algebra is abelian and 3-dimensional.

C_4 : The Lie algebra $2\mathfrak{aff}_2$. Its derived algebra is abelian and 2-dimensional.

The *derived algebra* of a Lie algebra is generated by all commutators. As a subspace of $\mathbb{C}^4$, the derived algebra is the image of our 4×6 matrix A. Hence the 4×4-minors of A vanish on all components, while the 3×3-minors of A vanish only on C_4.

Parametrization

Representatives for the four components:

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -2 & 0 \\ 0 & -1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & x & 1 & 0 & 0 & 0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & x & 0 & 0 & 0 & 0 \\ 0 & 0 & y & 0 & 0 & 0 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Parametrization

Representatives for the four components:

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -2 & 0 \\ 0 & -1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & x & 1 & 0 & 0 & 0 \end{bmatrix},$$

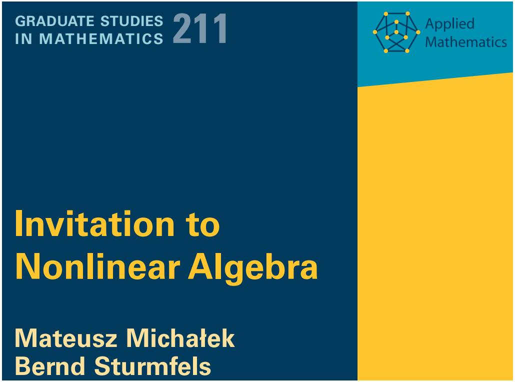
$$A_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & x & 0 & 0 & 0 & 0 \\ 0 & 0 & y & 0 & 0 & 0 \end{bmatrix}, \qquad A_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Each variety C_i is rational. Namely, C_i is the closure of the G-orbit of A_i where x and y range over $\mathbb{C}$. For instance,

$$C_2 \;=\; \text{closure of } \left\{ g^{-1} \cdot A_2(x) \cdot \wedge_2(g) \;:\; x \in \mathbb{C},\, g \in G \right\} \;\subset\; \mathbb{P}^{23}.$$

Can't we just use implicitization to compute the prime ideal I_{C_i} ?

For Students



Section 1.3: Degree, Hilbert Polynomial
Section 3.2: Primary Decomposition
Section 4.2: Implicitization

Chapter 9: Tensors
Chapter 10: Representation Theory

Resolution of Singularities

The key idea (also in Manivel '16) is to linearize the Jacobi identity.

[R. Basili: **Resolutions of singularities** *of varieties of Lie algebras of dimensions 3 and 4*, Journal of Lie Theory **12** (2002)]

Example: For C_2 we use $A = \wedge_2 g \cdot B \cdot \det(g) \cdot g^{-1}$, where

$$g = \begin{bmatrix} 1 & f_1 & f_2 & f_3 \\ 0 & 1 & 0 & f_4 \\ 0 & 0 & 1 & f_5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ k_1 & k_2 & 0 & 0 & 0 & 0 \\ k_3 & k_4 & 0 & 0 & 0 & 0 \\ k_5 & k_6 & k_1+k_4 & m & 0 & 0 \end{bmatrix}.$$

Algebraic Geometry

The key idea (also in Manivel '16) is to linearize the Jacobi identity.

Example: For C_2 we use $A = \wedge_2 g \cdot B \cdot \det(g) \cdot g^{-1}$, where

$$g = \begin{bmatrix} 1 & f_1 & f_2 & f_3 \\ 0 & 1 & 0 & f_4 \\ 0 & 0 & 1 & f_5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ k_1 & k_2 & 0 & 0 & 0 & 0 \\ k_3 & k_4 & 0 & 0 & 0 & 0 \\ k_5 & k_6 & k_1+k_4 & m & 0 & 0 \end{bmatrix}.$$

The f_i are local coordinates on the flag variety $\mathrm{Fl}(1,3,\mathbb{C}^4)$. The flag amounts to inclusion of the second derived algebra, spanned by the last column of g, into the derived algebra, which is spanned by the last three columns of g. The matrix B is linear in $k_1, \ldots, k_6, m$. It defines a rank seven vector bundle F over $\mathrm{Fl}(1,3,\mathbb{C}^4)$, and we have $\mathrm{image}(B) \simeq \mathfrak{h}_3$. The projective bundle $\mathbb{P}(F)$ is a nonsingular variety of dimension $11 = 6 + 5$ that maps birationally onto C_2.

[R. Basili: **Resolutions of singularities** *of varieties of Lie algebras of dimensions 3 and 4*, Journal of Lie Theory **12** (2002)]

Say it again for Macaulay2

"The projective bundle $\mathbb{P}(F)$ … maps birationally onto C_2"

```
R = QQ[f1,f2,f3,f4,f5,k1,k2,k3,k4,k5,k6,m, a121,a122,a123,a124,
   a131,a132,a133,a134,a141,a142,a143,a144,a231,a232,a233,a234,
   a241,a242,a243,a244,a341,a342,a343,a344, MonomialOrder => Eliminate 12];
I = ideal(a121-f1*f4*k5-f2*f5*k5+f1*k1+f2*k3+f3*k5, a122+f4*k5-k1, a132+f4*k6-k2,
a142+f4^2*k5+f4*f5*k6+f4*k4-f5*k2, a232+f1*f4*k6-f2*f4*k5-f1*k2+f2*k1+f4*m,
a242+f1*f4^2*k5+f1*f4*f5*k6+f1*f4*k4-f1*f5*k2-f3*f4*k5+f4*f5*m+f3*k1,
a342+f2*f4^2*k5+f2*f4*f5*k6+f2*f4*k4-f2*f5*k2-f3*f4*k6-f4^2*m+f3*k2,
a123+f5*k5-k3,  a133+f5*k6-k4, a143+f4*f5*k5+f5^2*k6-f4*k3+f5*k1,
a131-f1*f4*k6-f2*f5*k6+f1*k2+f2*k4+f3*k6, a233+f1*f5*k6-f2*f5*k5-f1*k4+f2*k3+f5*m,
a243+f1*f4*f5*k5+f1*f5^2*k6-f1*f4*k3+f1*f5*k1-f3*f5*k5+f5^2*m+f3*k3,
a343+f2*f4*f5*k5+f2*f5^2*k6-f2*f4*k3+f2*f5*k1-f3*f5*k6-f4*f5*m+f3*k4,
a144-f4*k5-f5*k6-k1-k4, a244-f1*f4*k5-f1*f5*k6-f1*k1-f1*k4+f3*k5-f5*m,
a234-f1*k6+f2*k5-m, a344-f2*f4*k5-f2*f5*k6-f2*k1-f2*k4+f3*k6+f4*m,
a141-f1*f4^2*k5-f1*f4*f5*k6-f2*f4*f5*k5-f2*f5^2*k6 -f1*f4*k4+f1*f5*k2+f2*f4*k3
       -f2*f5*k1+f3*f4*k5+f3*f5*k6+f3*k1+f3*k4,
a231-f1^2*f4*k6+f1*f2*f4*k5-f1*f2*f5*k6+f2^2*f5*k5+f1^2*k2-f1*f2*k1
    +f1*f2*k4+f1*f3*k6-f1*f4*m-f2^2*k3-f2*f3*k5-f2*f5*m+f3*m, a124-k5, a134-k6,
a241-f1^2*f4^2*k5-f1^2*f4*f5*k6-f1*f2*f4*f5*k5-f1*f2*f5^2*k6-f1^2*f4*k4+f1^2*f5*k2
    +f1*f2*f4*k3-f1*f2*f5*k1+2*f1*f3*f4*k5+f1*f3*f5*k6-f1*f4*f5*m+f2*f3*f5*k5
    -f2*f5^2*m+f1*f3*k4-f2*f3*k3-f3^2*k5+f3*f5*m,
a341-f1*f2*f4^2*k5-f1*f2*f4*f5*k6-f2^2*f4*f5*k5-f2^2*f5^2*k6-f1*f2*f4*k4
    +f1*f2*f5*k2+f1*f3*f4*k6+f1*f4^2*m+f2^2*f4*k3-f2^2*f5*k1+f2*f3*f4*k5
    +2*f2*f3*f5*k6+f2*f4*f5*m-f1*f3*k2+f2*f3*k1-f3^2*k6-f3*f4*m);
C2 = ideal selectInSubring( 1, gens gb(I) )
codim C2, degree C2, betti mingens C2
```

Not Radical

Corollary

The radical ideal of Lie_4 *is minimally generated by* 16 *quadrics and* 15 *quartics. The quadrics are the entries of the matrix* Θ*, and the quartics are the* 4×4 *minors of the matrix* A*.*

MATHREPO
MATHEMATICAL RESEARCH-DATA REPOSITORY

for C_1, C_3, C_4 are prime. To show that C_1 and C_3 are prime it is enough to run the *isPrime* command in Macaulay2. To show C_4 is prime we run

```
# minimalPrimes C4;
radical C4 == C4
```

Since we get the output 1 and *true* we see that C_4 is prime. Finally we take the intersection of these components to get the radical ideal of Lie_4 and calculate its dimension, degree and Betti numbers.

In the file C2 prime we verify that our ideal for C_2 is prime. We do this by representing the birational parametrization of C_2 mentioned in Section 5 of our paper.

What is a Polynomial?

4 Polynomials: Explicit versus Invariant

We now examine our ideal generators through the lens of the G-action. We identify the Schur modules $S_\lambda(\mathbb{C}^4)$ that generate the ideals I_{C_i}, and we identify polynomials that serve as highest weight vectors. The space $\mathbb{C}[A]_2 \simeq \mathbb{C}^{300}$ decomposes into five isotypical components:

λ	$(3,1,-1,-1)$	$(2,1,0,-1)$	$(2,0,0,0)$	$(1,1,1,-1)$	$(1,1,0,0)$
dim	126	64	10	10	6
mult	1	2	2	2	1

The last two columns were seen already in (5). The space of cubics $\mathbb{C}[A]_3 \simeq \mathbb{C}^{2600}$ decomposes into 11 isotypical components. We display the four components that are relevant for us:

λ	$(2,1,1,-1)$	$(3,0,0,0)$	$(2,1,0,0)$	$(1,1,1,0)$	$\cdots$
dim	36	20	20	4	$\cdots$
mult	5	3	4	3	$\cdots$

One ingredient in an invariant description of these G-modules is the *adjoint* $\mathrm{ad}(u)$ of an element u in our Lie algebra. This is the endomorphism $\mathbb{C}^4 \to \mathbb{C}^4$ given by $v \mapsto [u,v]$. For any index $i \in \{1,2,3,4\}$, the adjoint of e_i is represented by the 4×4 matrix $(a_{ijk})_{1\le j,k\le 4}$. The traces of the matrices $\mathrm{ad}(e_i)$ are the four linear forms that span $S_{(1,0,0,0)}(\mathbb{C}^4) \subset I_{C_1}$. To be explicit, $\mathrm{ad}(e_1)$ is obtained by prepending a zero column to the left three columns of A.

We begin with the module $S_{(3,0,0,0)}(\mathbb{C}^4)$. This has dimension 20 and occurs with multiplicity three in the space of cubics $\mathbb{C}[A]_3$. A highest weight vector for one embedding is

$$\begin{aligned} f_{3000} \;=\; & a_{122}^3 - a_{122}^2 a_{133} - a_{122}^2 a_{144} + 4a_{122}a_{123}a_{132} + 4a_{122}a_{124}a_{142} - a_{122}a_{133}^2 + 2a_{122}a_{133}a_{144} \\ & -4a_{122}a_{134}a_{143} - a_{122}a_{144}^2 + 4a_{123}a_{132}a_{133} - 4a_{123}a_{132}a_{144} + 8a_{123}a_{134}a_{142} + 8a_{124}a_{132}a_{143} \\ & -4a_{124}a_{133}a_{142} + 4a_{124}a_{142}a_{144} + a_{133}^3 - a_{133}^2 a_{144} + 4a_{133}a_{134}a_{143} - a_{133}a_{144}^2 + 4a_{134}a_{143}a_{144} + a_{144}^3. \end{aligned}$$

This module generates the ideal I_{C_1}, together with the linear forms and quadrics seen in Theorem 2. The module Gf_{3000} is also contained in I_{C_2}. But this ideal has 24 additional cubic generators. These additional cubics for C_2 are given by two irreducible G-modules:

$$S_{(2,1,0,0)}(\mathbb{C}^4) \,\oplus\, S_{(1,1,1,0)}(\mathbb{C}^4) \;\simeq\; \mathbb{C}^{20} \oplus \mathbb{C}^4.$$

The highest weight vectors for these two irreducible G-modules in $\mathbb{C}[A]_3$ are

$$\mathrm{trace}\big(\mathrm{ad}(e_1)\cdot\mathrm{ad}(e_2)\cdot\mathrm{ad}(e_3)\big) \;-\; \mathrm{trace}\big(\mathrm{ad}(e_2)\cdot\mathrm{ad}(e_1)\cdot\mathrm{ad}(e_3)\big)$$

5 From Vector Bundles to Degrees

The approach in [7] is based on desingularization of each component C_i, following Basili [1]. The key idea is to *linearize the Jacobi identity*, i.e. to express (2) by linear equations in A over a variety derived from $\mathrm{GL}(4,\mathbb{C})$. This yields alternative parametrizations which allow for the computation of degrees using Chern classes. We now correctly derive the numbers in Theorem 1 by this method. In what follows we present this for C_2 and then for C_1. The derivation for C_4 is similar to that for C_2, and that for C_3 was correct in [7, Section 3.3].

The linearized parametrization for C_2 was shown in the `Macaulay2` code in the proof of Theorem 2. It is given by a vector bundle F of rank 7 over the 5-dimensional flag variety $\mathrm{Fl}(1,3,\mathbb{C}^4)$. The degree of C_2 in $\mathbb{P}^{23}$ is computed using Chern classes and Segre classes:

$$\deg(C_2) \;=\; \int_{C_2} c_1(O(1))^{11} \;=\; \int_{\mathbb{P}(F)} c_1(O_F(1))^{11} \;=\; \int_{\mathrm{Fl}(1,3,V)} s_5(F). \tag{9}$$

From now set $V=\mathbb{C}^4$. The pair $(L,U)\in \mathrm{Fl}(1,3,V)$ gives the second and first derived algebra. We compute the Segre class in (9), as in [7, Section 3.2]. This uses the exact sequence

$$0 \longrightarrow K \simeq \mathrm{Hom}(V/U, \mathrm{End}^0_L(U)) \longrightarrow F \longrightarrow M = \mathrm{Hom}(\wedge^2(U/L), L) \longrightarrow 0.$$

Here, M is a line bundle and K is a rank six vector bundle. Note the matching parameters m and k_1,k_2,k_3,k_4,k_5,k_6 in the proof of Theorem 2. The kernel K fits into the exact sequence

$$0 \longrightarrow K \longrightarrow \mathrm{Hom}\big(V/U, \mathrm{End}(U)\big) \longrightarrow \mathrm{Hom}\big(V/U, \mathrm{Hom}(L,U)\big) \longrightarrow 0.$$

From the convention that the Segre class and Chern class are inverse to each other we get

$$s(F) \;=\; s(M)\cdot s(K) \;=\; s(M)\cdot s\big(\mathrm{Hom}(V/U,\mathrm{End}(U))\cdot c(\mathrm{Hom}(V/U,\mathrm{Hom}(L,U))\big). \tag{10}$$

This is a rational generating function $s(x,y)$ in $x=-c_1(L)$ and $y=-c_1(U)=c_1(V/U)$. These classes satisfy $x^4=y^4=0$ because they are induced from $\mathbb{P}(V)$ and $\mathbb{P}(V^\vee)$ respectively.

Let $s_5(x,y)$ denote the component of degree 5 in $s(x,y)$. We need to integrate it on the flag variety. Since $\mathrm{Fl}(1,3,V)$ is a divisor of type $(1,1)$ in $\mathbb{P}(V)\times\mathbb{P}(V^\vee)$, we can write

$$\deg(C_2) \;=\; \int s_5(F) \;=\; \int (x+y)s_5(x,y).$$

Conclusion: $55+361+121+295 = 832$

Read the math literature !!!

Never trust anything just because it's published

Check carefully using computational tools

.... like **HomotopyContinuation.jl**