Computations using higher-level correlations of the zeros of the Riemann $\zeta$-function

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The Riemann $\zeta$-function and conjectures

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{for } \Re(s) > 1 \]
Montgomery’s pair correlation approach

Let $0 < \gamma_1 \leq \gamma_2 \leq \ldots$ denote the imaginary parts of the non-trivial zeros.
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$$N(T) = \sum_{\gamma_j \leq T} 1$$

Then the fraction of simple zeros is bounded as follows:

$$\frac{N_s(T)}{N(T)} \geq 1 - M_2(T) \frac{N(T)}{N(T)}$$
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and

\[
M_2(T) = \sum_{i \neq j} \delta(\gamma_i, \gamma_j) = \sum_{\gamma_j \leq T} m_j - 1,
\]

where \( m_j \) is the multiplicity of \( \gamma_j \).
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where $m_j$ is the multiplicity of $\gamma_j$.

Then the fraction of simple zeros is bounded as follows:

$$\frac{N_s(T)}{N(T)} \geq 1 - \frac{M_2(T)}{N(T)}.$$
Montgomery’s pair correlation distribution

Let $f$ be a function with Fourier transform supported in $[-1, 1]$, and let

$$M_2(f, T) = \sum_{\gamma_i, \gamma_j \in [0, T], \atop i \neq j} f\left( (\gamma_i - \gamma_j) \frac{\log T}{2\pi} \right).$$

Assuming the Riemann hypothesis, we have

$$M_2(f, T) = N(T) \left( \int_{-\infty}^{\infty} f(x) \left( 1 - \frac{\text{sinc}(x)}{x^2} \right) \, dx + o(1) \right).$$

Here $1 - \frac{\text{sinc}(x)}{x^2}$ is the pair correlation function of the Gaussian Unitary Ensemble model.
Montgomery’s pair correlation distribution

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Assuming the Riemann hypothesis, we have

$$M_2(f, T) = N(T) \left( \int_{-\infty}^{\infty} f(x)(1 - \text{sinc}(x)^2)dx + o(1) \right).$$

Here $1 - \text{sinc}(x)^2$ is the pair correlation function of the Gaussian Unitary Ensemble model.
Montgomery’s pair correlation approach

\[ M_2(T) = \sum_{\gamma_i,\gamma_j \in [0, T], i \neq j} \delta(\gamma_i, \gamma_j) \]
Montgomery’s pair correlation approach

For a function $f$ with $f(0) = 1$ and $f(x) \geq 0$, we have

$$M_2(T) = \sum_{\gamma_i, \gamma_j \in [0, T], \ i \neq j} \delta(\gamma_i, \gamma_j) \leq \sum_{\gamma_i, \gamma_j \in [0, T], \ i \neq j} f((\gamma_i - \gamma_j) \frac{\log T}{2\pi}) = M_2(f, T)$$

Suppose that $\hat{f}$ is supported in $[-1, 1]$, then

$$M_2(f, T) = N(T) \left( \int_{-\infty}^{\infty} f(x) \left( 1 - \text{sinc}(x)^2 \right) \, dx + o(1) \right)$$

So the fraction of simple zeros is bounded by

$$\frac{N_s(T)}{N(T)} \geq 1 - \int_{-\infty}^{\infty} f(x) \left( 1 - \text{sinc}(x)^2 \right) \, dx + o(1).$$

In particular, the fraction of simple zeros is at least $\frac{2}{3}$. 

$\frac{5}{12}$
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\]

Suppose that \( \hat{f} \) is supported in \([-1, 1]\), then

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\frac{N_s(T)}{N(T)} \geq 1 - \int_{-\infty}^{\infty} f(x)(1 - \text{sinc}(x)^2)dx + o(1).
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In particular, the fraction of simple zeros is at least \( 2/3 \).
Generalizing to \( n \)-level correlations

Let

\[
M_n(T) = \sum_{\gamma_j \in [0, T], \text{distinct indices}} \delta(\gamma_{j_1}, \ldots, \gamma_{j_n}) = \sum_{\gamma_j \leq T} (m_j - 1) \cdots (m_j - n + 1)
\]

and \( Z_n(T) \) the number of zeros \( \frac{1}{2} + \gamma i \) with multiplicity \( \geq n \) and \( \gamma \leq T \).
Generalizing to \( n \)-level correlations

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M_n(T) = \sum_{\gamma_{j_1}, \ldots, \gamma_{j_n} \in [0, T] \text{ distinct indices}} \delta(\gamma_{j_1}, \ldots, \gamma_{j_n}) = \sum_{\gamma_j \leq T} (m_j - 1) \cdots (m_j - n + 1)
\]

and \( Z_n(T) \) the number of zeros \( \frac{1}{2} + \gamma i \) with multiplicity \( \geq n \) and \( \gamma \leq T \).

Then the fraction of zeros with multiplicity at least \( n \) is bounded as follows:

\[
\frac{Z_n(T)}{N(T)} \leq \frac{1}{(n-1)!} \frac{M_n(T)}{N(T)}.
\]
Rudnick & Sarnak’s generalization to \( n \)-level correlations

Let \( f \) be a function with Fourier transform restricted to \( 2H_n \subset \mathbb{R}^{n-1} \), and let

\[
M_n(f, T) = \sum_{\gamma_1, \ldots, \gamma_n \in [0, T]} f(\tilde{\gamma}_1 - \tilde{\gamma}_n, \ldots, \tilde{\gamma}_{n-1} - \tilde{\gamma}_n).
\]

Assuming the Riemann hypothesis, we have

\[
M_n(f, T) = N(T) \left( \int_{\mathbb{R}^{n-1}} f(x) W_n(x, 0) \, dx + o(1) \right)
\]

Here \( W_n(x) = \det(\text{sinc}(x_i - x_j))_{i,j=1}^n \) is the limiting \( n \)-level correlation density for the Gaussian Unitary Ensemble model.
Optimization problem

\[
\begin{align*}
\text{minimize} \quad & \nu_n(f) = \int f(x) W_n(x, 0) \, dx \\
\text{subject to} \quad & f(0) = 1 \\
& f(x) \geq 0 \\
& \text{support}(\hat{f}) \subseteq 2H_n
\end{align*}
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Optimization problem

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subject to \[ f(0) = 1 \]
\[ f(x) \geq 0 \]
\[ \text{support}(\hat{f}) \subseteq 2H_n \]

Parametrization:
\[ f(x) = \sum_{i,j} X_{ij} g_i(x) g_j(x) \]

with \( X \geq 0 \) and \( \text{support}(\hat{g}_i) \subseteq H_n \).
Parametrizations

Let

\[ f(x) = \sum_{\alpha, \alpha'} X_{\alpha, \alpha'} g_{\alpha}(x) g_{\alpha'}(x) \]

with \( X \geq 0 \), and thus

\[ \hat{f}(x) = \sum_{\alpha, \alpha'} X_{\alpha, \alpha'} \hat{g}_\alpha \ast \hat{g}_{\alpha'}(x). \]
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with \( X \geq 0 \), and thus

\[ \tilde{f}(x) = \sum_{\alpha, \alpha'} X_{\alpha, \alpha'} \tilde{g}_{\alpha} \ast \tilde{g}_{\alpha'}(x). \]

We choose

- \( \tilde{g}_{\alpha}(x) = p_{\alpha}(x)1_{H_n}(x) \), where \( p_{\alpha} = x^\alpha \) for a multi-index \( \alpha \).
Parametrizations

Let

\[ f(x) = \sum_{\alpha, \alpha'} X_{\alpha, \alpha'} g_\alpha(x) g_{\alpha'}(x) \]

with \( X \succeq 0 \), and thus

\[ \tilde{f}(x) = \sum_{\alpha, \alpha'} X_{\alpha, \alpha'} \tilde{g}_\alpha \star \tilde{g}_{\alpha'}(x). \]

We choose

• \( \tilde{g}_\alpha(x) = p_\alpha(x) 1_{H_n}(x) \), where \( p_\alpha = x^\alpha \) for a multi-index \( \alpha \).
• \( \tilde{g}_\alpha(x) = e^{-2\pi i x \cdot \lambda} 1_{H_n}(x) \) for \( \lambda \in \frac{1}{2} \mathbb{Z}^n \) (i.e., \( g_\lambda(x) = 1_{H_n}(x - \lambda) \)).
Solving the semidefinite program

Lemma

Let $A$ be a positive semidefinite matrix and let $x$ be a solution to the system $Ax = b$. Then $xx^T / (x^T b)^2$ is an optimal solution to the semidefinite program

$$\begin{align*}
\text{minimize} & \quad \langle A, X \rangle \\
\text{subject to} & \quad \langle bb^T, X \rangle = 1 \\
& \quad X \succeq 0.
\end{align*}$$
Results

Assuming the Riemann hypothesis, we prove that at most 3.86% of the non-trivial zeros have multiplicity $\geq 3$. We are working on the proof that at most about 0.5% of the non-trivial zeros have multiplicity $\geq 4$. 
Thank you!