The Chvátal-Gomory Procedure for Integer SDPs

Frank de Meijer, Renata Sotirov

Tilburg University, The Netherlands

Outline of the talk

- Integer Semidefinite Programs (ISDPs)
- Chvátal-Gomory procedure for ISDPs
- A Branch-and-Cut algorithm for ISDPs
- Case study: the Quadratic Traveling Salesman Problem

Integer semidefinite programs

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where $\mathbf{C}, \mathbf{A}_i \in \mathcal{S}^n$, $\mathbf{b} \in \mathbb{R}^m$.

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  - in combinatorial optimization

Example: BQPs and ISDPs

Binary Quadratic Problems & ISDPs

Given

- cost matrix $\mathbf{Q} \in \mathcal{S}^n$
- $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{b} = (b_i) \in \mathbb{R}^m$

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Examples:

- max-cut, stable set problem, quadratic assignment problem, graph coloring, graph partition problem, bandwidth problem, etc.

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- rewrite the objective function:

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- reformulated BQP:

$$\begin{aligned} \min \quad & \langle \mathbf{Q}, \mathbf{X} \rangle \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{diag}(\mathbf{X}) = \mathbf{x} \\ & \begin{pmatrix} 1 & \mathbf{x}^\top \\ \mathbf{x} & \mathbf{X} \end{pmatrix} \succeq \mathbf{0}, \ \mathrm{rank} \begin{pmatrix} 1 & \mathbf{x}^\top \\ \mathbf{x} & \mathbf{X} \end{pmatrix} = 1, \end{aligned}$$

  where $\mathbf{diag} : \mathcal{S}^n \to \mathbb{R}^n$ maps a matrix to a vector containing its diag. entries.

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Let $\mathbf{X} \in \{0,1\}^{n \times n}$ be a symmetric matrix.

Then $\mathbf{X} \succeq \mathbf{0}$ if and only if $\mathbf{X} = \sum_{i=1}^{k} \mathbf{x_i} \mathbf{x_i}^\top$ for some $\mathbf{x_i} \in \{0,1\}^n$, $i \in [k]$.

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Lemma (de Meijer, S.)

Let $\bar{\mathbf{X}} = \begin{pmatrix} 1 & \mathrm{diag}(\mathbf{X})^\top \\ \mathrm{diag}(\mathbf{X}) & \mathbf{X} \end{pmatrix} \succeq \mathbf{0}$. Then $\mathrm{rank}(\bar{\mathbf{X}}) = 1 \;\Leftrightarrow\; \mathbf{X} \in \{0,1\}^{n\times n}$.

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$$\text{BQP} \quad \Leftrightarrow \quad \text{BSDP} \begin{cases} \min & \langle \mathbf{Q}, \mathbf{X} \rangle \\ \text{s.t.} & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathrm{diag}(\mathbf{X}) = \mathbf{x} \\ & \mathbf{X} \in \{0,1\}^{n\times n}, \quad \begin{pmatrix} 1 & \mathbf{x}^\top \\ \mathbf{x} & \mathbf{X} \end{pmatrix} \succeq \mathbf{0} \end{cases}$$

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$$c(C) := \bigcap_{\substack{(\mathbf{c},d) \in \mathbb{Z}^m \times \mathbb{R} \\ C \subseteq \{\mathbf{x} : \mathbf{c}^\top \mathbf{x} \leq d\}}} \left\{ \mathbf{x} \in \mathbb{R}^m : \mathbf{c}^\top \mathbf{x} \leq \lfloor d \rfloor \right\}$$

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- the smallest k for which $C_I = C^{(k)}$ is known as the Chvátal rank of C
- finite Chvátal rank is proven for:
  - bounded real polyhedra – Chvátal
  - unbounded rational polyhedra – Schrijver
  - irrational polytopes – Dunkel, Schulz
  - bounded conic representable sets – Çezik, Iyengar
  - rational ellipsoids Dey and Vielma
  - strictly convex bodies – Dadush, Dey, Vielma
  - compact convex sets – Braun, Pokutta and Dadush, Dey, Vielma

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Then, $\exists\mathbf{U}\in\mathcal{S}_+^n$ such that $\langle\mathbf{A_i},\mathbf{U}\rangle = c_i$ for all $i\in[m]$ and $\langle\mathbf{C},\mathbf{U}\rangle\le d$.

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- the Chvátal-Gomory closure of P:

$$c(P) = \bigcap_{\substack{\mathbf{U}\in\mathcal{S}^n_+ \text{ s.t.} \\ \langle\mathbf{A_i},\mathbf{U}\rangle\in\mathbb{Z},\ i\in[m]}} \left\{\mathbf{x}\in\mathbb{R}^m \;:\; \sum_{i=1}^m x_i\langle\mathbf{A_i},\mathbf{U}\rangle \le \lfloor\langle\mathbf{C},\mathbf{U}\rangle\rfloor\right\}$$

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- Separation of CG cuts for conic problems is posted as an **open problem** by Çezik–Iyengar.

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$$c(P \cap H) = c(P) \cap H.$$

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- A **closed-form expression** for the CG closure of some bounded spectrahedra

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Total dual integrality for SDPs ?

De Carli Silva and Tunçel. A notion on total dual integrality for convex, semidefinite, and extended formulations. SIAM J. Discrete Math., 34, 2018.

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Total dual integrality for SDPs ?

Definiton (Property $(\mathrm{P}\mathbb{Z})$)

A matrix $\mathbf{X} \in \mathcal{S}_n^+$ satisfies integrality property $(\mathrm{P}\mathbb{Z})$ if

$$\mathbf{X} = \sum_{S \subseteq [n]} \mathbf{y_S} \mathbb{1}_\mathbf{S} \mathbb{1}_\mathbf{S}^\top \quad \text{for some } \mathbf{y} : \mathcal{P}([n]) \to \mathbb{Z}_+,$$

where $\mathbb{1}_\mathbf{S}$ is the indicator vector of S, and $\mathcal{P}([n])$ the set of all subsets of $[n]$.

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A matrix $\mathbf{X} \in \mathcal{S}_n^+$ satisfies integrality property $(\mathrm{P}\mathbb{Z})$ if

$$\mathbf{X} = \sum_{S \subseteq [n]} \mathbf{y_S} \mathbb{1}_\mathbf{S} \mathbb{1}_\mathbf{S}^\top \quad \text{for some } \mathbf{y} : \mathcal{P}([n]) \to \mathbb{Z}_+,$$

where $\mathbb{1}_\mathbf{S}$ is the indicator vector of S, and $\mathcal{P}([n])$ the set of all subsets of $[n]$.

Definiton (Total dual integrality for SDPs)

An LMI $\mathbf{C} - \sum_{i=1}^m \mathbf{A_i} x_i \succeq \mathbf{0}$ is called totally dual integral if, for every $\mathbf{b} \in \mathbb{Z}^m$, the SDP dual to $\sup \left\{ \mathbf{b}^\top \mathbf{x} \, : \, \mathbf{C} - \sum_{i=1}^m \mathbf{A_i} x_i \succeq 0 \right\}$ has an optimal solution satisfying property $(\mathrm{P}\mathbb{Z})$ whenever it has an optimal solution.

A closed-form expression for $c(P)$ (cont.)

Theorem (de Meijer and S.)

Let $P = \left\{ \mathbf{x} \in \mathbb{R}^m \; : \; \mathbf{C} - \sum_{i=1}^m \mathbf{A_i} x_i \succeq \mathbf{0} \right\}$ with $\mathbf{A_i} \in \mathbb{Z}^{n \times n} \cap \mathcal{S}^n$ for all $i \in [m]$ s.t. $\mathbf{C} - \sum_{i=1}^m \mathbf{A_i} x_i \succeq \mathbf{0}$ is totally dual integral and satisfies Slater's condition.

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- For any rational polyhedron there exists a TDI system that describes P. Is there such analogue for spectrahedra?

on exploiting CG cuts for spectrahedra . . .

A CG-based branch-and-cut algorithm for ISDPs

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A CG-based branch-and-cut algorithm for ISDPs

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- our algorithm extends the work of:
  Kobayashi and Takano. A B&C algorithm for solving mixed-integer semidefinite optimization problems. Comput. Optim. Appl., 75, 2020.

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- **Kobayashi–Takano**: Let $\mathbf{d}$ be an eigenvector corresponding to the smallest eigenvalue of $\mathbf{C} - \sum_{i=1}^{m} \mathbf{A_i}\hat{x}_i$. Then, add the cut:

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Can one do even better?

A CG-based branch-and-cut algorithm for ISDPs (cont.)

- let $\mathbf{U} \in \mathcal{S}^n_+$, $\langle \mathbf{A_i}, \mathbf{U} \rangle \in \mathbb{Z}$, $\forall i \in [m]$

A CG-based branch-and-cut algorithm for ISDPs (cont.)

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- S-CG cuts are introduced for rational polyhedra:
  Dash, Günlük, Lee. On a generalization of the CG closure. Math. Program., 192, 2022

A CG-based branch-and-cut algorithm for ISDPs (cont.)

Algorithm 1: CG-based B&C algorithm for solving (D_{ISDP})

Input: $\mathbf{C}, \mathbf{A_i}, i \in [m], S$

1 Initialize $\mathcal{F} = \left\{ \mathbf{x} \in \mathbb{R}^m \ : \ \mathrm{diag}\left(\mathbf{C} - \sum_{i=1}^m \mathbf{A_i} x_i\right) \geq 0 \right\}$.

2 **B&B procedure:** Start or continue the B&B algorithm for solving the MILP $\max \left\{ \mathbf{b}^\top \mathbf{x} \ : \ \mathbf{x} \in \mathcal{F} \cap \mathbb{Z}^m \right\}$ using the callback function at each node in the tree.

3 **Callback procedure:** **if** *an integer point* $\hat{\mathbf{x}} \in \mathcal{F}$ *is found* **then**

4   **if** $\lambda_{\min}\left(\mathbf{C} - \sum_{i=1}^m \mathbf{A_i} \hat{x}_i\right) < 0$ **then**

5   Call SeparationRoutine$(\mathbf{C}, \mathbf{A_1}, \ldots, \mathbf{A_m}, S, \hat{\mathbf{x}})$ which provides $\mathbf{U_j}$, $j \in [K]$.

6   Add the cuts

$$\sum_{i=1}^m \langle \mathbf{A_i}, \mathbf{U_j} \rangle x_i \leq \lfloor \langle \mathbf{C}, \mathbf{U_j} \rangle \rfloor_S \quad \text{for } j \in [K] \text{ to } \mathcal{F}.$$

7   **else**

8   Use $\hat{\mathbf{x}}$ to cut off other nodes in the branching tree.

9   **end**

10   Return to Step 2

11 **end**

Output: $\hat{\mathbf{x}}, OPT := \mathbf{b}^\top \mathbf{x}$

Separation routines?

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- . . . a **generic** routine for binary SDPs, based on

Thm (Djukanović and Rendl, Letchford and Sørensen)

Let $\mathbf{X} \in \{0,1\}^{n \times n}$. Then $\mathbf{X} \succeq \mathbf{0} \Leftrightarrow \mathbf{X} = \sum_{i=1}^{k} \mathbf{x_i}\mathbf{x_i}^\top$ for $\mathbf{x_i} \in \{0,1\}^n$, $i \in [k]$.

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- . . . a **problem-specific** routine for constructing **S-CG cuts** for given a optimization problem.

application of the CG-based B&C algorithm . . .

ISDP for the Quadratic Traveling Salesman Problem

- $G = (V, A)$ ...directed simple graph
  - V ...vertex set, $n = |V|$
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$$\mathcal{T}_n(G) := \left\{ \mathbf{X}^{\mathcal{C}} \in \{0,1\}^{n \times n} \ : \ x_{ij}^{\mathcal{C}} = 1 \text{ iff } (i,j) \in \mathcal{C} \text{ for Hamiltonian cycle } \mathcal{C} \right\}$$

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- the set of the **2-arcs** of G:

$$\mathcal{A} := \{(i,j,k) \ : \ (i,j),(j,k) \in A, \ |\{i,j,k\}| = 3\}$$

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$$\mathcal{A} := \{(i,j,k) \;:\; (i,j),(j,k) \in A,\; |\{i,j,k\}| = 3\}$$

- $\mathbf{Q} = (q_{ijk}) \in \mathbb{R}^{n \times n \times n}$ s.t. $q_{ijk} = 0$ if $(i,j,k) \notin \mathcal{A}$ ... cost matrix

ISDP for the Quadratic Traveling Salesman Problem

the Quadratic Traveling Salesman Problem:

$$QTSP(\mathbf{Q}, G) := \min_{\mathbf{X} \in \mathcal{T}_n(G)} \sum_{i,j,k=1}^{n} q_{ijk} x_{ij} x_{jk}$$

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- formulation of the QTSP:

$$\begin{aligned} QTSP(\mathbf{Q}, G) := \min \ & \textstyle\sum_{i,j,k=1}^{n} q_{ijk} y_{ijk} \\ \text{s.t.} \ & \text{coupling constraints} \\ & y_{ijk} \geq 0 \quad \forall (i,j,k) \in \mathcal{A} \\ & \mathbf{X} \in \mathcal{T}_n(G) \end{aligned}$$

ISDP for the QTSP

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\min \quad & \sum_{i,j,k=1}^{n} q_{ijk} y_{ijk} \\
\text{s.t.} \quad & \text{coupling constriants} \\
& \beta \mathbf{I_n} + \alpha \mathbf{J_n} - \frac{1}{2}\left((\mathbf{X} + \mathbf{X}^{(2)}) + (\mathbf{X} + \mathbf{X}^{(2)})^\top \right) \succeq \mathbf{0} \\
& y_{ijk} \geq 0 \quad \forall (i,j,k) \in \mathcal{A} \\
& \mathbf{X},\ \mathbf{X}^{(2)} \in \Pi_n,
\end{aligned}
$$

where $\alpha \geq (2 - \cos\left(\frac{2\pi}{n}\right))/n$, $\cos\left(\frac{2\pi}{n}\right) \leq \beta < 2$

and Π_n is the set of permutation matrices

Chvátal-Gomory cuts for the ISDPs of the QTSP

- the SDP constraint:

$$\beta \mathbf{I_n} + \alpha \mathbf{J_n} - \frac{1}{2}\left((\mathbf{X} + \mathbf{X^{(2)}}) + (\mathbf{X} + \mathbf{X^{(2)}})^\top\right) \succeq \mathbf{0}$$

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- Our B&C algorithm starts optimizing over:

$$\mathcal{F} := \left\{(\mathbf{y}, \mathbf{X}, \mathbf{X^{(2)}}) \in \mathbb{R}_+^{\mathcal{A}} \times \Pi_n \times \Pi_n \ : \ \text{coupling constriants}\right\}$$

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- If an integer solution $(\hat{\mathbf{y}}, \hat{\mathbf{X}}, \hat{\mathbf{X}}^{(2)})$ is found in the branching tree, then check $\lambda_{\min}\left(\beta \mathbf{I_n} + \alpha \mathbf{J_n} - \frac{1}{2}\left((\hat{\mathbf{X}} + \hat{\mathbf{X}}^{(2)}) + (\hat{\mathbf{X}} + \hat{\mathbf{X}}^{(2)})^\top\right)\right) \geq 0.$

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IS THIS AN EFFICIENT PROCEDURE FOR SOLVING GENERAL ISDPS?

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The talk is based on:

de Meijer and Sotirov, The Chvátal-Gomory-Gomory Procedure for Integer SDPs with Applications in Combinatorial Optimization, https://arxiv.org/abs/2201.10224

THANK YOU FOR YOUR ATTENTION

Chvátal-Gomory cuts for the ISDPs of the QTSP (cont.)

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$\beta \mathbf{I_n} + \alpha \mathbf{J_n} - \frac{1}{2}\left((\hat{\mathbf{X}} + \hat{\mathbf{X}}^{(2)} + (\hat{\mathbf{X}} + \hat{\mathbf{X}}^{(2)})^\top\right)$

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- Let $\{S_1, \ldots, S_k\}$ be the partition of V implied by the cycle covers in $\hat{\mathbf{X}}$, then

$$v_i^j := \begin{cases} n - |S_j| & \text{if } i \in S_j \\ -|S_j| & \text{if } i \notin S_j, \end{cases} \qquad \forall j \in [k]$$

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- CG cut of with dual multiplier $\mathbf{U} = \mathbb{1}_{\mathbf{S}} \mathbb{1}_{\mathbf{S}}^{\top}$:

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- S-CG cut with dual multiplier $\mathbf{U} = \mathbb{1}_\mathbf{S}\mathbb{1}_\mathbf{S}^\top$ and $\sum_{k\in N:(i,k,j)\in\mathcal{A}} y_{ikj} - x_{ij}^{(2)} = 0$, $\forall i,j \in S$, each with dual multiplier 1, and $-y_{ikj} \leq 0$ $\forall (i,k,j) \in \mathcal{A}$, each with dual multiplier 1

$$\sum_{\substack{i\in S\\ j\in S}} x_{ij} + \sum_{\substack{i\in S\\ j\in S}} \sum_{\substack{k\in N\setminus S:\\ (i,k,j)\in\mathcal{A}}} y_{ikj} \leq |S| - 1, \quad \forall S \subset N, 2 \leq |S| < \frac{1}{2}n$$