Stable-Set and Coloring bounds based on k-ones 0-1 optimization

Franz Rendl
franz.rendl@aau.at

Alpen-Adria-Universität Klagenfurt

joint work with Dunja Pucher, Klagenfurt

ALPEN-ADRIA UNIVERSITÄT KLAGENFURT | WIEN GRAZ

The high level picture

Stable-Set and Coloring are difficult combinatorial optimization problems.

The ϑ-number introduced by Lovász is a key ingredient to analyze these problems.

The main purpose of this presentation is to provide bounds for Stable-Set and Coloring which avoid the use of ϑ.

An upper bound for the stability number

Let G be a graph and $k \in \mathbb{N}$ be given.

$$s(k) := \min \frac{1}{2} x^T A x \text{ such that } x_i \in \{0,1\}, e^T x = k$$

e is the all-ones vector.

An upper bound for the stability number

Let G be a graph and $k \in \mathbb{N}$ be given.

$$s(k) := \min \frac{1}{2} x^T A x \text{ such that } x_i \in \{0,1\}, e^T x = k$$

e is the all-ones vector.

Key obserervation: If $s(k) > 0$ then G has no stable set of size k, hence

$$\alpha(G) < k.$$

An upper bound for the stability number

Let G be a graph and $k \in \mathbb{N}$ be given.

$$s(k) := \min \frac{1}{2} x^T A x \text{ such that } x_i \in \{0, 1\}, e^T x = k$$

e is the all-ones vector.
Key obserervation: If $s(k) > 0$ then G has no stable set of size k, hence

$$\alpha(G) < k.$$

This upper bound on $\alpha(G)$ is not useful directly, because finding $s(k)$ is NP-hard.

k-partitions

Let $V = \{1, \dots, n\}$ denote the vertices of a graph G.
We consider partitions of V into k partition blocks.
Partition matrix X: $n \times k$ binary matrix with

$$Xe = e.$$

Each row has exactly one nonzero entry equal to 1.

k-partitions

Let $V = \{1, \ldots, n\}$ denote the vertices of a graph G.
We consider partitions of V into k partition blocks.
Partition matrix X: $n \times k$ binary matrix with

$$Xe = e.$$

Each row has exactly one nonzero entry equal to 1.
Inner product of rows i and $j = 1$ iff i and j in same partition block

$$\sum_{[ij] \in E(G)} \sum_r x_{ir} x_{jr} = \frac{1}{2} \langle X, AX \rangle = \text{ \# edges in same blocks}$$

A lower bound for $\chi(G)$

We consider the following (intractable) minimization problem

$$c(k) := \min \frac{1}{2}\langle X, AX\rangle \text{ such that } X \text{ is } k\text{-partition matrix}$$

If $c(k) > 0$ then G has no k-partition into stable sets because each partition contains edges within the same blocks.
Therefore we have the lower bound $k < \chi(G)$.

A lower bound for $\chi(G)$

We consider the following (intractable) minimization problem

$$c(k) := \min \frac{1}{2}\langle X, AX\rangle \text{ such that } X \text{ is } k\text{-partition matrix}$$

If $c(k) > 0$ then G has no k-partition into stable sets because each partition contains edges within the same blocks.
Therefore we have the lower bound $k < \chi(G)$.

Practical use: If we find a tractable positive lower bound for $c(k)$, then we may still conclude $k < \chi(G)$.

SDP relaxation for $c(k)$

Given the k-partition matrix X, we consider $\tilde{X} := \begin{pmatrix} X \\ e^T \end{pmatrix}$ and observe that

$$\tilde{X}\tilde{X}^T = \begin{pmatrix} XX^T & e \\ e^T & k \end{pmatrix}$$

using $Xe = e$ and $e^T e = k$. Moreover $diag(XX^T) = e$.

$$c(k) \geq C(k) = \min\{\frac{1}{2}\langle A, Y\rangle : \begin{pmatrix} Y & e \\ e^T & k \end{pmatrix} \succeq 0, diag(Y) = e, Y \geq 0\}.$$

Properties of relaxation $C(k)$

$c(k)$ is defined only for integers k but $C(k)$ for any $k \geq 1$.

Lemma

The function $C(t)$ is monotonically decreasing for $t > 1$.

If $\begin{pmatrix} Y & e \\ e^T & t \end{pmatrix} \succeq 0$ then this also holds for $t' > t$.

Properties of relaxation $C(k)$

$c(k)$ is defined only for integers k but $C(k)$ for any $k \geq 1$.

Lemma

The function $C(t)$ is monotonically decreasing for $t > 1$.

If $\begin{pmatrix} Y & e \\ e^T & t \end{pmatrix} \succeq 0$ then this also holds for $t' > t$.

Lemma

$C(n) = 0$ and $C(1) > 0$ if $E(G) \neq \emptyset$.

$\begin{pmatrix} I & e \\ e^T & n \end{pmatrix} \succeq 0$, therefore $C(n) = 0$.

Properties of relaxation $C(k)$

$c(k)$ is defined only for integers k but $C(k)$ for any $k \geq 1$.

Lemma

The function $C(t)$ is monotonically decreasing for $t > 1$.

If $\begin{pmatrix} Y & e \\ e^T & t \end{pmatrix} \succeq 0$ then this also holds for $t' > t$.

Lemma

$C(n) = 0$ and $C(1) > 0$ if $E(G) \neq \emptyset$.

$\begin{pmatrix} I & e \\ e^T & n \end{pmatrix} \succeq 0$, therefore $C(n) = 0$.

Note also that $C(1) = 0$ only holds for the all-ones matrix, so $C(1) > 0$, if $E(G) \neq \emptyset$.

More Properties of $C(t)$

Lemma

If $C(t) > 0$ then $\chi(G) \geq \lceil t \rceil$.

A coloring with $k = \lceil t \rceil - 1$ colors would imply that $C(k) = 0$ and therefore $C(t) = 0$.

Using $C(t)$

From the previous results we conclude:

Theorem

If G is a nontrivial graph, then there exists a unique value $t^*(G) > 1$ such that $C(t) = 0$ for $t \leq t^*(G)$ and $C(t) > 0$ for $t > t^*(G)$.

Note that $C(t) > 0$ holds for the open set $t > t^*(G)$.
Using $t^*(G) + \delta$ gives the tightest lower bound
$\chi(G) \geq \lceil t^*(G) + \delta \rceil$.

What can we say about $t^*(G)$ if we know the graph G?

A connection to Szegedy's bound

We consider Szegedy's improvement $\theta_{Sz}(\overline{G})$ of the Lovász-number $\vartheta(\overline{G})$ of the complement graph $\overline{G}$ for which we have

$$\vartheta(\overline{G}) \leq \vartheta_{Sz}(\overline{G}) \leq \chi(G).$$

$$\vartheta_{Sz}(\overline{G}) = \min t \text{ such that } \begin{pmatrix} Y & e \\ e^T & t \end{pmatrix} \succeq 0,$$

$$diag(Y) = e, Y \geq 0, y_{ij} = 0 \text{ for } [ij] \in E(G).$$

A surprising result

Theorem

Let G be a nontrivial graph. Then $t^*(G) = \vartheta_{Sz}(\overline{G})$.

A surprising result

Theorem

Let G be a nontrivial graph. Then $t^*(G) = \vartheta_{Sz}(\overline{G})$.

This follows from

Lemma

$$C(\vartheta_{Sz}(\overline{G})) = 0$$

Lemma

$$C(t) > 0 \text{ for } t > \vartheta_{Sz}(\overline{G})$$

Stable Set Problem

For given A (adjacency matrix of a graph G) and k we consider

$$s(k) := \min \frac{1}{2} x^T Ax \text{ such that } x_i \in \{0, 1\} \text{ and } \sum_i x_i = k.$$

We recall from before the following key observation: If $s(k) > 0$, then G has no stable set of size k, and therefore $\alpha(G) < k$

SDP relaxation for $s(k)$

$s(k) := \min \frac{1}{2} x^T A x$ such that $x_i \in \{0, 1\}$ and $\sum_i x_i = k$.

Here is the generic SDP relaxation.

$$\min \frac{1}{2} \langle A, Y \rangle \text{ such that } Y - yy^T \succeq 0, diag(Y) = y, e^T y = k.$$

Lifting a linear equation into SDP

We have SDP

$$\min\langle C, Y\rangle \text{ such that } Y - yy^T \succeq 0, diag(Y) = y$$

How should $a^T y = a_0 > 0$ be modeled in the relaxation? (Here $a = e$ and $a_0 = k$.) We have $a^T y = a_0$ and $\langle J, Y\rangle = a_0^2$.

Lifting a linear equation into SDP

We have SDP

$$\min\langle C, Y\rangle \text{ such that } Y - yy^T \succeq 0, diag(Y) = y$$

How should $a^T y = a_0 > 0$ be modeled in the relaxation? (Here $a = e$ and $a_0 = k$.) We have $a^T y = a_0$ and $\langle J, Y\rangle = a_0^2$.
With $Y - yy^T \succeq 0$ we get

$$\langle aa^T, Y\rangle \geq (a^T y)^2$$

So it is plausible to ask for equality here.

SDP relaxation has no interior

Define

$$F_1 := \{(Y, y) : y = diag(Y), Y - yy^T \succeq 0, \langle aa^T, Y \rangle = a_0^2, a^T y = a_0\}$$

Lemma

The set F_1 has no interior

We have

$$\begin{pmatrix} a \\ -a_0 \end{pmatrix}^T \begin{pmatrix} Y & y \\ y^T & 1 \end{pmatrix} \begin{pmatrix} a \\ -a_0 \end{pmatrix} = a^T Y a - 2a_0 a^T y + a_0^2 = 0.$$

Since the matrix is also positive semidefinite, it follows that $\begin{pmatrix} a \\ -a_0 \end{pmatrix}$ is in the nullspace and $Ya = a_0 y$.

Flat extension

$$F_1 := \{(Y, y) : y = diag(Y), Y - yy^T \succeq 0, \langle aa^T, Y\rangle = a_0^2, a^T y = a_0\}$$

We also introduce

$$F_2 := \{(Y, y) : y = diag(Y), Y \succeq 0, Ya = a_0 y, a^T y = a_0\}$$

Lemma

$$F_1 = F_2$$

Lemma

$Z = \begin{pmatrix} Y & y \\ y^T & 1 \end{pmatrix}$ *and* Y *have same rank. (*Z *is flat extension of* Y*.)*

SDP relaxation $S(k)$ of $s(k)$

$$s(k) := \min \frac{1}{2} x^T A x \text{ such that } x_i \in \{0,1\} \text{ and } \sum_i x_i = k.$$

$$S(k) = \min \frac{1}{2}\langle A, Y\rangle \text{ such that } Y \succeq 0, diag(Y) = y, e^T y = k, Ye = ky.$$

The SDP relaxation $S(k)$ is defined for any $k > 1$ and we would like to find the smallest t such that $S(t) > 0$ yielding the upper bound $\alpha(G) \leq \lfloor t \rfloor$.

Properties of $S(t)$

We recall the tightening of $\vartheta(G)$ towards $\alpha(G)$ introduced by Lovász and Schrijver (1991) denoted $\vartheta_{LS}(G)$.

$$\vartheta_{LS}(G) = \max e^T y \text{ such that } \operatorname{diag}(Y) = y,\ Y - yy^T \succeq 0,$$

$$Y \geq 0, y_{ij} = 0 \text{ for } [ij] \in E(G)$$

Properties of $S(t)$

We recall the tightening of $\vartheta(G)$ towards $\alpha(G)$ introduced by Lovász and Schrijver (1991) denoted $\vartheta_{LS}(G)$.

$$\vartheta_{LS}(G) = \max e^T y \text{ such that } \operatorname{diag}(Y) = y,\; Y - yy^T \succeq 0,$$

$$Y \geq 0, y_{ij} = 0 \text{ for } [ij] \in E(G)$$

Theorem

$$S(t) > 0 \textit{ iff } t \geq \vartheta_{LS}(G)$$

Proof

In the proof we use that

$$Ye = \vartheta_{LS}(G)y$$

holds for any optimizer Y, y of $\vartheta_{LS}(G)$.
This is used to show that Y, y is feasible for $S(\vartheta_{LS}(G))$. Since $y_{ij} = 0$ on G, we have $\langle A, Y \rangle = 0$ and therefore $S(\vartheta_{LS}(G)) = 0$.

If $S(t) = 0$ then it also holds that $S(t') = 0$ for $1 < t' < t$.

Finally, $S(t) > 0$ for $t > \vartheta_{LS}(G)$

How use these observations?

If we have a value $k \in \mathbb{N}$ such that

$$S(k + 1 - \delta) > 0 \text{ then } \alpha(G) \leq k.$$

How use these observations?

If we have a value $k \in \mathbb{N}$ such that

$$S(k+1-\delta) > 0 \text{ then } \alpha(G) \le k.$$

If in addition

$$S(k+\delta) = 0 \text{ then } \alpha(G) \le k = \lfloor \vartheta_{LS}(G) \rfloor.$$

We get this conclusion without actually computing $\vartheta_{LS}(G)$.

Similar for Coloring

If we have a value $k \in \mathbb{N}$ such that

$$C(k - 1 + \delta) > 0 \text{ then } \chi(G) \geq k.$$

Similar for Coloring

If we have a value $k \in \mathbb{N}$ such that

$$C(k-1+\delta) > 0 \text{ then } \chi(G) \geq k.$$

If in addition

$$C(k-\delta) = 0 \text{ then } \chi(G) \geq k = \lceil \vartheta_{Sz}(\overline{G}) \rceil.$$

We get this conclusion without actually computing $\vartheta_{Sz}(\overline{G})$.

Practical aspects

Computing $\vartheta(G)$ involves solving an SDP with at least $m = |E(G)|$ equations. Once m gets large ($m > 10000$) this is not practical, even if n is relatively small $n \leq 200$.

Computing $C(t)$ or $S(t)$ requires only $2n$ equations, independent of m, but we may have to check several choices for t.

We use $C(t)$ and $S(t)$ as a basic relaxation, and want to add further constraints.

We are interested in graphs where $|E(G)|$ is too large to work with $\vartheta(G)$.

Exact subgraph constraints

A systematic way to tighten these relaxations consists in asking that all subgraphs of size r for $r = 2, 3, \ldots$ are exact, meaning that the respective submatrices are in the respective stable set polytope of size r,

- see Adams, Anjos, Rendl, Wiegele (2015) for theoretical aspects,
- see Gaar, Rendl (IPCO 2019) for computational issues.

A new look at small subgraphs

Pucher (dissertation in progress) suggests to enumerate small subgraphs G' by considering $\chi(G') = r$ for $r = 1, 2, \ldots$.

We first enumerate all maximal cliques in G.

If graph is too large, limit enumeration to clique sizes up to fixed t, then this is still polynomial time.

Identify violated subgraphs

Subgraphs whith clique number 1 (=subgraph is itsself a clique) can be shown to be exact (=subgraph lies in convex hull of its stable sets).

First interesting case: (maximal) clique + (another) clique

$$Y_I = \begin{pmatrix} y_1 & \dots & 0 & Y_{1,k+1} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & y_k & Y_{k,k+1} \\ Y_{1,k+1} & \dots & Y_{k,k+1} & y_{k+1} \end{pmatrix}$$

Here clique of sizes k and 1.

Simple violation condition

We can show

Theorem

G_I is exact iff

$$0 \leq \ Y_{i,k+1} \leq y_i \text{ for } i = 1, \ldots, k$$

$$\sum_i Y_{i,k+1} \leq y_{k+1}$$

$$\sum_i y_i \leq 1 + \sum_i Y_{i,k+1}$$

This generalizes naturally if second clique has sizes > 1.

Practical use

Having list of all maximal cliques, we make pairwise comparison.

Add only the most violated pairs of cliques, and require previous linear inequalities to hold for them.

After several such iterations, no pairs of cliques have significant violation.

Moving to triples of cliques leads to much more complicated violation conditions, and to not improve bounds significantly.

Stability Number and bounds for random graphs

n	$\alpha(G)$	ϑ-bound	our bound	max cliques
100	46	46.48	46	211
100	30	32.94	30	338
100	23	25.55	23	527
100	21	22.98	21	793
200	59	66.75	63	817
200	40	48.27	46	1499
200	32	39.97	38	2914
200	25	32.59	31	5546

Last Slide

- Approach for Stable-Set looks promising.
- Next steps: Apply to Coloring
- Combine with standard exact subgraphs?