Computations using higher-level correlations of the zeros of the Riemann ζ-function

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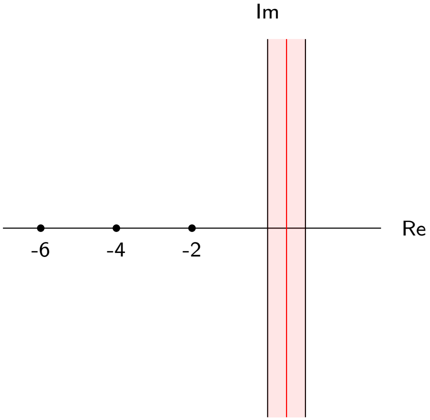
The Riemann ζ-function and conjectures

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{for } \mathrm{Re}(s) > 1$$



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and

$$M_2(T) = \sum_{\substack{i \neq j \\ \gamma_i, \gamma_j \le T}} \delta(\gamma_i, \gamma_j) = \sum_{\gamma_j \le T} m_j - 1,$$

where m_j is the multiplicity of γ_j.

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where m_j is the multiplicity of γ_j.

Then the fraction of simple zeros is bounded as follows:

$$\frac{N_s(T)}{N(T)} \geq 1 - \frac{M_2(T)}{N(T)}.$$

Montgomery's pair correlation distribution

Let f be a function with Fourier transform supported in $[-1, 1]$, and let

$$M_2(f, T) = \sum_{\substack{\gamma_i, \gamma_j \in [0, T] \\ i \neq j}} f\left((\gamma_i - \gamma_j)\frac{\log T}{2\pi}\right).$$

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Assuming the Riemann hypothesis, we have

$$M_2(f, T) = N(T)\Big(\int_{-\infty}^{\infty} f(x)(1 - \operatorname{sinc}(x)^2)\,dx + o(1)\Big).$$

Here $1 - \operatorname{sinc}(x)^2$ is the pair correlation function of the Gaussian Unitary Ensemble model.

Montgomery's pair correlation approach

$$M_2(T) = \sum_{\substack{\gamma_i, \gamma_j \in [0, T] \\ i \neq j}} \delta(\gamma_i, \gamma_j)$$

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For a function f with $f(0) = 1$ and $f(x) \geq 0$, we have

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Suppose that $\hat{f}$ is supported in $[-1, 1]$, then

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So the fraction of simple zeros is bounded by

$$\frac{N_s(T)}{N(T)} \geq 1 - \int_{-\infty}^{\infty} f(x)(1 - \operatorname{sinc}(x)^2)dx + o(1).$$

In particular, the fraction of simple zeros is at least 2/3.

Generalizing to n-level correlations

Let

$$M_n(T) = \sum_{\substack{\gamma_{j_1},\ldots,\gamma_{j_n}\in[0,T] \\ \text{distinct indices}}} \delta(\gamma_{j_1},\ldots,\gamma_{j_n}) = \sum_{\gamma_j \le T} (m_j - 1)\cdots(m_j - n + 1)$$

and $Z_n(T)$ the number of zeros $\frac{1}{2} + \gamma i$ with multiplicity $\geq n$ and $\gamma \leq T$.

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Then the fraction of zeros with multiplicity at least n is bounded as follows:

$$\frac{Z_n(T)}{N(T)} \leq \frac{1}{(n-1)!} \frac{M_n(T)}{N(T)}.$$

Rudnick & Sarnak's generalization to *n*-level correlations

Let f be a function with Fourier transform restricted to $2H_n \subset \mathbb{R}^{n-1}$, and let

$$M_n(f,T) = \sum_{\substack{\gamma_{j_1},\dots,\gamma_{j_n} \in [0,T] \\ \text{distinct indices}}} f(\tilde{\gamma}_{j_1} - \tilde{\gamma}_{j_n}, \dots, \tilde{\gamma}_{j_{n-1}} - \tilde{\gamma}_{j_n}).$$

Assuming the Riemann hypothesis, we have

$$M_n(f,T) = N(T)\Big(\int_{\mathbb{R}^{n-1}} f(x) W_n(x,0)\,dx + o(1) \Big)$$

Here $W_n(x) = \det\big(\operatorname{sinc}(x_i - x_j)\big)_{i,j=1}^{n}$ is the limiting n-level correlation density for the Gaussian Unitary Ensemble model.

Optimization problem

$$
\begin{aligned}
\text{minimize} \quad & \nu_n(f) = \int f(x) W_n(x,0)\,dx \\
\text{subject to} \quad & f(0) = 1 \\
& f(x) \geq 0 \\
& \text{support}(\hat{f}) \subseteq 2H_n
\end{aligned}
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Parametrization:

$$f(x) = \sum_{i,j} X_{ij} g_i(x) g_j(x)$$

with $X \succeq 0$ and $\mathrm{support}(\hat{g}_i) \subseteq H_n$.

Parametrizations

Let

$$f(x) = \sum_{\alpha,\alpha'} X_{\alpha,\alpha'} g_\alpha(x) g_{\alpha'}(x)$$

with $X \succeq 0$, and thus

$$\widehat{f}(x) = \sum_{\alpha,\alpha'} X_{\alpha,\alpha'} \widehat{g_\alpha} * \widehat{g_{\alpha'}}(x).$$

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- $\widehat{g_\alpha}(x) = p_\alpha(x) 1_{H_n}(x)$, where $p_\alpha = x^\alpha$ for a multi-index α.
- $\widehat{g_\alpha}(x) = e^{-2\pi i x \cdot \lambda} 1_{H_n}(x)$ for $\lambda \in \frac{1}{2}\mathbb{Z}^n$ (i.e., $g_\lambda(x) = \widehat{1_{H_n}}(x - \lambda)$).

Solving the semidefinite program

Lemma

Let A be a positive semidefinite matrix and let x be a solution to the system $Ax = b$. Then $xx^\mathsf{T}/(x^\mathsf{T}b)^2$ is an optimal solution to the semidefinite program

$$\begin{aligned} \text{minimize} \quad & \langle A, X \rangle \\ \text{subject to} \quad & \langle bb^\mathsf{T}, X \rangle = 1 \\ & X \succeq 0. \end{aligned}$$

Results

Assuming the Riemann hypothesis, we prove that at most 3.86% of the non-trivial zeros have multiplicity ≥ 3. We are working on the proof that at most about 0.5% of the non-trivial zeros have multiplicity ≥ 4.

Thank you!