Bounding the set of classical correlations of a many-body system

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Google Research

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Outline

- A quantum information scientist meets non-negative polynomials
- Sum-of-squares representations
- Characterizing many-body correlations
- An application of sum-of-quares in quantum information: self-testing

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- A bit of history

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Who among us would not be happy to lift the veil behind which is hidden the future; to gaze at the coming developments of our science and at the secrets of its development in the centuries to come? What will be the ends toward which the spirit of future generations of mathematicians will tend? What methods, what new facts will the new century reveal in the vast and rich field of mathematical thought?

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Riemann Hypothesis and other number theory conjectures

Given a multivariate non-negative polynomial, does it admit a sum-of-squres representation?

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Given $p(\vec{x}) \in \mathbb{R}[\vec{x}]$ satisfying $p(\vec{x}) \geq 0 \ \forall \vec{x} \in \mathbb{R}^n$, does $p(\vec{x})$ admit a sum-of-squares (s.o.s.) representation $p(\vec{x}) = \sum_i q_i(\vec{x})^2$?

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- Converse is trivial. When does equivalence hold?

- How powerful is this representation?

- Can one extend it to subsets of $\mathbb{R}^n$?
  - Semialgebraic sets $\begin{cases} f_i(\vec{x}) & = & 0 \\ g_j(\vec{x}) & \geq & 0 \end{cases}$

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Non-constructive proof

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But $(x^2+y^2)(x^4y^2+x^2y^4+1-3x^2y^2)$ does admit a sos decomposition!!

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$$Q\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

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Example: Cauchy-Schwarz Inequality

$$||\vec{x}||^2 \cdot ||\vec{y}||^2 - \langle \vec{x}, \vec{y} \rangle^2 = \sum_{i<j} (x_i y_j - x_j y_i)^2$$

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- *Primal-dual formulation*

$$
\begin{array}{cc}
\min_X & \langle C, X \rangle \\
\text{s.t.} & \langle A_i, X \rangle = b_i \\
 & X \succcurlyeq 0
\end{array}
$$

Semidefinite Programming

- Primal-dual formulation

$$\begin{array}{ccccc} \min_X & \langle C, X\rangle & & \max_y & b^T y \\ \text{s.t.} & \langle A_i, X\rangle = b_i & & \text{s.t.} & \sum_i A_i y_i \preccurlyeq C \\ & X \succcurlyeq 0 & & & \end{array}$$

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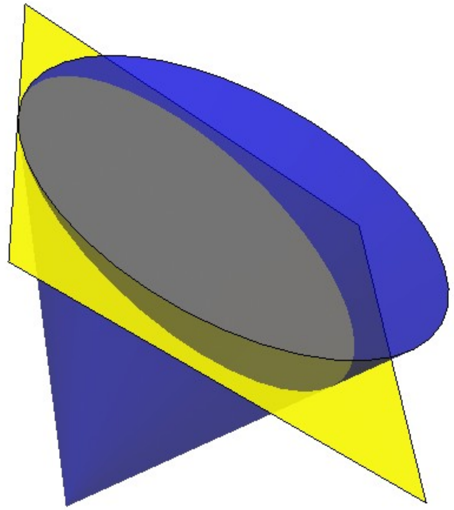

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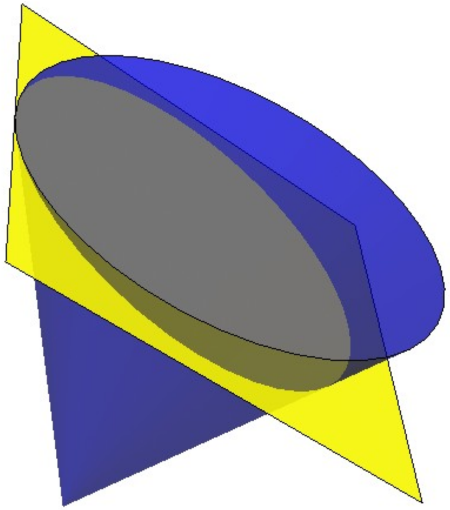

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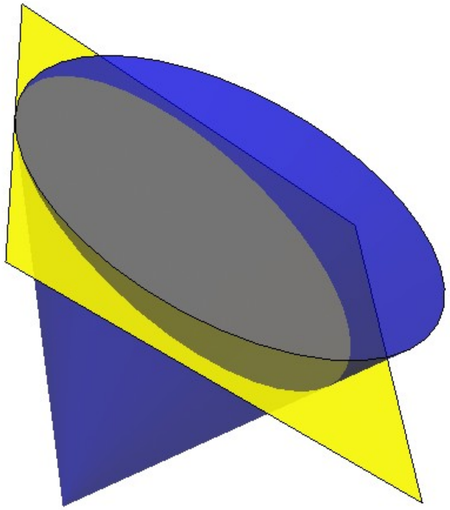

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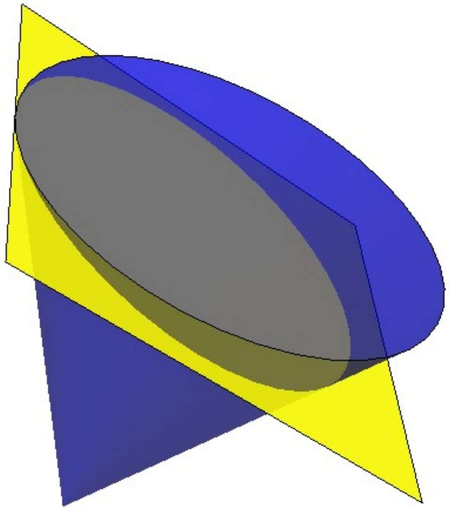

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Deciding if $p(\vec{x})$ admits a sos representation is simply an SdP in disguise

An easy one, since one can assume $\deg(q_i) \leq d$

$$p(\vec{x}) = \sum_i q_i^2(\vec{x}) \qquad (\deg(p) = 2d)$$

Semidefinite Programming

- Writing a polynomial as a sos

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$$p(\vec{x}) = \begin{pmatrix} x^2 \\ y^2 \\ xy \end{pmatrix}^T \begin{pmatrix} q_0 & q_1 & q_2 \\ & q_3 & q_4 \\ & & q_5 \end{pmatrix} \begin{pmatrix} x^2 \\ y^2 \\ xy \end{pmatrix}$$

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$$Q = \begin{pmatrix} 2 & -3 & 1 \\ -3 & 5 & 0 \\ 1 & 0 & 5 \end{pmatrix} = L^T L$$

Semidefinite Programming

- Writing a polynomial as a sos

$$p(x,y) = 2x^4 + 5y^4 - x^2y^2 + 2x^3y$$

$$p(\vec{x}) = \begin{pmatrix} x^2 \\ y^2 \\ xy \end{pmatrix}^T \begin{pmatrix} q_0 & q_1 & q_2 \\ & q_3 & q_4 \\ & & q_5 \end{pmatrix} \begin{pmatrix} x^2 \\ y^2 \\ xy \end{pmatrix}$$

$$= q_0x^4 + q_3y^4 + (2q_1 + q_5)x^2y^2 + \dots$$

Finding $Q \succcurlyeq 0$ is a semidefinite program!!

$$Q = \begin{pmatrix} 2 & -3 & 1 \\ -3 & 5 & 0 \\ 1 & 0 & 5 \end{pmatrix} = L^T L \quad L = \frac{1}{\sqrt{2}} \begin{pmatrix} 2 & -3 & 1 \\ 0 & 1 & 3 \end{pmatrix}$$

Semidefinite Programming

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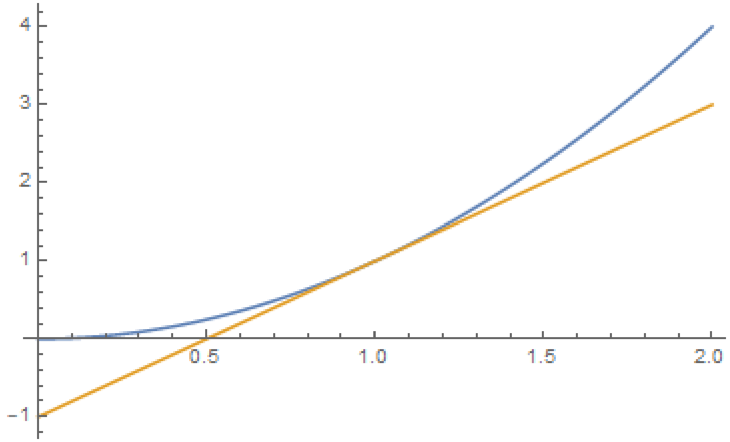

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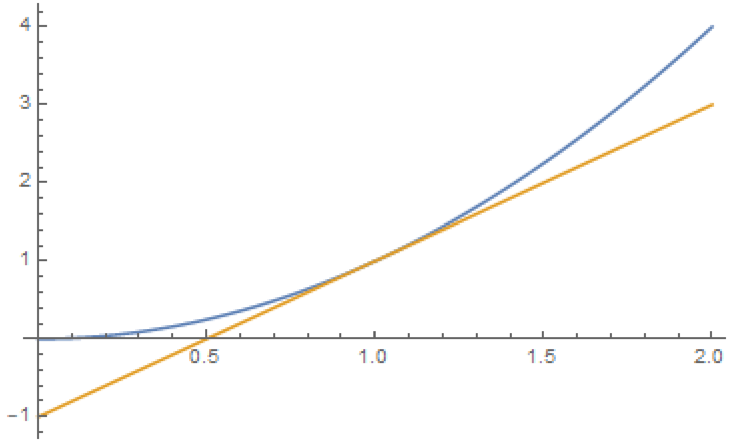

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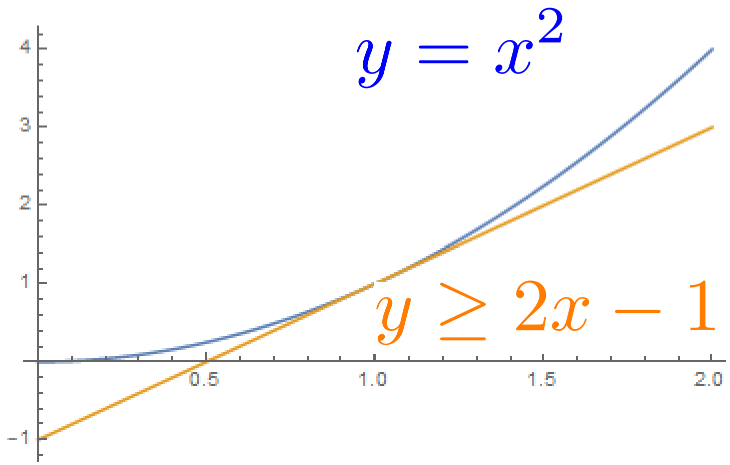

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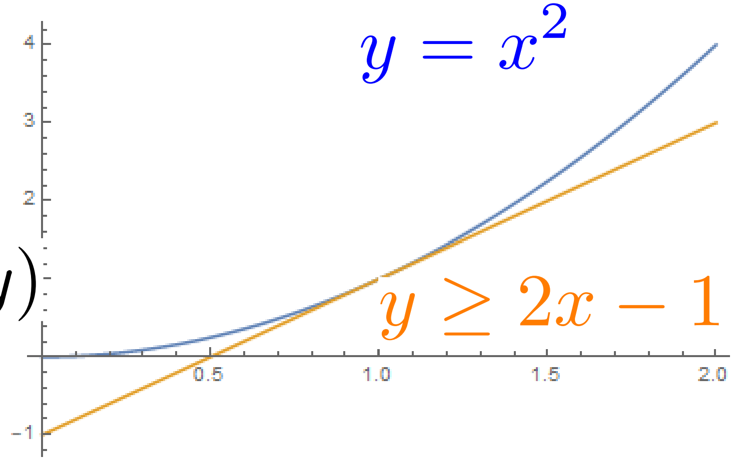

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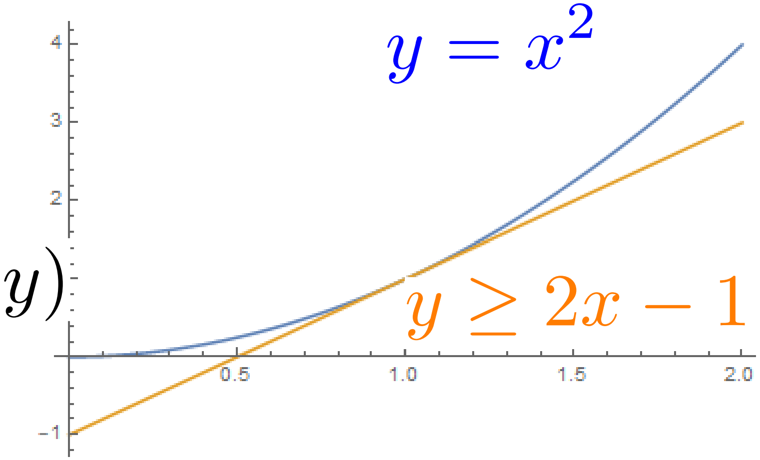

Sums-of-squares modulo ideals are powerful!

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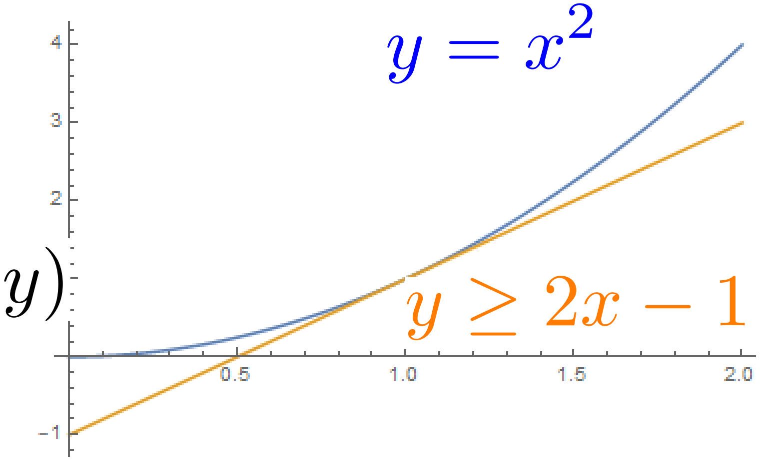

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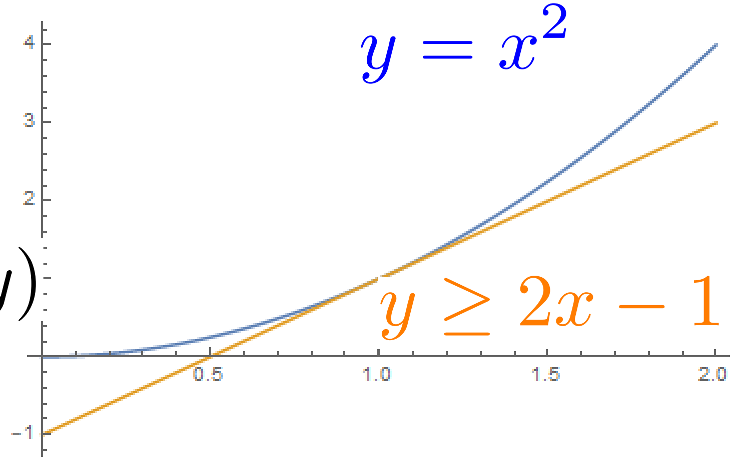

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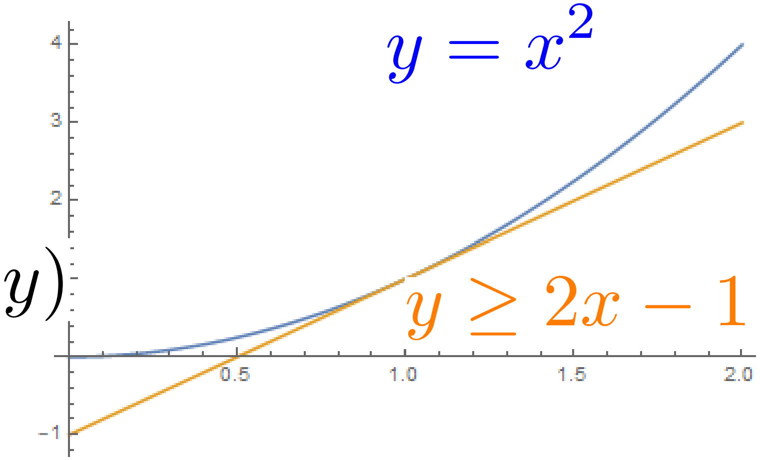

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Ideal generated by $\{f_i\}$

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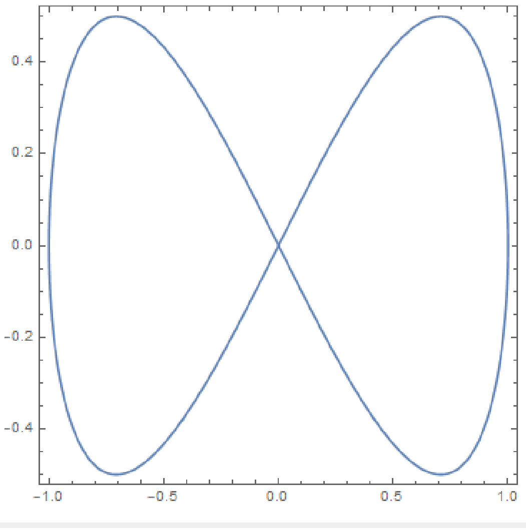

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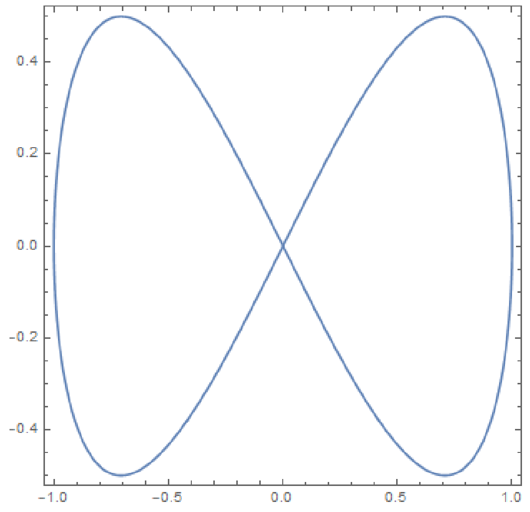

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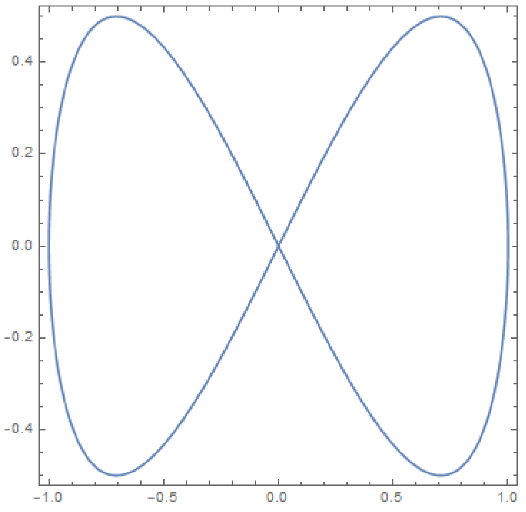

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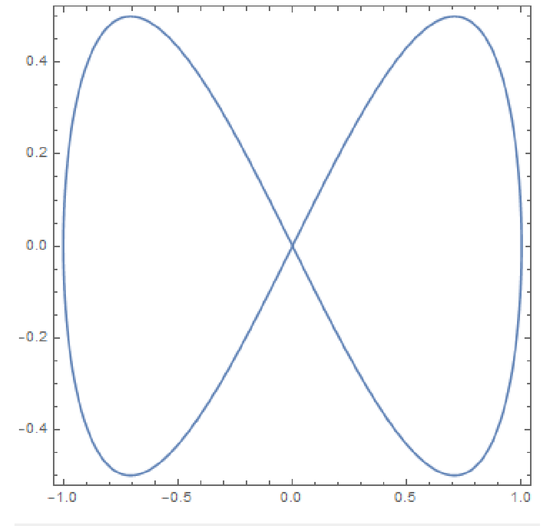

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Moment matrix, 2nd order

[Gouveia, Thomas, Convex Hulls of semialgebraic sets, 2012]
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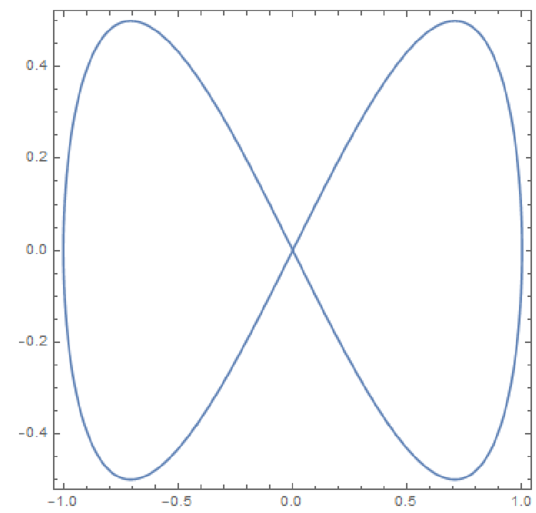

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w_i^j linearizes $x^i y^j$

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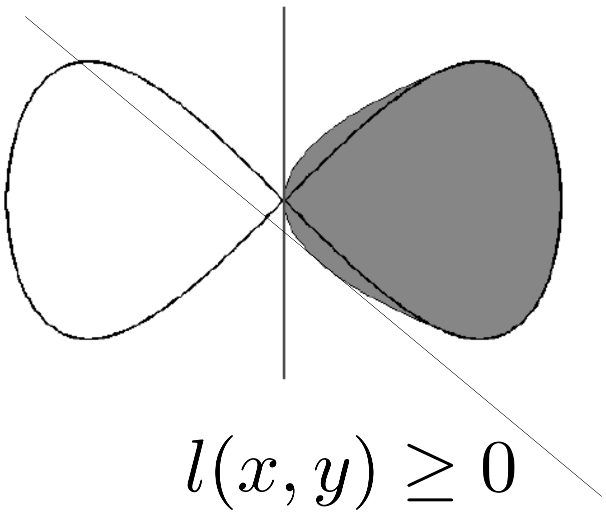

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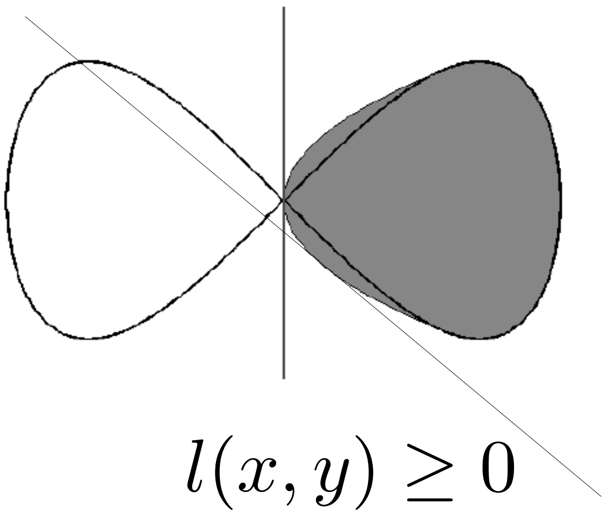

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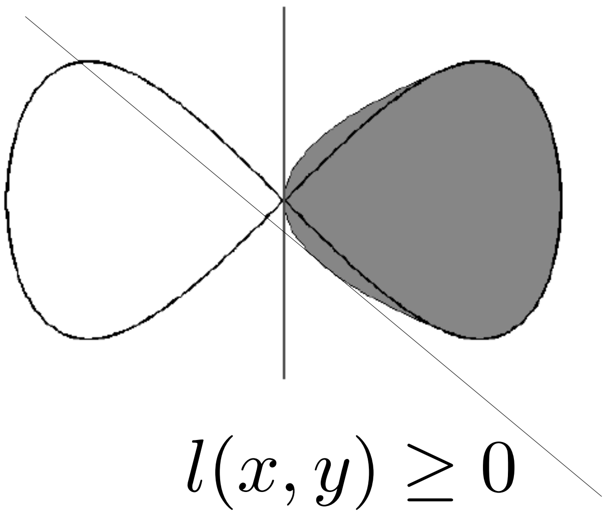
Shifted moment matrix by x, 1[st] order



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sums-of squares

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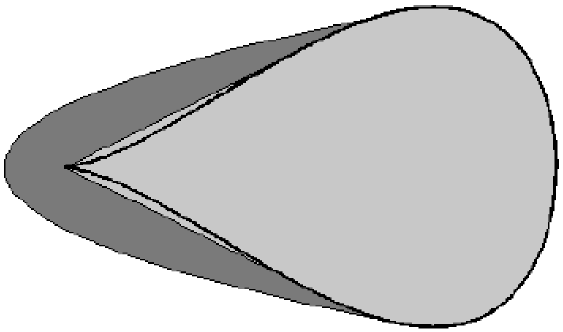

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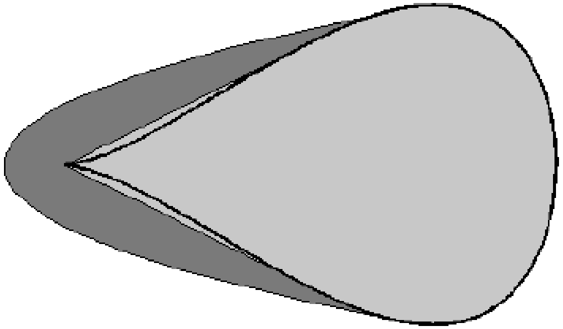

Increasing the degree of the sos σ_i gives more representability power

Semidefinite Programming

- A more physical example

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Classical Hamiltonian

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⟨aQa^L⟩

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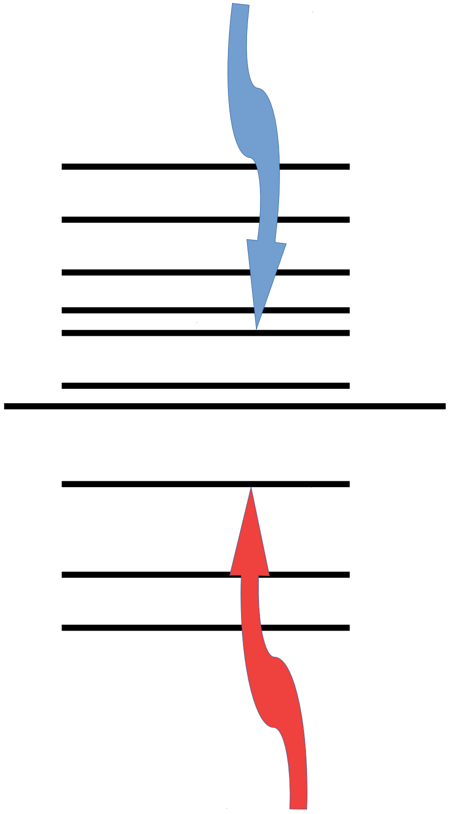

Outline

- A quantum information scientist meets non-negative polynomials
- Sum-of-squares representations
- Characterizing many-body correlations
- An application of sum-of-quares in quantum information: self-testing

A bit of Quantum Info

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    - Too many degrees of freedom
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  - Let's try to characterize **only the statistics** arising from quantum physics

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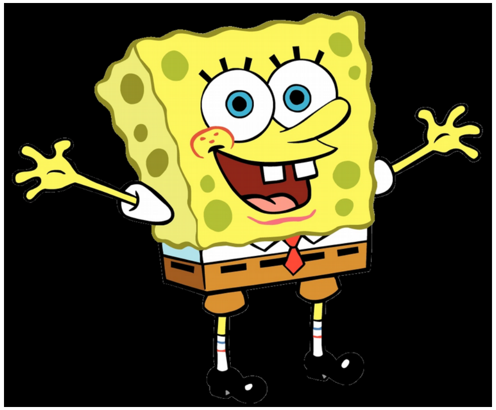

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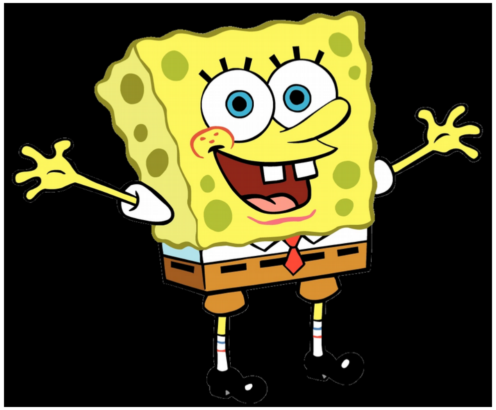

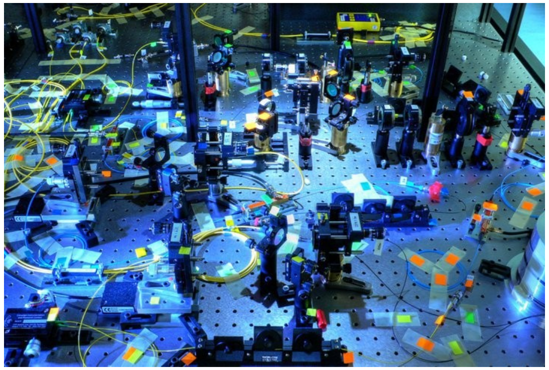

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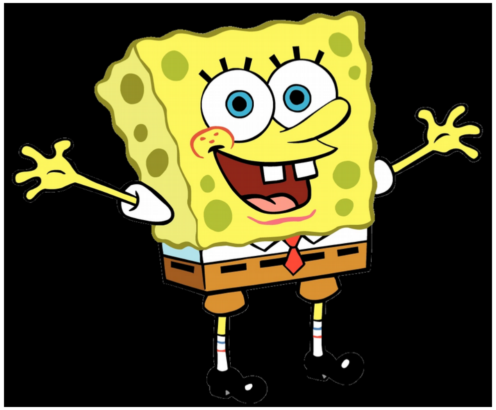

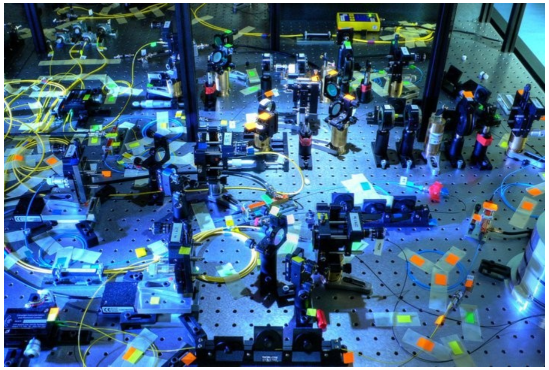

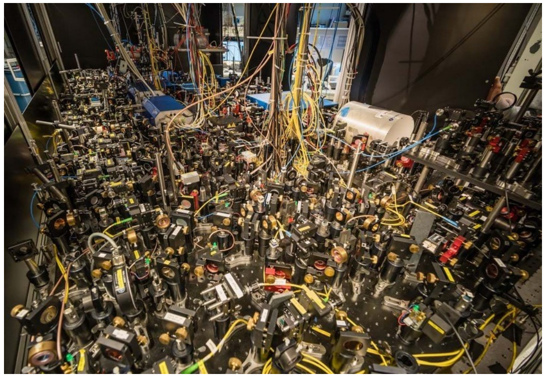

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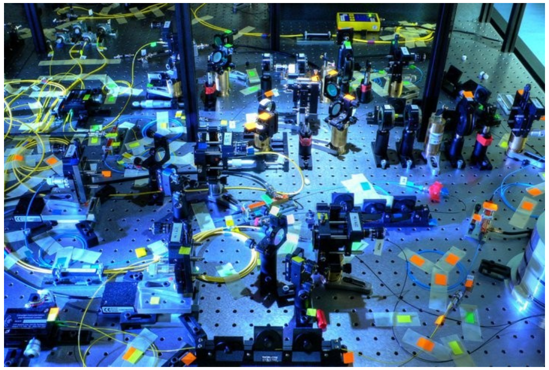

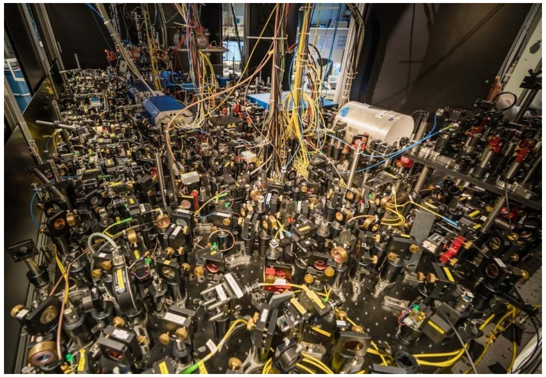

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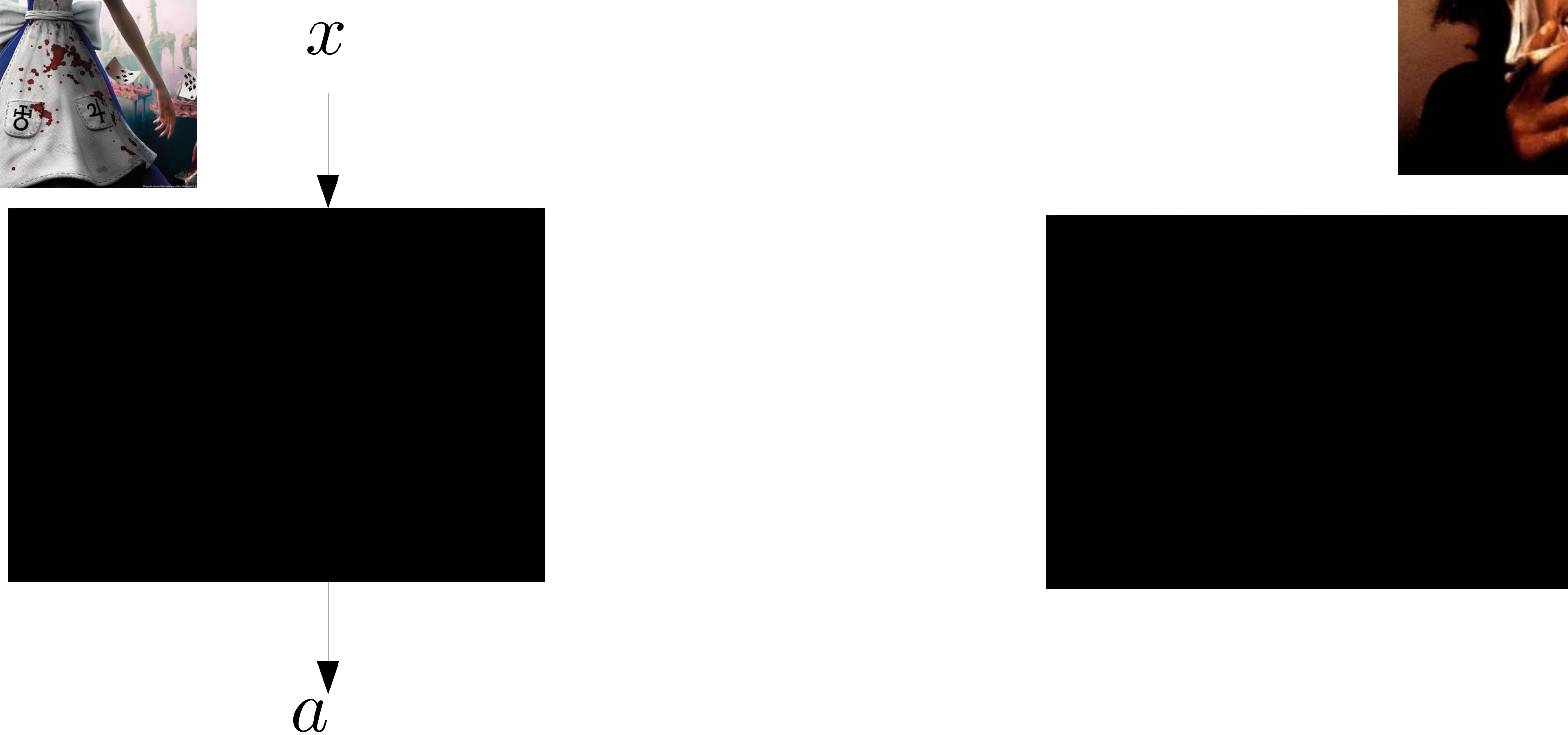

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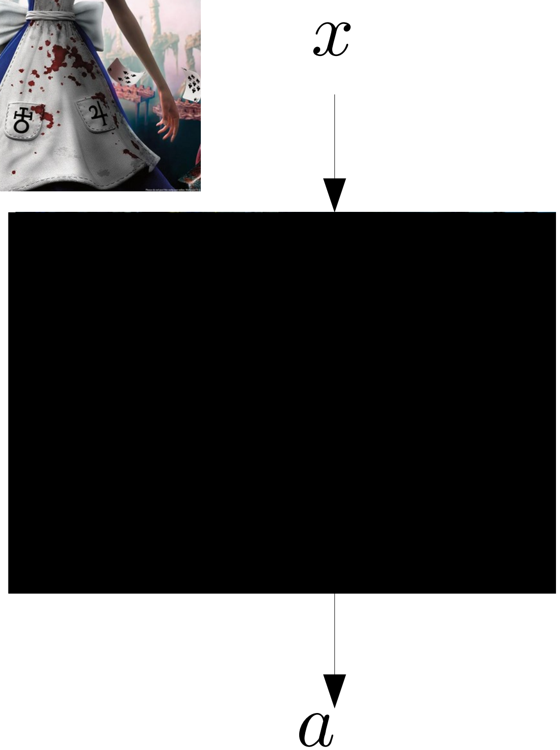

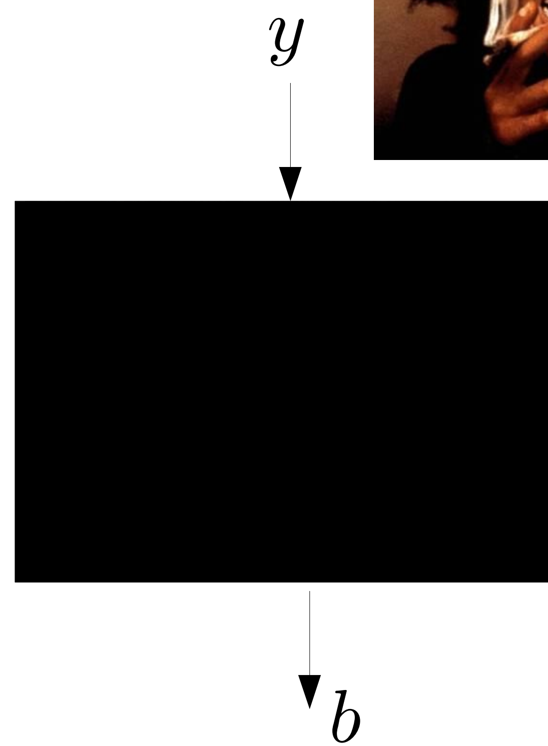

A bit of Quantum Info

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x $\quad$ y

a $\quad$ $p(ab|xy)$ $\quad$ b

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⟨aQa^L⟩

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$$p(ab|xy) = \int_{\Lambda} \mathrm{d}\lambda p(\lambda) p(a|x\lambda) p(b|y\lambda)$$

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$$\sum_a p(ab|xy) = \sum_a p(ab|x'y) \equiv p(b|y), \ldots$$

A bit of Quantum Info

- A perhaps surprising fact

If Alice and Bob can simulate statistics locally, perhaps assisted by shared randomness: LHVM

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$$LHVM \subsetneq Q \subsetneq NS$$

A bit of Quantum Info

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Popescu-Rohrlich 94, Brassard et al 06, Linden et al 07,
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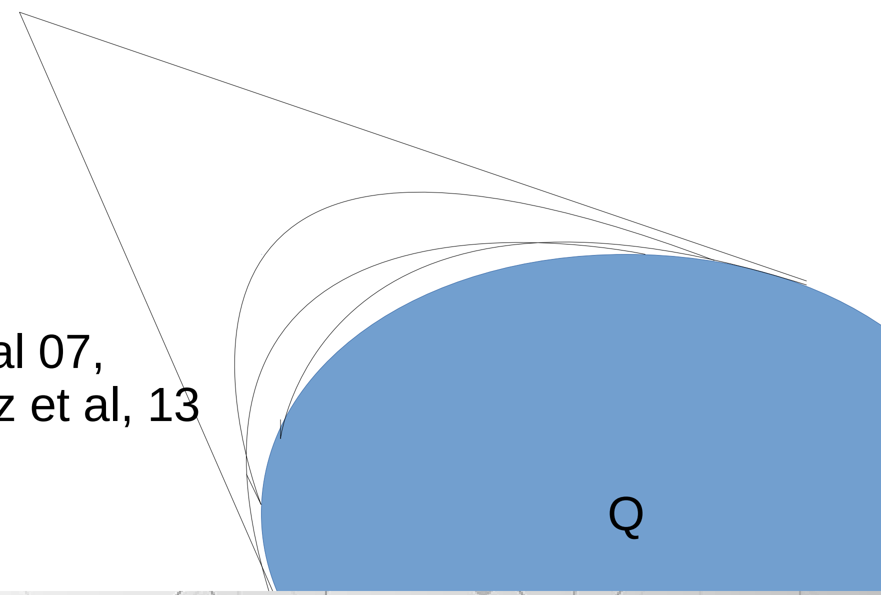

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$$Q_1 \supseteq Q_{1+AB} \supseteq Q_2 \supseteq \ldots \supseteq Q$$

Navascués-Pironio-Acín, PRL, 2007

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Sos of degree 2 "mod I"

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"Almost quantum" correlations

Navascués et al, NatComms 2014

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Includes most of the operational principles

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A bit of Quantum Info

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$$2 - CHSH = 2 - A_0(B_0 + B_1) - A_1(B_0 - B_1) \geq 0$$

A bit of Quantum Info

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- Other cases: PPT states do not violate CHSH

$$(2 - CHSH)^2 + [(2 - CHSH^\Gamma)^2]^\Gamma \propto 2 - CHSH \geq_{|PPT} 0$$

⟨aQa⟩

Nonlocality in many-body quantum systems

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- The complexity of the problem

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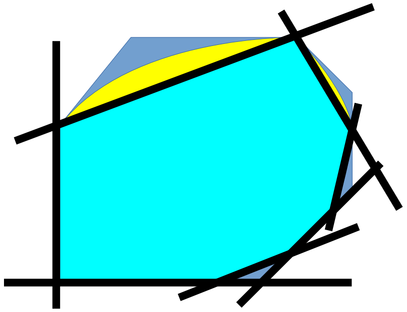
Finding all Bell inequalities ⟷ Convex Hull problem



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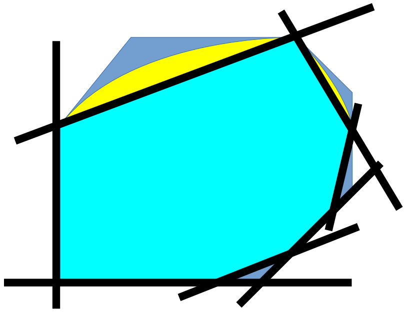
(n, m, d) scenario



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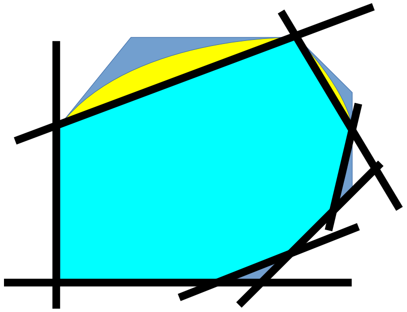

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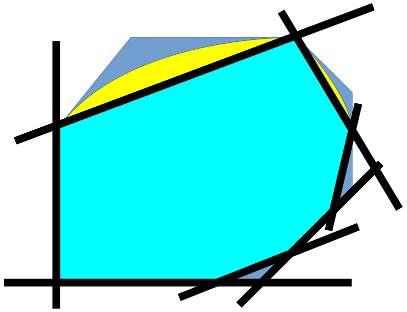

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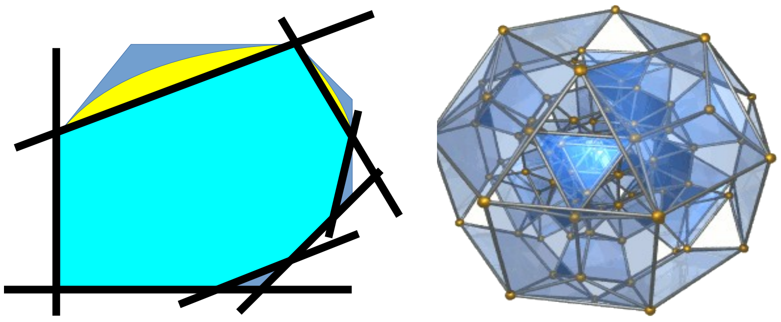

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[B. Chazelle, *An optimal convex hull algorithm in any fixed dimension, Discrete Comput. Geom.* **10** 377409 (1993)]

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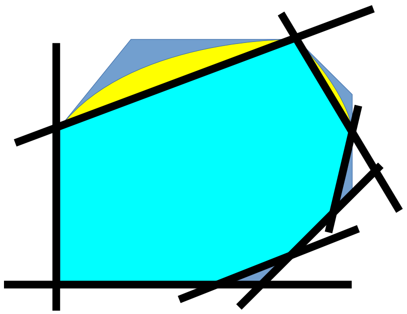

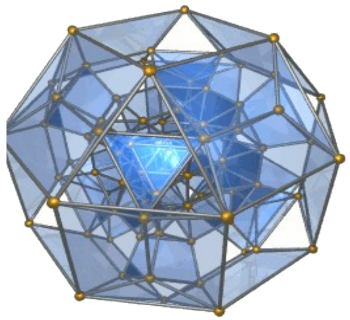

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Examples

[redacted]

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Finding all Bell inequalities $\longleftrightarrow$ Convex Hull problem

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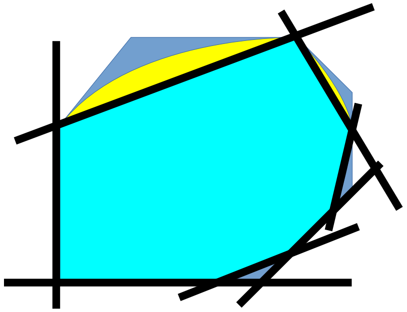

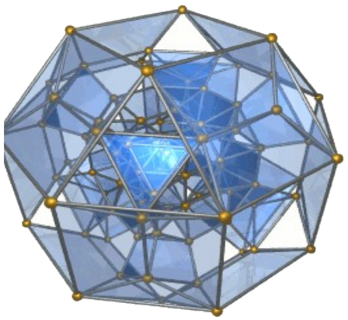

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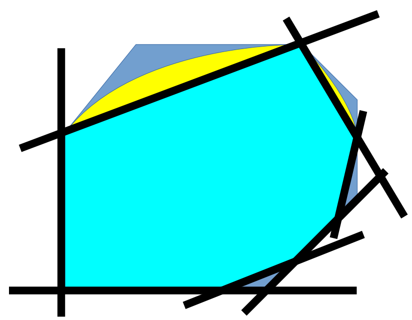

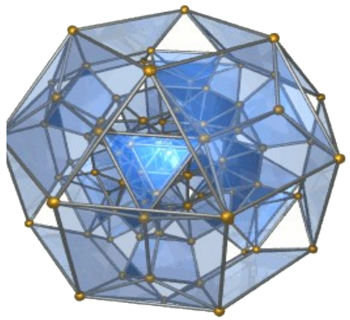

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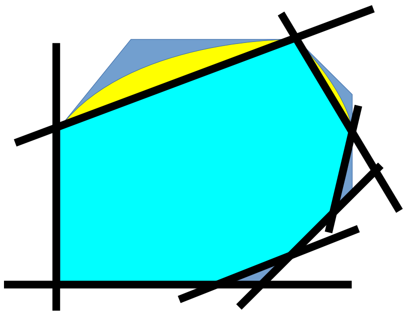

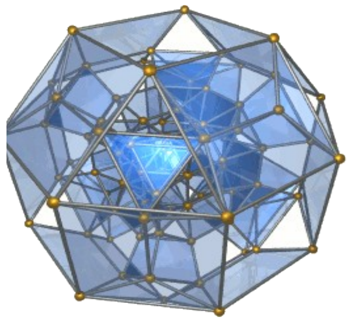

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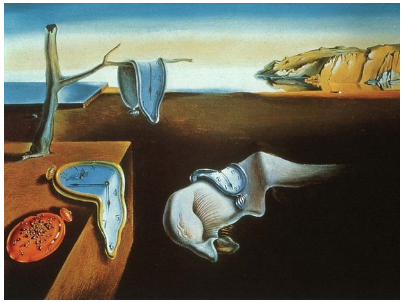
Examples

$$(2,2,2) \longrightarrow O(\mathrm{ms})$$
$$(3,2,2) \longrightarrow 5'$$
$$(4,2,2) \longrightarrow 10^{67} \ \text{years}$$
$$\vdots$$



[S. Dalí *The persistence of memory (1931)*]

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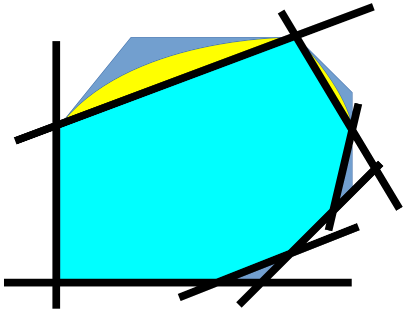

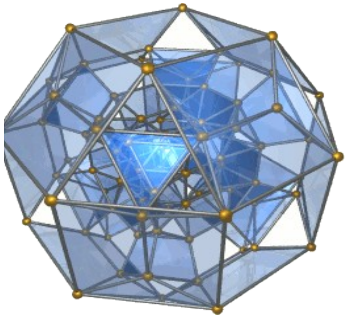

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Examples

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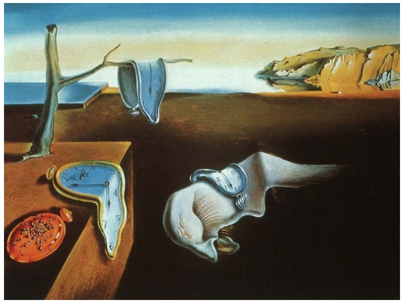
$(3, 2, 2) \longrightarrow 5'$

$(4, 2, 2) \longrightarrow 10^{67}$ years

$\vdots$

$(10^4, 2, 2) \longrightarrow 10^{10^{10^{4.67867\ldots}}}$ basically any timescale you want



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- Reducing the mathematical complexity

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Polytope				
Dimension				
Vertices				

Nonlocality in many-body quantum systems

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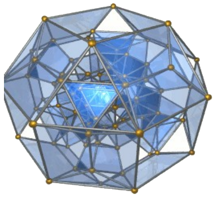
Polytope	$\mathbb{P}_n$			
Dimension	$3^n - 1$			
Vertices	2^{2n}			

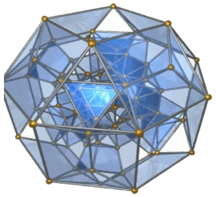


Nonlocality in many-body quantum systems

- Reducing the mathematical complexity

2-body

Polytope	$\mathbb{P}_n$ Lower order correlators			
Dimension	$3^n - 1$			
Vertices	2^{2n}			



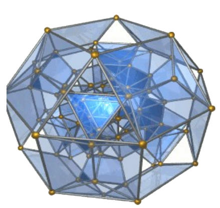
Nonlocality in many-body quantum systems

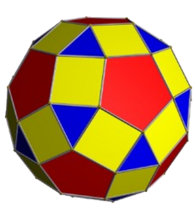
- Reducing the mathematical complexity

2-body

Lower order correlators

Polytope	$\mathbb{P}_n$	$\mathbb{P}_2$		
Dimension	$3^n - 1$	$2n^2$		
Vertices	2^{2n}	2^{2n}		





Nonlocality in many-body quantum systems

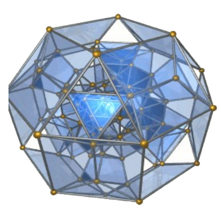
- Reducing the mathematical complexity

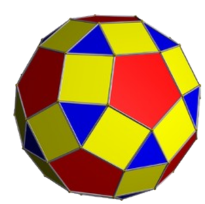
2-body

Lower order correlators

Action of a symmetry group

Polytope	$\mathbb{P}_n$	$\mathbb{P}_2$		
Dimension	$3^n - 1$	$2n^2$		
Vertices	2^{2n}	2^{2n}		





Nonlocality in many-body quantum systems

- Reducing the mathematical complexity

2-body

Cyclic group

Lower order correlators

Action of a symmetry group

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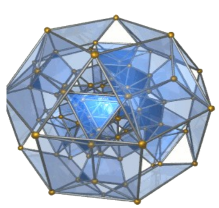
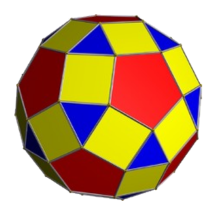
Dimension	$3^n - 1$	$2n^2$		
Vertices	2^{2n}	2^{2n}		





[JT, A. B. Sainz, T. Vértesi, M. Lewenstein, A. Acín, R. Augusiak
J. Phys. A: Math. Theor. **47** 424024 (2014)]

Nonlocality in many-body quantum systems

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2-body

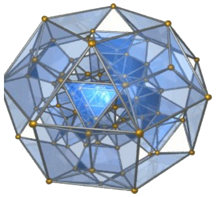
Cyclic group

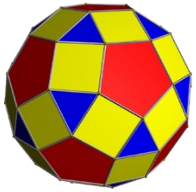
Symmetric group

Lower order correlators

Action of a symmetry group

Polytope	$\mathbb{P}_n$	$\mathbb{P}_2$		
Dimension	$3^n - 1$	$2n^2$		
Vertices	2^{2n}	2^{2n}		





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[JT, R. Augusiak, A. B. Sainz, T. Vértesi, M. Lewenstein, A. Acín *Science* **344** 1256 (2014)]

Nonlocality in many-body quantum systems

- Reducing the mathematical complexity

2-body

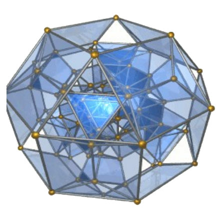
Cyclic group

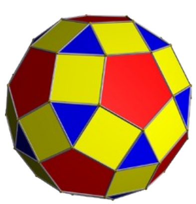
Symmetric group

Lower order correlators

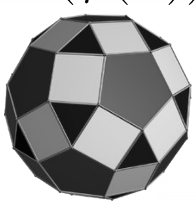
Action of a symmetry group

Polytope	$\mathbb{P}_n$	$\mathbb{P}_2$	$\mathbb{P}_2^{\mathcal{S}}$	
Dimension	$3^n - 1$	$2n^2$	5	
Vertices	2^{2n}	2^{2n}	$\binom{n+3}{3}$	





$\mathrm{CH}(\varphi(\mathbb{T}))$



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2-body

Cyclic group

Symmetric group

Deterministic Local Strategies Parametrization

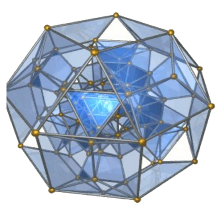
Lower order correlators

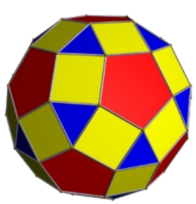
Action of a symmetry group

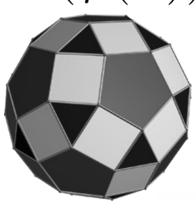
Extremality analysis

Polytope	$\mathbb{P}_n$	$\mathbb{P}_2$	$\mathbb{P}_2^S$	
Dimension	$3^n - 1$	$2n^2$	5	
Vertices	2^{2n}	2^{2n}	$\binom{n+3}{3}$	

$\mathrm{CH}(\varphi(\mathbb{T}))$

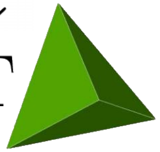






φ

Parameter space $\mathbb{T}$



[JT, A. B. Sainz, T. Vértesi, M. Lewenstein, A. Acín, R. Augusiak *J. Phys. A: Math. Theor.* **47** 424024 (2014)]

[JT, R. Augusiak, A. B. Sainz, T. Vértesi, M. Lewenstein, A. Acín *Science* **344** 1256 (2014)]

Nonlocality in many-body quantum systems

- Reducing the mathematical complexity

2-body

Cyclic group

Symmetric group

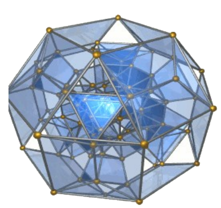
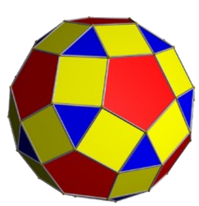
Deterministic Local Strategies Parametrization

Lower order correlators

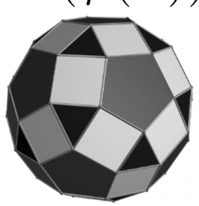
Action of a symmetry group

Extremality analysis

Polytope	$\mathbb{P}_n$	$\mathbb{P}_2$	$\mathbb{P}_2^S$	$\mathbb{P}_2^S$
Dimension	$3^n - 1$	$2n^2$	5	5
Vertices	2^{2n}	2^{2n}	$\binom{n+3}{3}$	$2(n^2+1)$

$\mathrm{CH}(\varphi(\mathbb{T}))$



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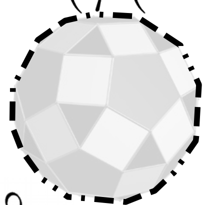

φ φ

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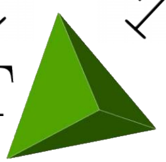

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Some applications

Some applications

PRL **119,** 230402 (2017) PHYSICAL REVIEW LETTERS week ending 8 DECEMBER 2017

Bounding the Set of Classical Correlations of a Many-Body System

Matteo Fadel[1,*] and Jordi Tura[2,3,†]

We present a method to certify the presence of Bell correlations in experimentally observed statistics, and to obtain new Bell inequalities. Our approach is based on relaxing the conditions defining the set of correlations obeying a local hidden variable model, yielding a convergent hierarchy of semidefinite programs (SDP's). Because the size of these SDP's is independent of the number of parties involved, this technique allows us to characterize correlations in many-body systems. As an example, we illustrate our method with the experimental data presented in Science **352**, 441 (2016).

DOI: 10.1103/PhysRevLett.119.230402

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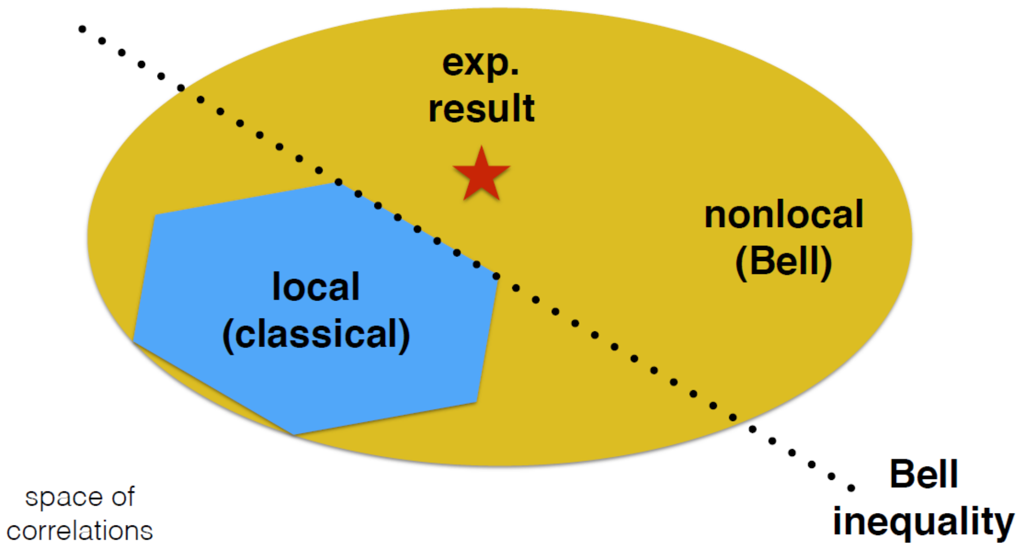

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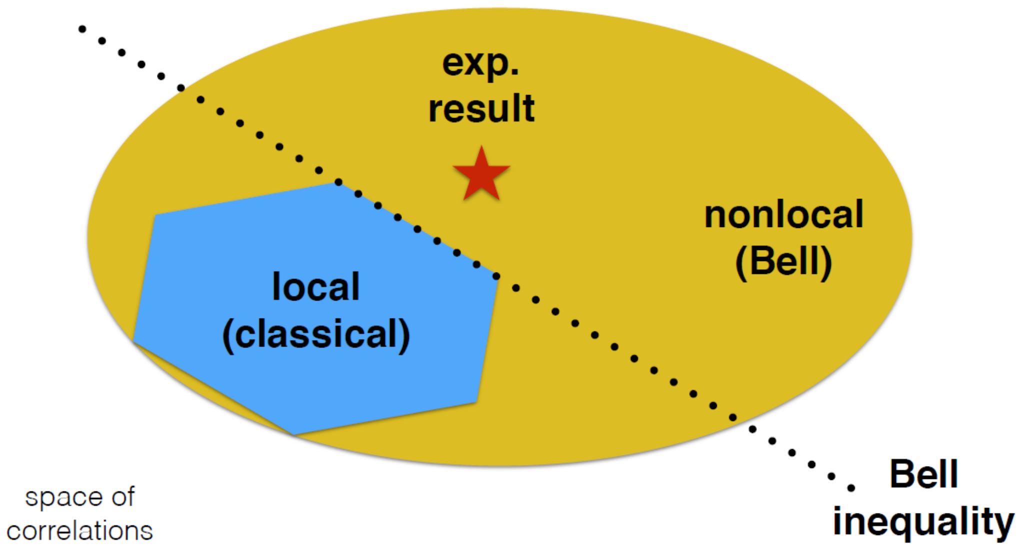

- *Certify Bell correlations from experiments*

Some applications

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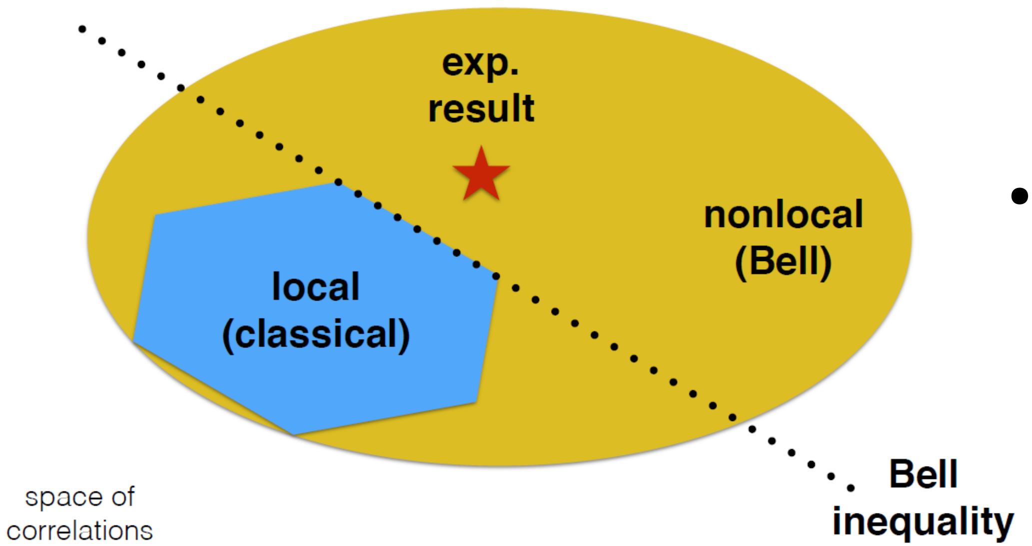

- *Certify Bell correlations from experiments*
- *Find new Bell Inequalities*

Some applications

Some applications

We consider Bell inequalities of the form

$$\boxed{\sum_{k} \sum_{j_1 \leq \ldots \leq j_k} \alpha_{j_1 \ldots j_k} \mathcal{S}_{j_1 \ldots j_k} + \beta_C \geq 0}$$

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with $$\mathcal{S}_{j_1 \ldots j_k} = \sum_{\substack{i_1, \ldots, i_k = 1 \\ \text{all } i\text{'s different}}}^{N} \langle \mathcal{M}_{j_1}^{(i_1)} \ldots \mathcal{M}_{j_k}^{(i_k)} \rangle$$

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Does NOT depend on

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Dimension depends on
- Order of the correlators
- #Measurements
- #outcomes

Does NOT depend on
- #Parties

Previous results

Previous results

Science **344**, 1256 (2014)

Detecting nonlocality in many-body quantum states

J. Tura,[1] R. Augusiak,[1*] A. B. Sainz,[1] T. Vértesi,[2] M. Lewenstein,[1,3] A. Acín[1,3]

Previous results

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Example $$-2\mathcal{S}_0 + \frac{1}{2}\mathcal{S}_{00} - \mathcal{S}_{01} + \frac{1}{2}\mathcal{S}_{11} + 2N \geq 0$$

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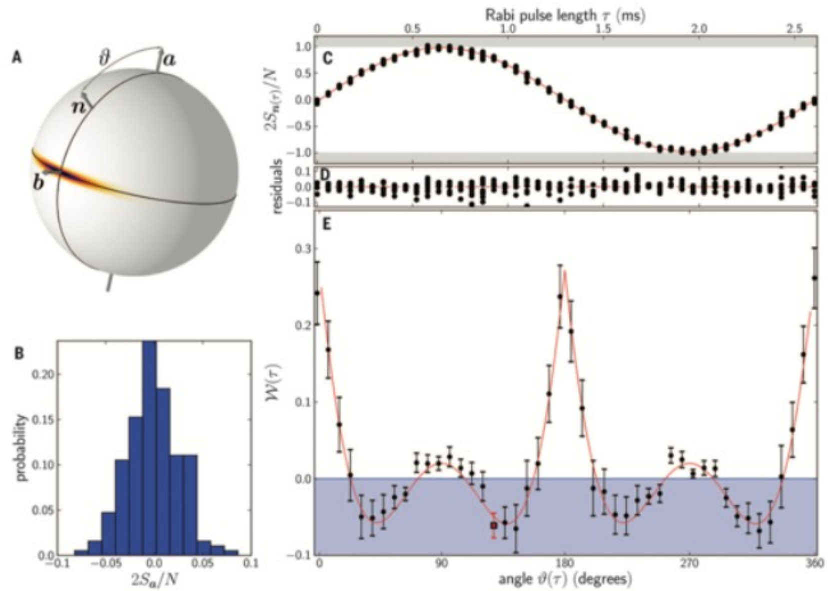

Science **352**, 441 (2016)

Bell correlations in a Bose-Einstein condensate

Roman Schmied,[1*] Jean-Daniel Bancal,[2,4*] Baptiste Allard,[1*] Matteo Fadel,[1]
Valerio Scarani,[2,3] Philipp Treutlein,[1†] Nicolas Sangouard[4†]

$$\hat{W} = -\left|\frac{\hat{S}_n}{N/2}\right| + (\boldsymbol{a}\cdot\boldsymbol{n})^2\frac{\hat{S}_a^2}{N/4} + 1 - (\boldsymbol{a}\cdot\boldsymbol{n})^2$$

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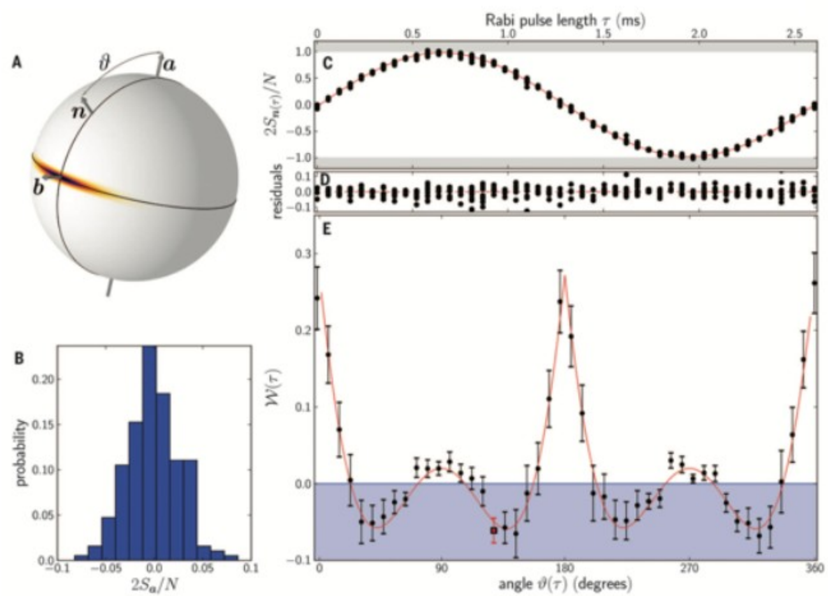

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PRL **118**, 140401 (2017) PHYSICAL REVIEW LETTERS week ending 7 APRIL 2017

Bell Correlations in Spin-Squeezed States of 500 000 Atoms

Nils J. Engelsen, Rajiv Krishnakumar, Onur Hosten, and Mark A. Kasevich*

The Local polytope

- Solving for a few values of N…

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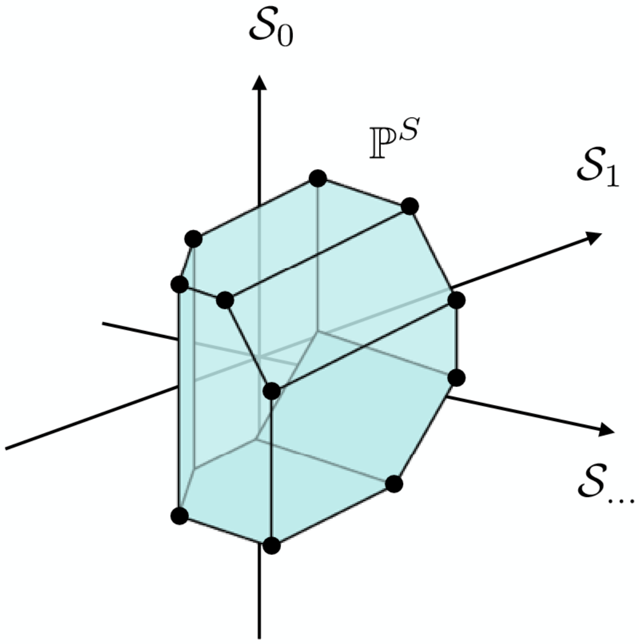

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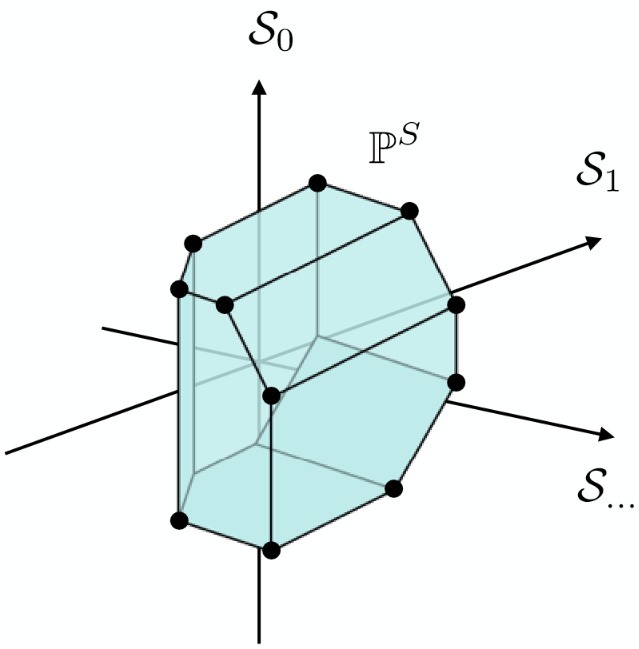


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12734 -1161 306 56 -17 1
9858 -981 270 53 -16 2
31754 -3128 867 167 -51 6
11138 -1083 308 57 -18 2
29470 -3021 970 171 -60 10
73560 -8307 3465 544 -245 70
125548 -14079 5791 912 -405 112
7830 -941 486 70 -39 15
53590 -6422 3297 475 -263 100
13808 -1473 213 84 -13 2
8712 -916 272 53 -17 3
7294 -737 154 40 -9 1
29820 -3667 3011 456 -409 280
68590 -1710 1900 1359 -1800 1900
34166 -4313 2892 399 -290 154
16086 430 -2123 3 -35 164
76470 1886 -10051 15 -151 760
7980 627 -1140 46 -110 190
53706 1418 -7063 9 -115 544
5674 138 -743 1 -11 56
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11490 437 -1620 7 -40 140
44970 1701 -6320 26 -155 545
9874 381 -1392 6 -35 121
661284 -84797 65197 9576 -7991 5096
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68050 1691 -8976 15 -136 680
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144796 2109 -17901 8 -153 1224
402610 7581 -50540 9 -570 3610
355188 5127 -43815 16 -371 2992
297910 5655 -37466 9 -426 2680
26410 152 -3040 -1 -10 190
8262 -884 253 52 -16 3
118598 5643 -7524 1831 -2970 3762
42404 -4335 867 238 -51 6
19860 -2413 1373 190 -117 50
37380 -4725 2901 406 -269 130
320450 -38322 18807 2785 -1473 540
434700 -49913 23293 3420 -1727 560
4012 -399 383 76 -81 68
117750 6426 -17640 179 -700 1820
80674 969 -9690 -3 -68 646
32490 0 0 599 -850 950
237630 -30519 23400 3437 -2860 1820
32250 2608 -4320 269 -560 880
120292 9627 -15735 1055 -2154 3333
109630 8151 -12540 1106 -2145 3135
88930 6461 -13650 327 -910 1820
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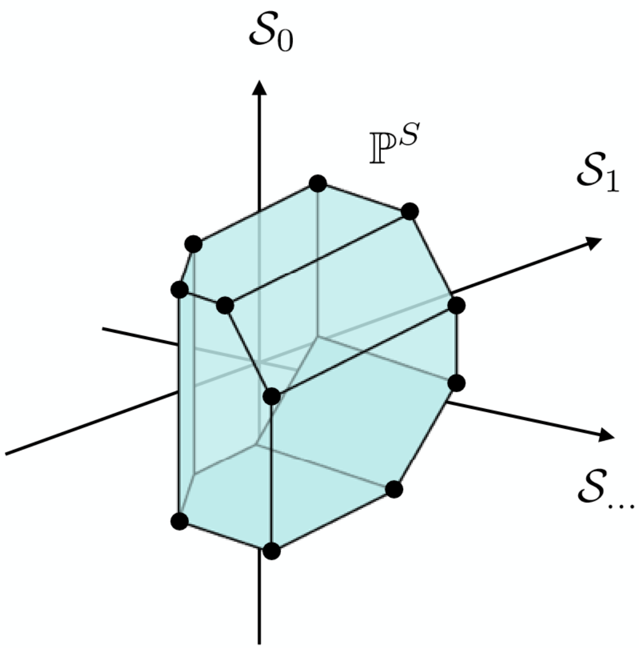
```

The Local polytope

- Solving for a few values of N…



25314	-2531	268	134	-15	1
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25994	-2584	287	136	-16	1
13264	-1197	325	57	-18	1
12734	-1161	306	56	-17	1
9858	-981	270	53	-16	2
31754	-3128	867	167	-51	6
11138	-1083	308	57	-18	2
29470	-3021	970	171	-60	10
73560	-8307	3465	544	-245	70
125548	-14079	5791	912	-405	112
7830	-941	486	70	-39	15
53590	-6422	3297	475	-263	100
13808	-1473	213	84	-13	2
8712	-916	272	53	-17	3
7294	-737	154	40	-9	1
29820	-3667	3011	456	-409	280
68590	-1710	1900	1359	-1800	1900
34166	-4313	2892	399	-290	154
16086	430	-2123	3	-35	164
76470	1886	-10051	15	-151	760
7980	627	-1140	46	-110	190
53706	1418	-7063	9	-115	544
5674	138	-743	1	-11	56
208810	13167	-18810	2597	-4620	6270
11490	437	-1620	7	-40	140
44970	1701	-6320	26	-155	545
9874	381	-1392	6	-35	121
661284	-84797	65197	9576	-7991	5096
456380	31122	-69160	1329	-4095	8645
93970	2243	-12258	15	-178	920
178980	4693	-23465	27	-380	1805
11286	912	-1520	91	-192	304
129948	3541	-17219	27	-290	1337
68050	1691	-8976	15	-136	680
4320	121	-575	1	-10	45
36138	1862	-5320	45	-196	532
363880	6861	-45701	9	-516	3265
322260	4679	-39799	16	-339	2720
109660	2073	-13781	3	-156	985
144796	2109	-17901	8	-153	1224
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42404	-4335	867	238	-51	6
19860	-2413	1373	190	-117	50
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434700	-49913	23293	3420	-1727	560
4012	-399	383	76	-81	68
117750	6426	-17640	179	-700	1820
80674	969	-9690	-3	-68	646
32490	0	0	599	-850	950
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120292	9627	-15735	1055	-2154	3333
109630	8151	-12540	1106	-2145	3135
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232524	13293	-34965	401	-1498	3731

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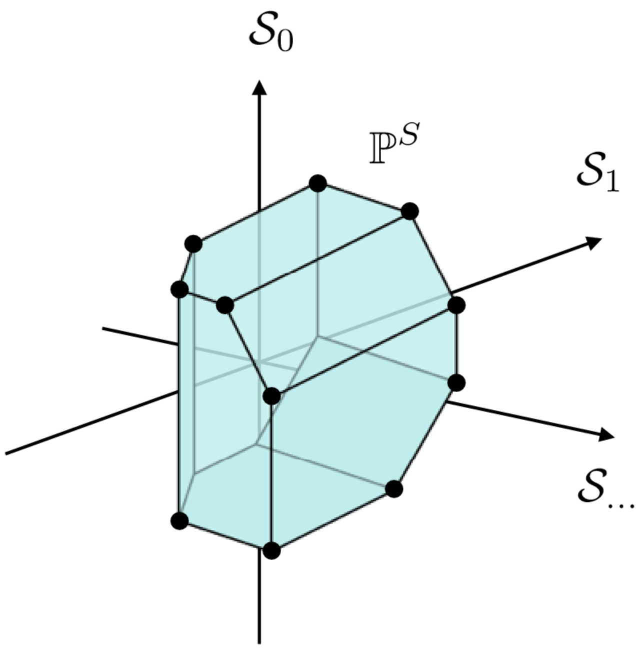

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26334	-2601	306	136	-17	1
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125548	-14079	5791	912	-405	112
7830	-941	486	70	-39	15
53590	-6422	3297	475	[illegible]	[illegible]
13808	-1473	213	84	-13	[illegible]
8712	-916	272	53	-17	[illegible]
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29820	-3667	3011	456	[illegible]	[illegible]
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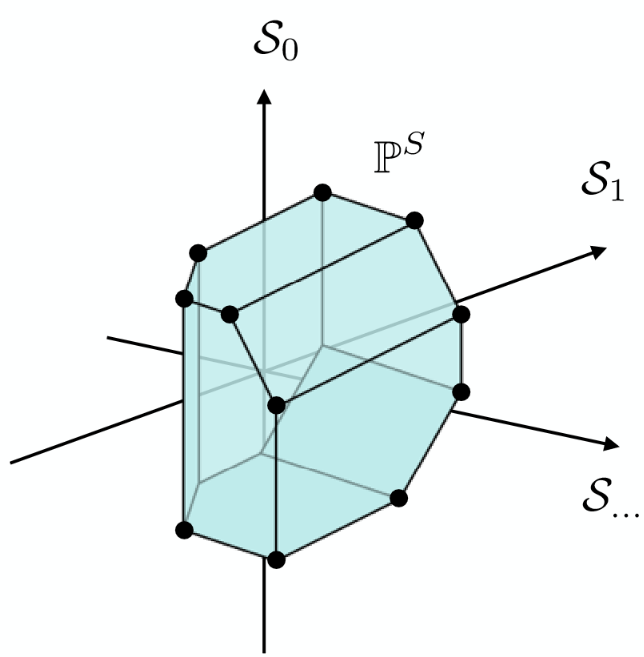
8262	-884	253	52	-16	3
118598	5643	-7524	1831	-2970	3762
42404	-4335	867	238	-51	6
19860	-2413	1373	190	-117	50
37380	-4725	2901	406	-269	130
320450	-38322	18807	2785	-1473	540
434700	-49913	23293	3420	-1727	560
4012	-399	383	76	-81	68
117750	6426	-17640	179	-700	1820
80674	969	-9690	-3	-68	646
32490	0	0	599	-850	950
237630	-30519	23400	3437	-2860	1820
32250	2608	-4320	269	-560	880
120292	9627	-15735	1055	-2154	3333
109630	8151	-12540	1106	-2145	3135
88930	6461	-13650	327	-910	1820
518130	31122	-77805	1004	-3640	8645
232524	13293	-34965	401	-1498	3731

The Local polytope

- Solving for a few values of N…



25314	-2531	268	134	-15	1
26334	-2601	306	136	-17	1
25994	-2584	287	136	-16	1
13264	-1197	325	57	-18	1
12734	-1161	306	56	-17	1
9858	-981	270	53	-16	2
31754	-3128	867	167	-51	6
11138	-1083	308	57	-18	2
29470	-3021	970	171	-60	10
73560	-8307	3465	544	-245	70
125548	-14079	5791	912	-405	112
7830	-941	486	70	-39	15
53590	-6422	3297	475	[illegible]	[illegible]
13808	-1473	213	84	-13	[illegible]
8712	-916	272	53	-17	[illegible]
7294	-737	154	40	-9	[illegible]
29820	-3667	3011	456	[illegible]	[illegible]
68590	-1710	1900	1359	-1800	1900
34166	-4313	2892	399	-290	154
16086	430	-2123	3	-35	164
76470	1886	-10051	15	-151	760
7980	627	-1140	46	-110	190
53706	1418	-7063	9	-115	544
5674	138	-743	1	-11	56
208810	13167	-18810	2597	-4620	6270
11490	437	-1620	7	-40	140
44970	1701	-6320	26	-155	545
9874	381	-1392	6	-35	121
661284	-84797	65197	9576	-7991	5096
456380	31122	-69160	1329	-4095	8645
93970	2243	-12258	15	-178	920
178980	4693	-23465	27	-380	1805
11286	912	-1520	91	-192	304
129948	3541	-17219	27	-290	1337
68050	1691	-8976	15	-136	680
4320	121	-575	1	-10	45
36138	1862	-5320	45	-196	532
363880	6861	-45701	9	-516	3265
322260	4679	-39799	16	-339	2720
109660	2073	-13781	3	-156	985
144796	2109	-17901	8	-153	1224
402610	7581	-50540	9	-570	3610
355188	5127	-43815	16	-371	2992
297910	5655	-37466	9	-426	2680
26410	152	-3040	-1	-10	190
8262	-884	253	52	-16	3
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32490	0	0	599	-850	950
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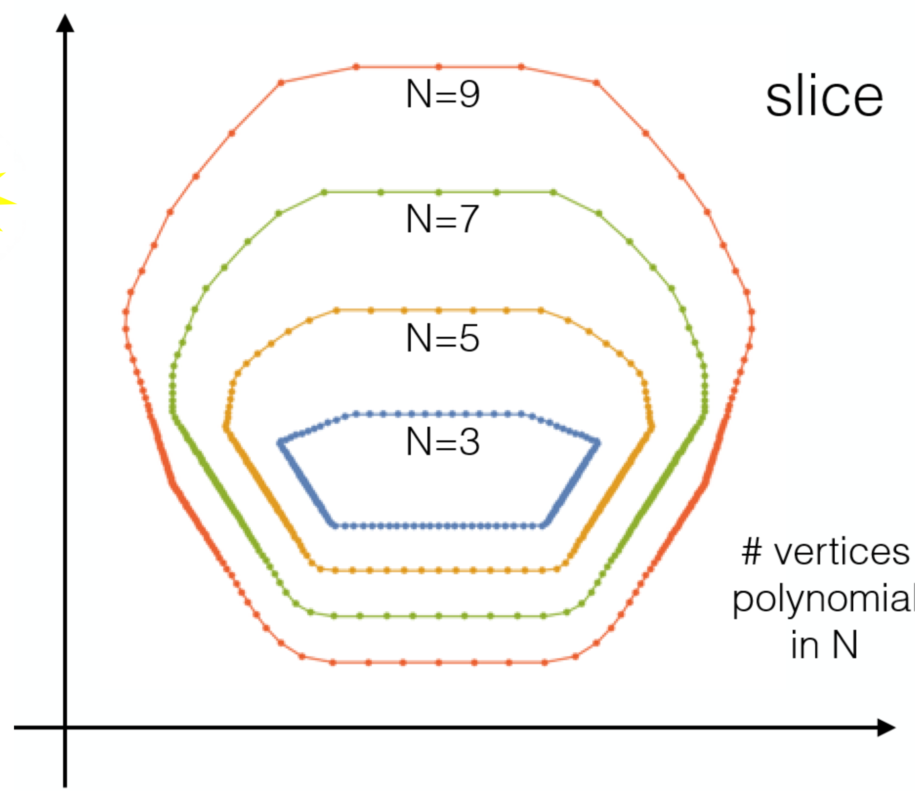
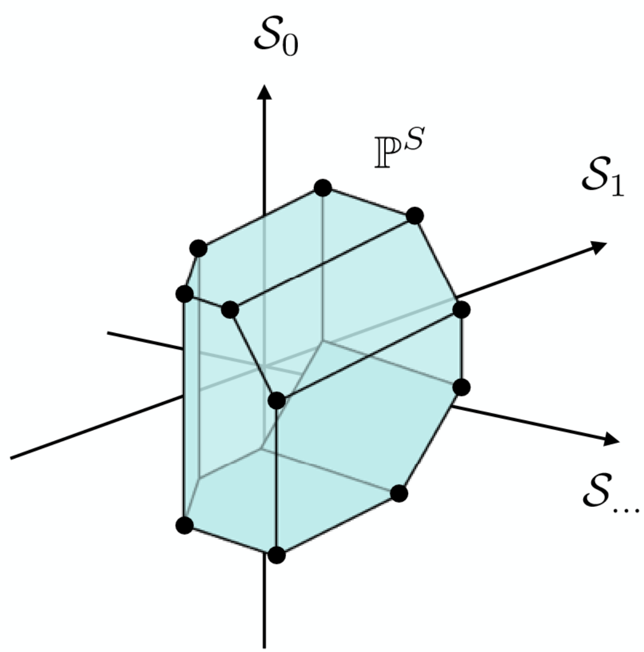


The Local polytope

- Solving for a few values of N…



What's the physical significance of each inequality?

25314	-2531	268	134	-15	1
26334	-2601	306	136	-17	1
25994	-2584	287	136	-16	1
13264	-1197	325	57	-18	1
12734	-1161	306	56	-17	1
9858	-981	270	53	-16	2
31754	-3128	867	167	-51	6
11138	-1083	308	57	-18	2
29470	-3021	970	171	-60	10
73560	-8307	3465	544	-245	70
125548	-14079	5791	912	-405	112
7830	-941	486	70	-39	15
53590	-6422	3297	475	[illegible]	[illegible]
13808	-1473	213	84	-13	[illegible]
8712	-916	272	53	-17	[illegible]
7294	-737	154	40	-9	1
29820	-3667	3011	456	-40[illegible]	[illegible]
68590	-1710	1900	1359	-1800	1900
34166	-4313	2892	399	-290	154
16086	430	-2123	3	-35	164
76470	1886	-10051	15	-151	760
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53706	1418	-7063	9	-115	544
5674	138	-743	1	-11	56
208810	13167	-18810	2597	-4620	6270
11490	437	-1620	7	-40	140
44970	1701	-6320	26	-155	545
9874	381	-1392	6	-35	121
661284	-84797	65197	9576	-7991	5096
456380	31122	-69160	1329	-4095	8645
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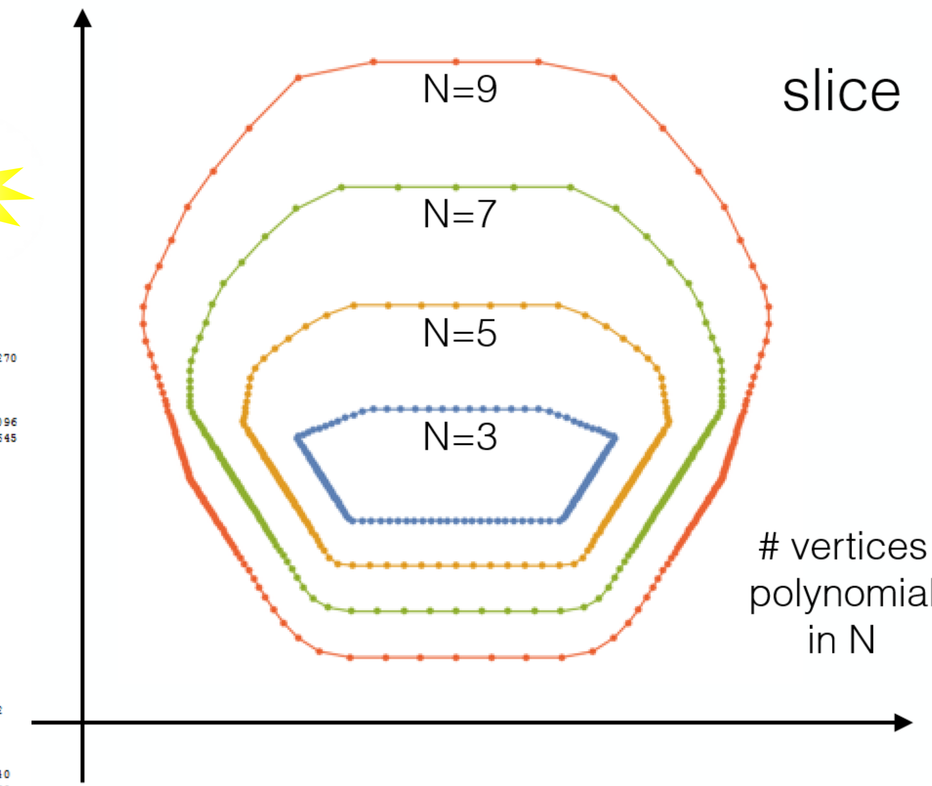
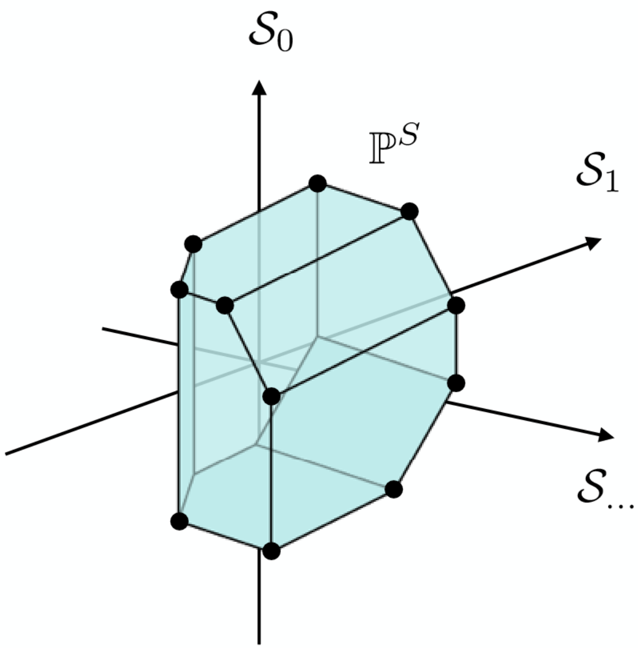


The Local polytope

- Solving for a few values of N...



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26334	-2601	306	136	-17	1
25994	-2584	287	136	-16	1
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53590	-6422	3297	475	[illegible]	[illegible]
13808	-1473	213	84	-13	[illegible]
8712	-916	272	53	-17	[illegible]
7294	-737	154	40	-9	1
29820	-3667	3011	456	-40[illegible]	[illegible]
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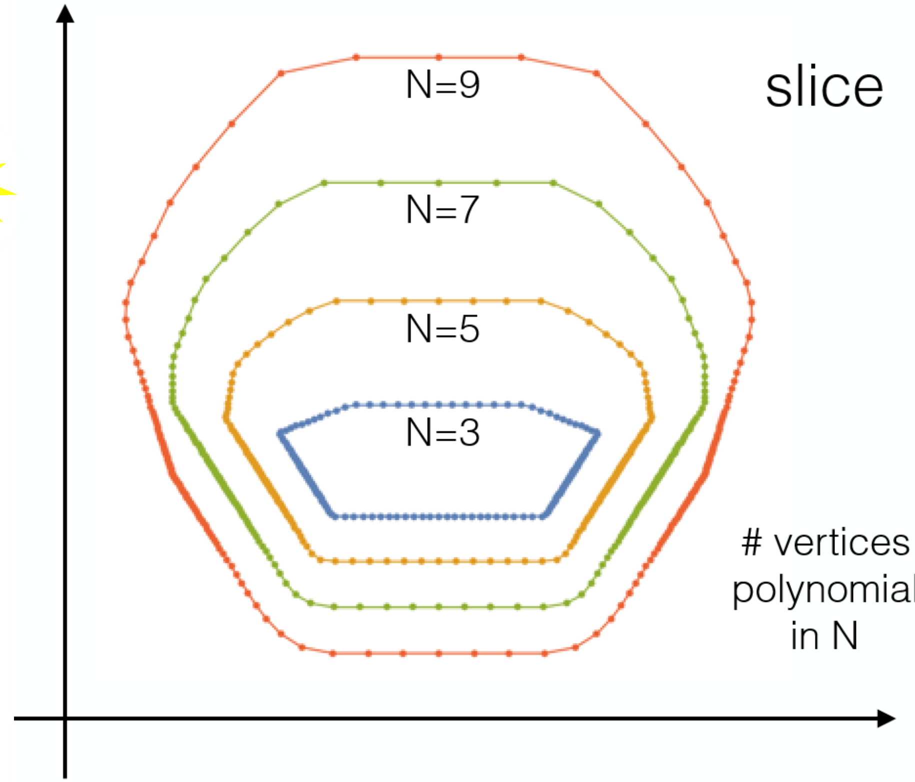


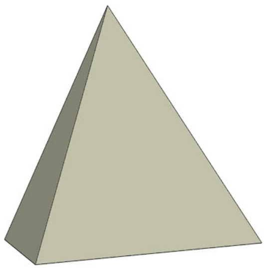
What's the physical significance of each inequality?
Are they equally relevant?

Bounding the LHVM set

- Main Observation

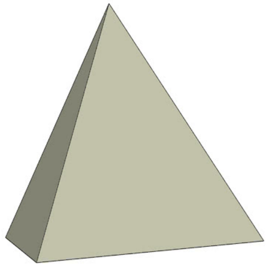
Bounding the LHVM set

- Main Observation



Bounding the LHVM set

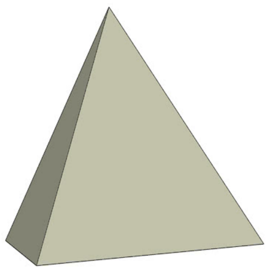
- Main Observation



As the system becomes larger

Bounding the LHVM set

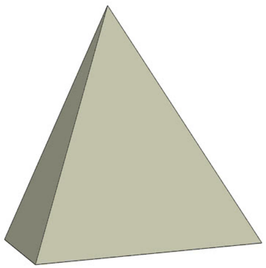
- Main Observation



As the system becomes larger

Bounding the LHVM set

- Main Observation

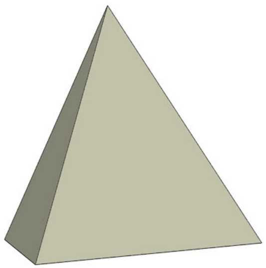


As the system becomes larger

Where does this extra structure come from?

Bounding the LHVM set

- Main Observation



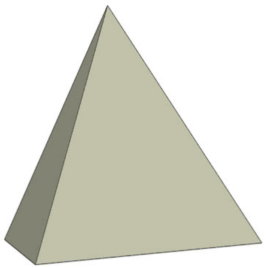
As the system becomes larger

Where does this extra structure come from?

Local
Deterministic
Strategy view:

Bounding the LHVM set

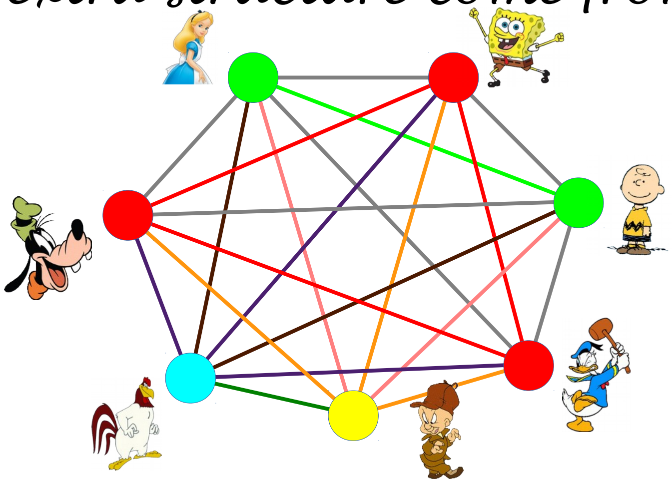
- Main Observation



As the system becomes larger

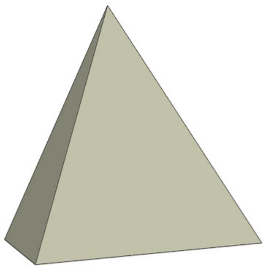
Where does this extra structure come from?

Local Deterministic Strategy view:



Bounding the LHVM set

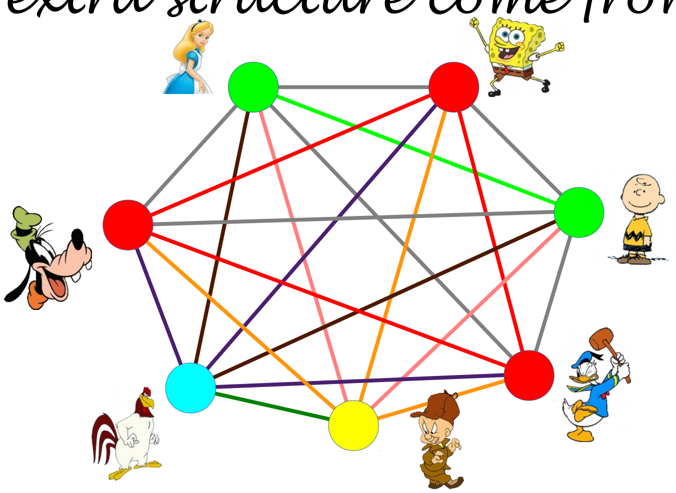
- Main Observation



As the system becomes larger

Where does this extra structure come from?

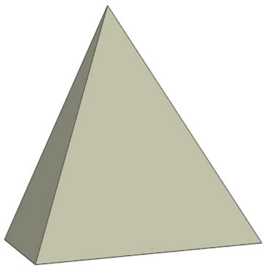
Local Deterministic Strategy view:



Permutational Invariance:

Bounding the LHVM set

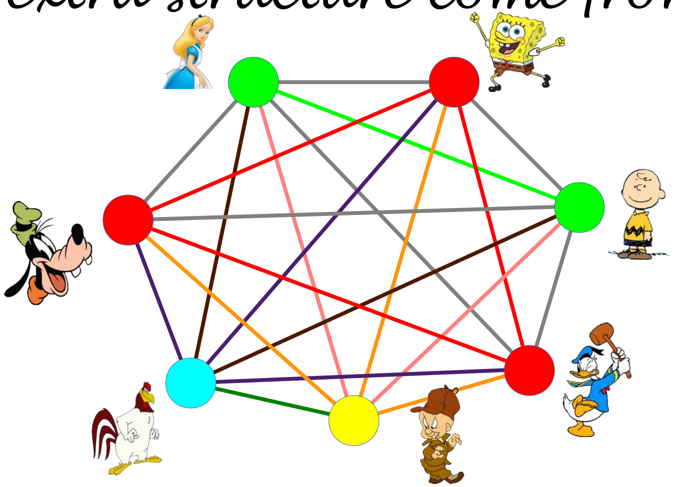
- Main Observation



As the system becomes larger

Where does this extra structure come from?

Local Deterministic Strategy view:

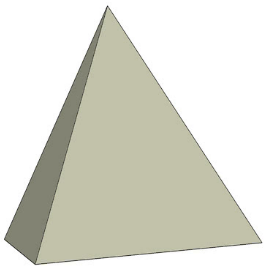


Permutational Invariance:

Only amount of each color becomes relevant

Bounding the LHVM set

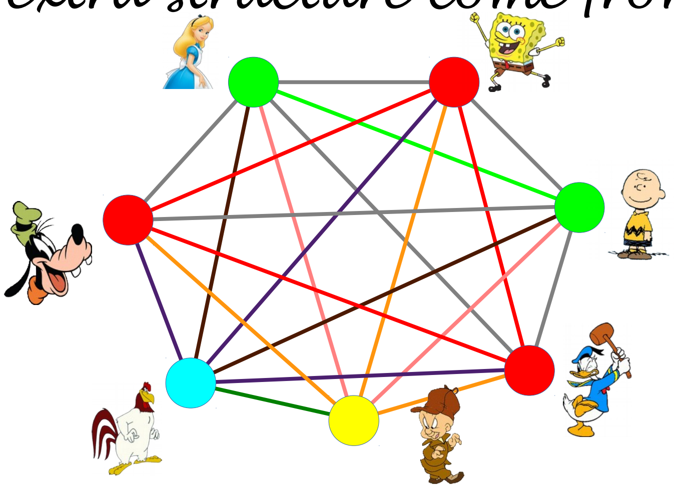
- Main Observation



As the system becomes larger

Where does this extra structure come from?

Local Deterministic Strategy view:



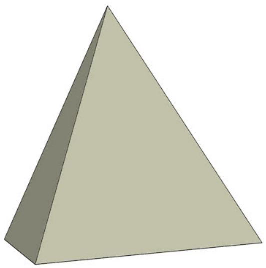
Permutational Invariance:

Only amount of each color becomes relevant

$x_i \geq 0$

Bounding the LHVM set

- Main Observation



As the system becomes larger

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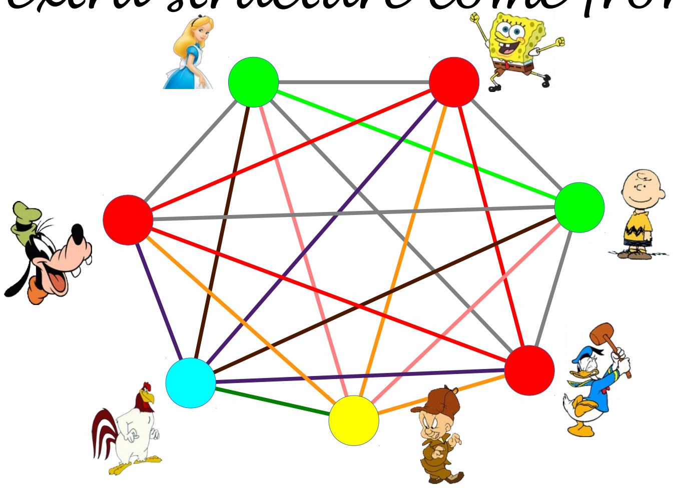
Local Deterministic Strategy view:



Permutational Invariance:

Only amount of each color becomes relevant

$$x_i \geq 0 \quad \sum_i x_i = N$$

Bounding the LHVM set

- Algebraic structure at every LDS

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$$\mathcal{S}_{kl} = \mathcal{S}_k \cdot \mathcal{S}_l - \mathcal{Z}_{kl}$$

⟨aQa^L⟩

Bounding the LHVM set

- Algebraic structure at every LDS

$$\mathcal{S}_{kl} = \mathcal{S}_k \cdot \mathcal{S}_l - \mathcal{Z}_{kl}$$

$$\begin{pmatrix} N \\ \mathcal{S}_1 \\ \mathcal{S}_0 \\ \mathcal{Z}_{01} \end{pmatrix} = 2H^{\otimes 2} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

Bounding the LHVM set

- Algebraic structure at every LDS

$$\mathcal{S}_{kl} = \mathcal{S}_k \cdot \mathcal{S}_l - \mathcal{Z}_{kl}$$

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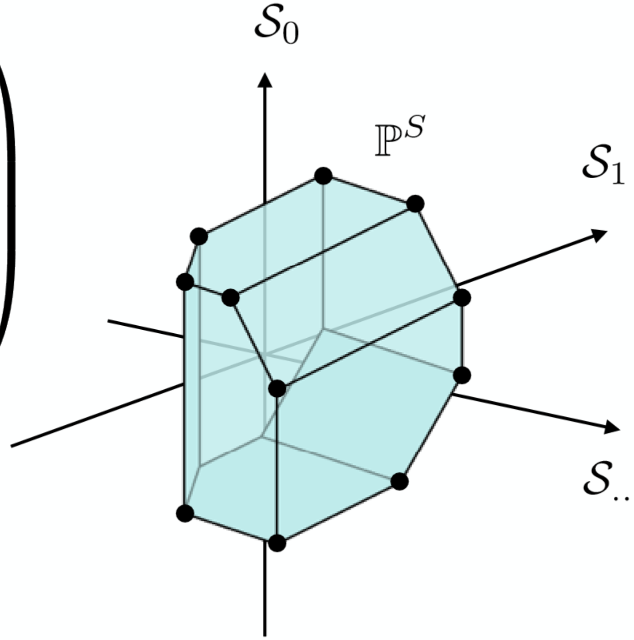

$$\mathbb{P}^{\mathrm{S}} = \mathrm{CH}\left\{\vec{\mathcal{S}}(\vec{x}) \ \text{s.t.} \ \sum_i x_i = N, \ x_i \in \mathbb{Z}_{\geq 0}\right\}$$

Bounding the LHVM set

- Algebraic structure at every LDS

$$\mathcal{S}_{kl} = \mathcal{S}_k \cdot \mathcal{S}_l - \mathcal{Z}_{kl}$$

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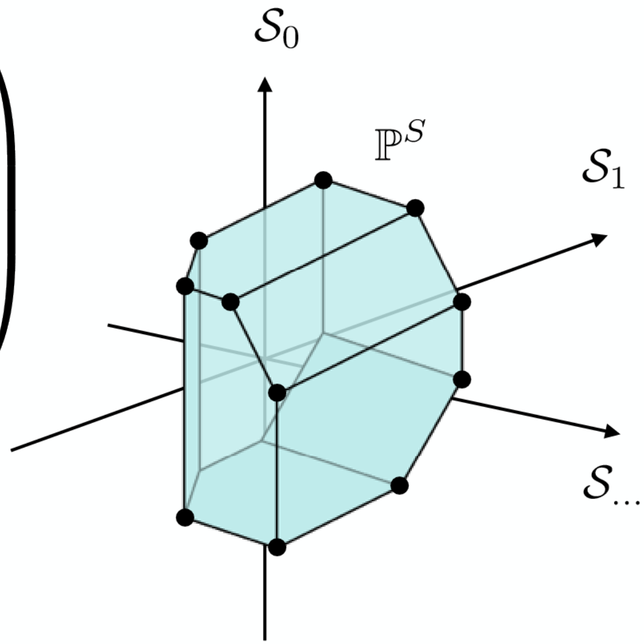


$$\mathbb{P}^{\mathrm{S}} = \mathrm{CH}\left\{ \vec{\mathcal{S}}(\vec{x}) \ \text{ s.t. } \sum_i x_i = N,\ x_i \in \mathbb{Z}_{\geq 0} \right\}$$

Goal: Define a manifold interpolating the vertices of the symmetric 2-body polytope, and compute its convex hull

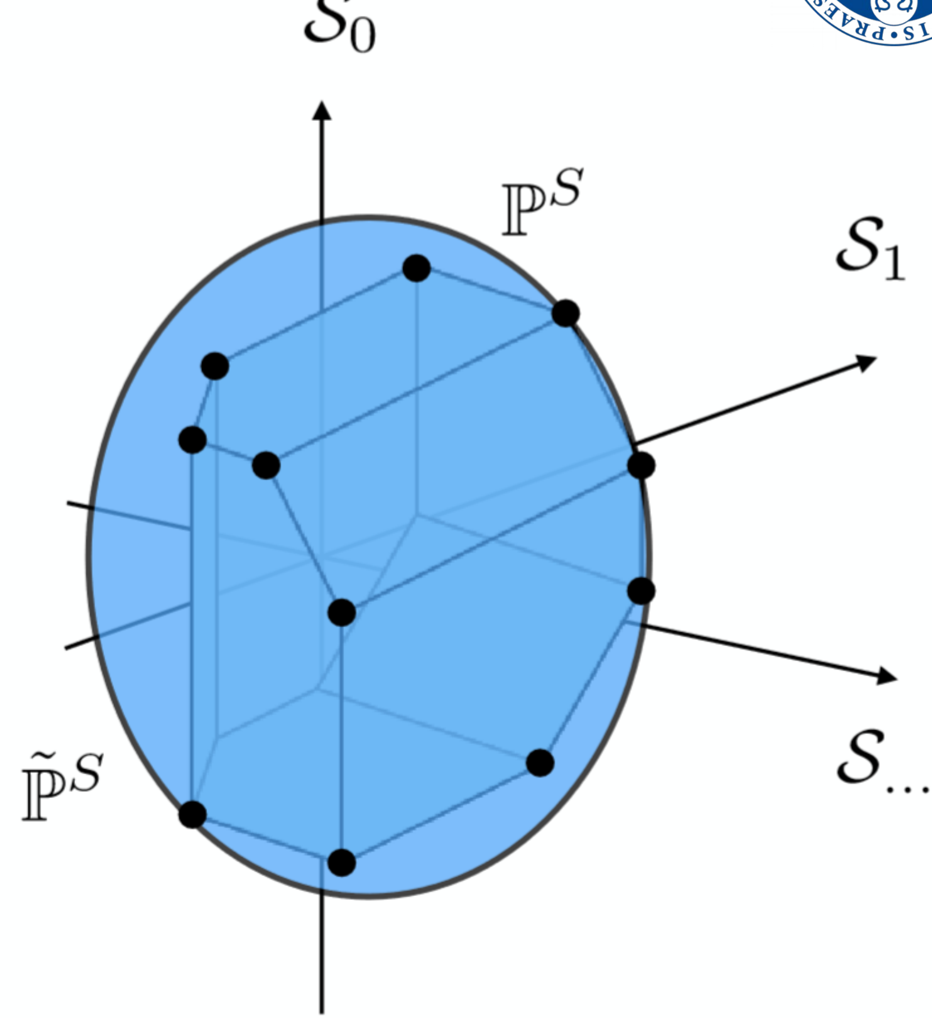
First relaxation

First relaxation

$$\mathbb{P}^{\mathrm{S}} = \mathrm{CH}\left\{ \vec{\mathcal{S}}(\vec{x}) \;\text{ s.t. }\; \sum_i x_i = N, x_i \in \mathbb{Z}_{\geq 0} \right\}$$

$$\widetilde{\mathbb{P}^{\mathrm{S}}} = \mathrm{CH}\left\{ \vec{\mathcal{S}}_K(\vec{x}) \;\text{ s.t. }\; \sum_i x_i = N, x_i \in \mathbb{R}_{\geq 0} \right\}$$

$$\tilde{\mathbb{P}}^S \supseteq \mathbb{P}^S$$



First relaxation

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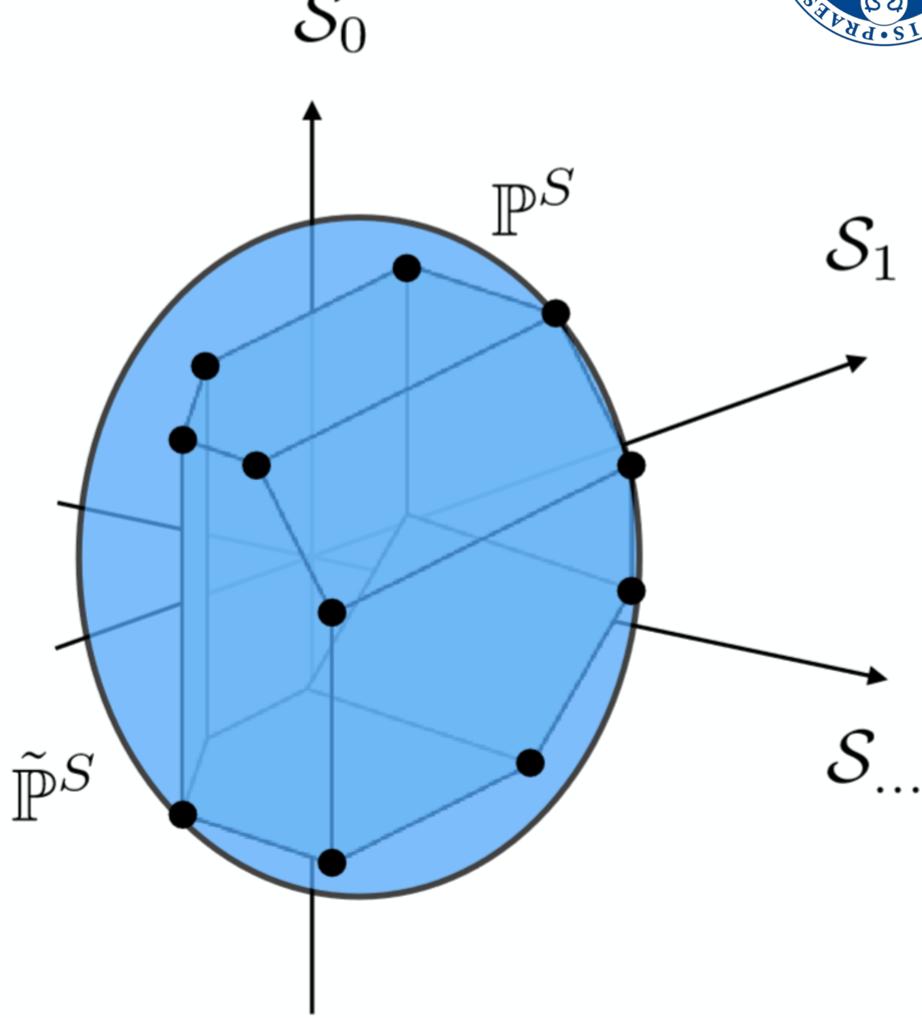
$$\widetilde{\mathbb{P}^{\mathrm{S}}} = \mathrm{CH}\left\{\vec{\mathcal{S}}_K(\vec{x}) \text{ s.t. } \sum_i x_i = N, x_i \in \mathbb{R}_{\geq 0}\right\}$$

$$\tilde{\mathbb{P}}^S \supseteq \mathbb{P}^S$$

$$\begin{cases} f_i(\vec{\mathcal{S}}_K) = 0 \\ g_j(\vec{\mathcal{S}}_K) \geq 0 \end{cases}$$



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$$\mathbb{P}^{\mathrm{S}} = \mathrm{CH}\left\{\vec{\mathcal{S}}(\vec{x}) \ \text{ s.t. } \ \sum_i x_i = N, x_i \in \mathbb{Z}_{\geq 0}\right\}$$

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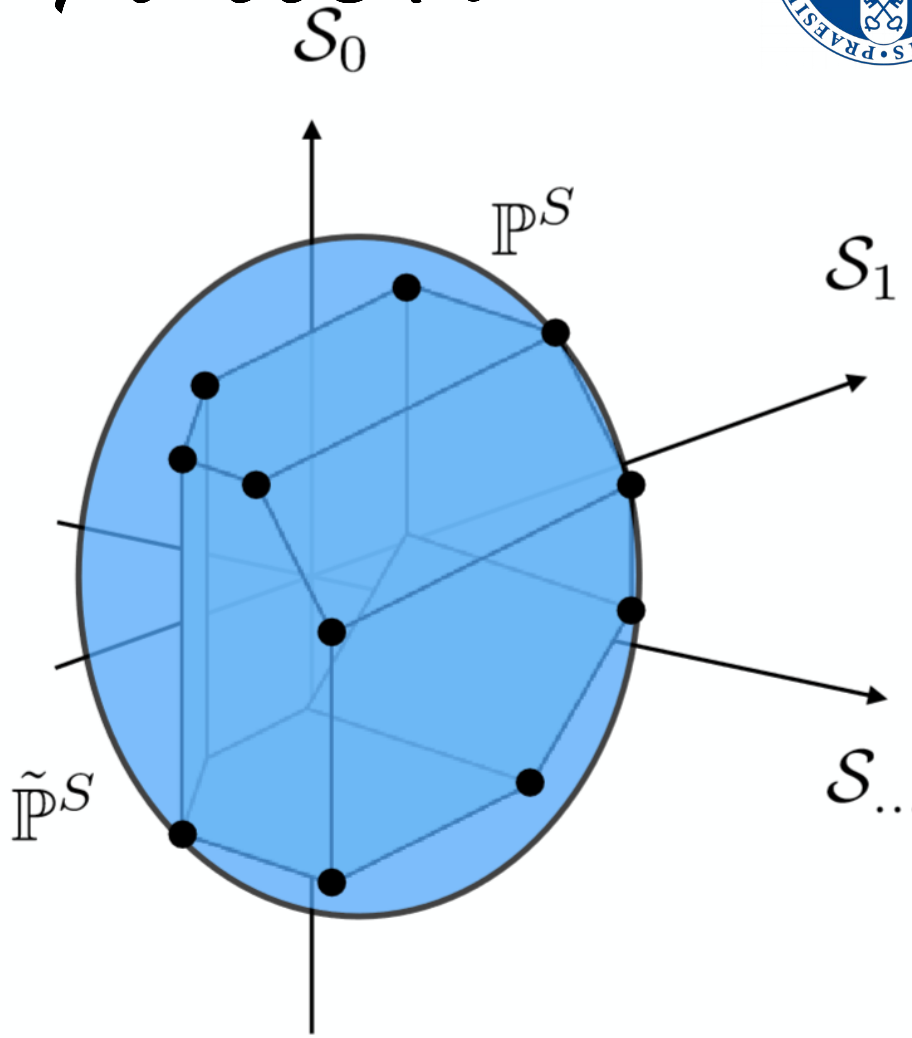

Computing the convex hull of a semialgebraic set is NP-hard

Second relaxation

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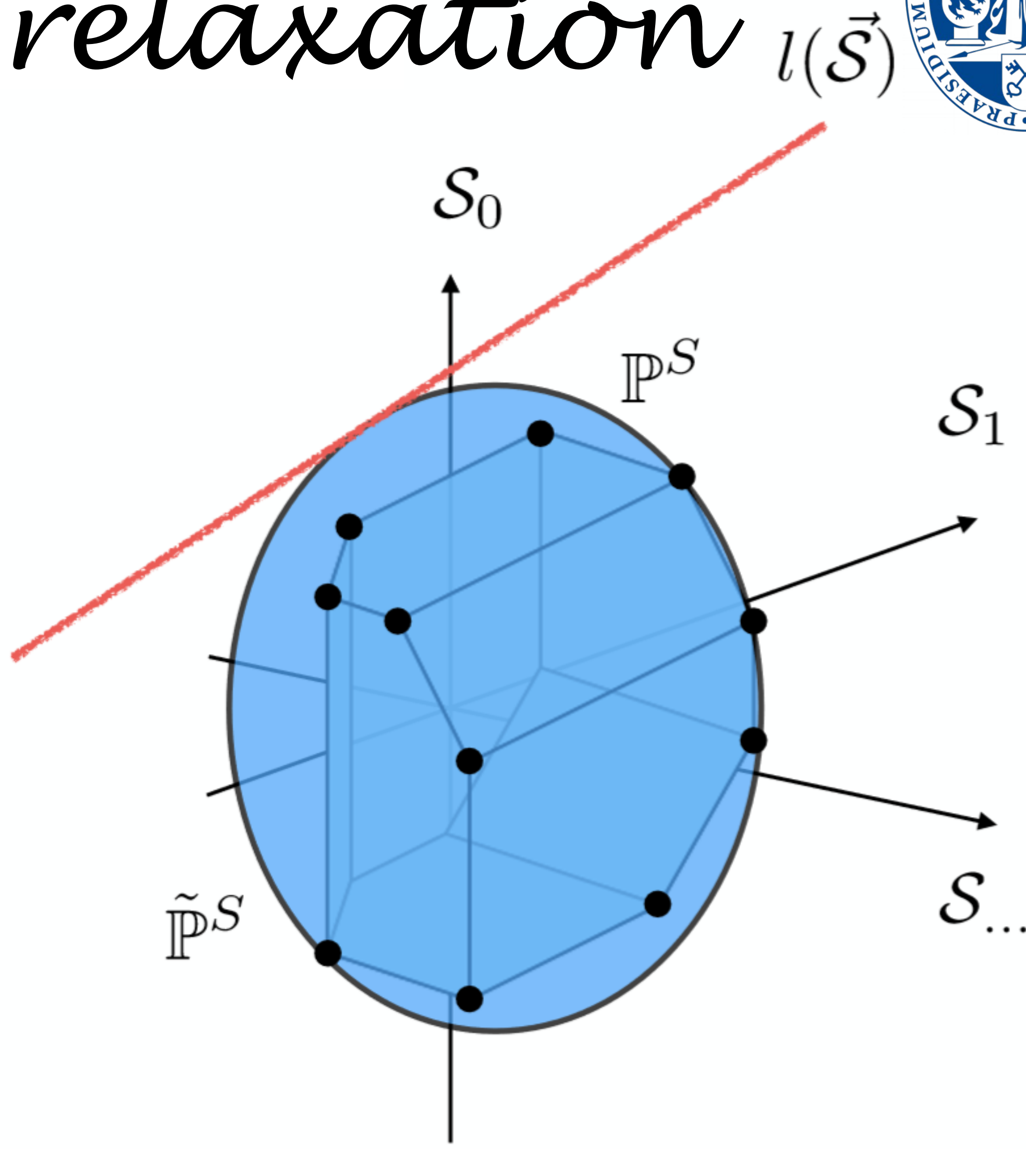
$\tilde{\mathbb{P}}^S$ is defined by $\begin{cases} f_i(\vec{\mathcal{S}}_K) = 0 \\ g_j(\vec{\mathcal{S}}_K) \geq 0 \end{cases}$

ansatz: $l(\vec{\mathcal{S}}) = \sum_{i=0}^{m} g_i(\vec{\mathcal{S}})\ \sigma_i(\vec{\mathcal{S}})$

$\sigma_i(\vec{\mathcal{S}})$ sos polynomials

NOTE: l is positive in $\tilde{\mathbb{P}}^S$



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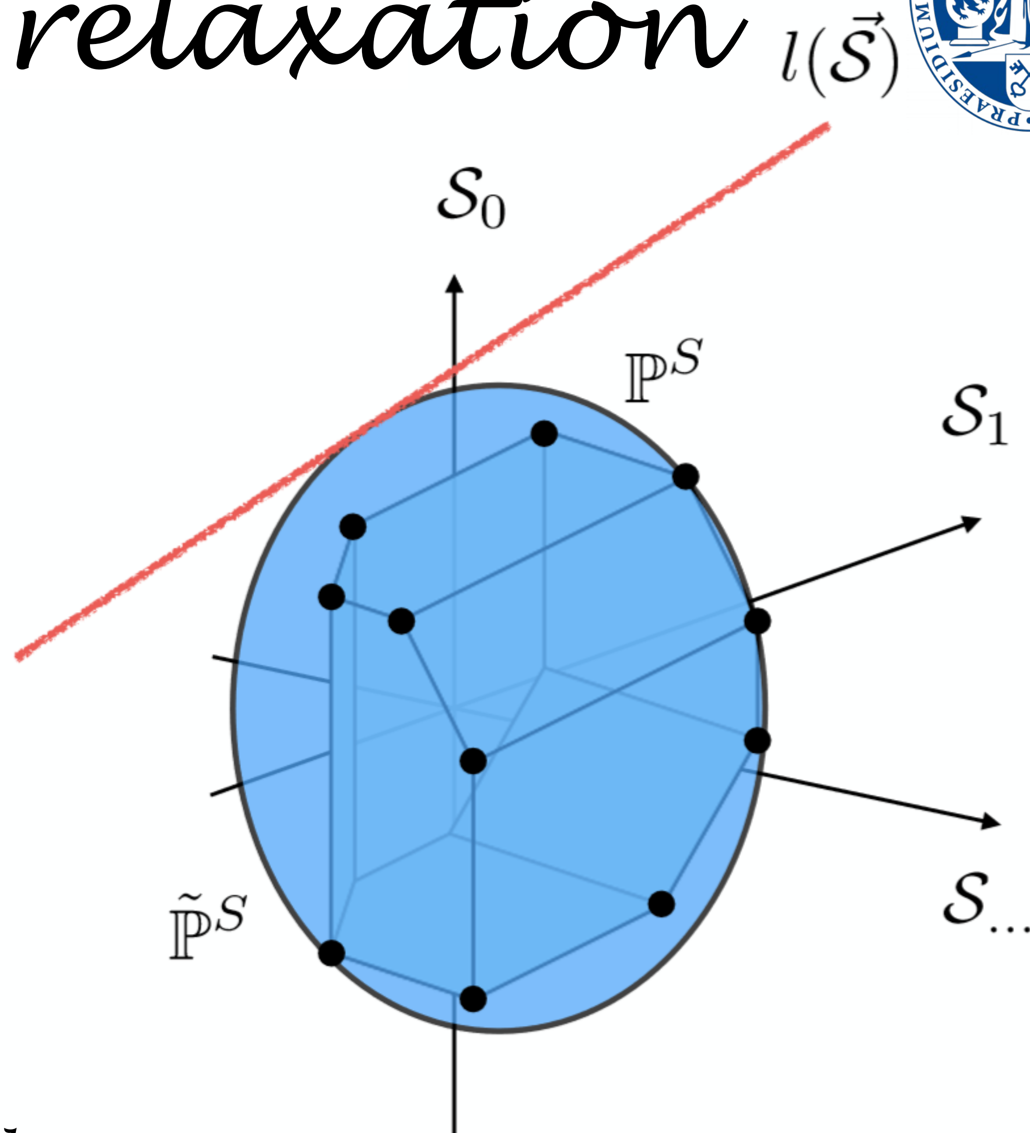


Solve the sos representation problem as an SDP!

The SDP

The SDP

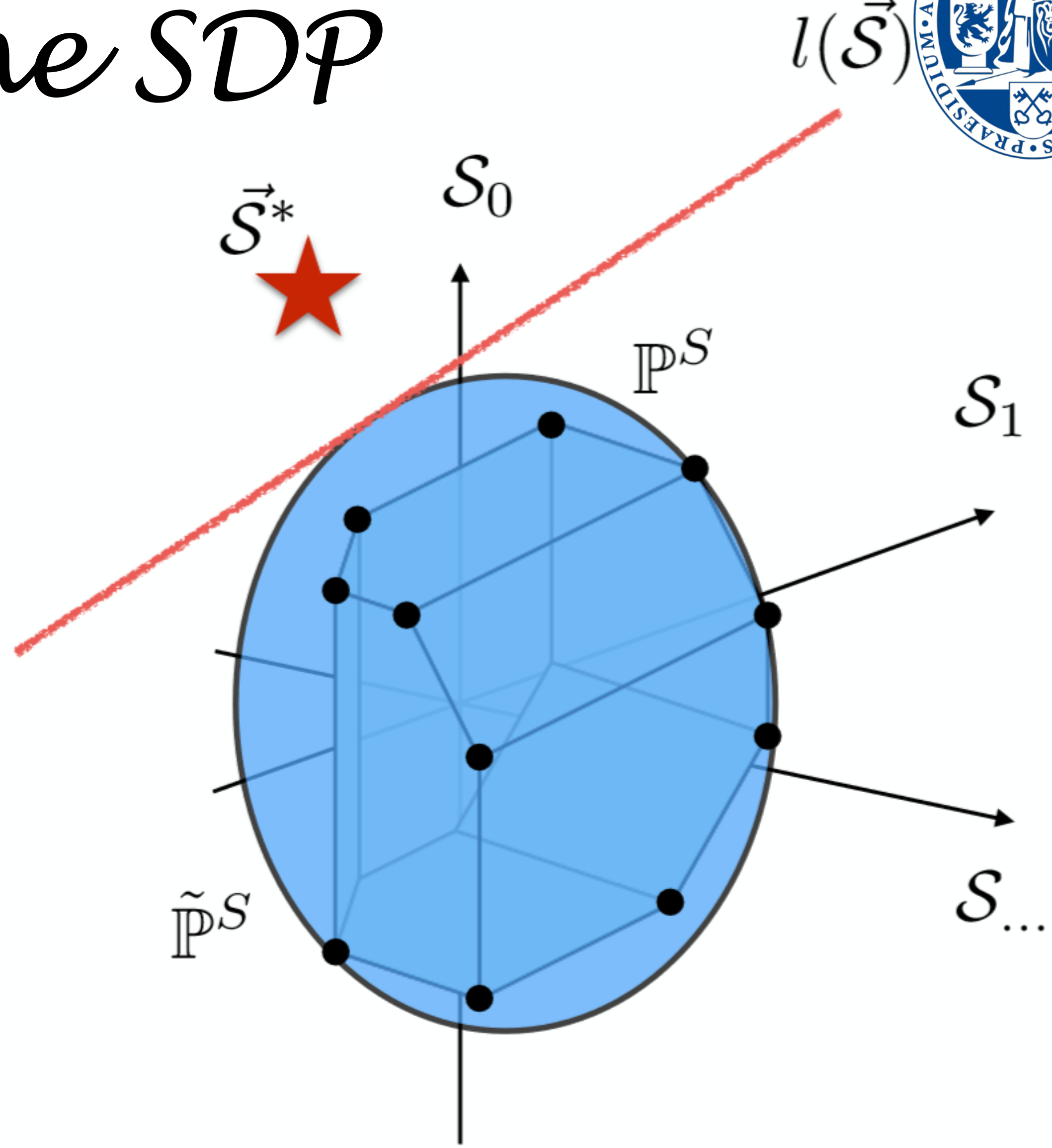
$$l(\vec{\mathcal{S}}) = \sum_{i=0}^{m} g_i(\vec{\mathcal{S}})\,\sigma_i(\vec{\mathcal{S}})$$
$$= \Gamma \cdot G$$

GOAL: prove $l(\vec{\mathcal{S}}^*) < 0$
for some $\sigma_i(\vec{\mathcal{S}})$

$G \succeq 0$ $\qquad \Gamma = \begin{pmatrix} 1 & \mathcal{S}_0 & \mathcal{S}_1 & \mathcal{S}_{00} & \dots \\ \mathcal{S}_0 & \mathcal{S}_0^2 & \mathcal{S}_0\mathcal{S}_1 & \mathcal{S}_0\mathcal{S}_{00} & \dots \\ \mathcal{S}_1 & \mathcal{S}_1\mathcal{S}_0 & \mathcal{S}_1^2 & \mathcal{S}_1\mathcal{S}_{00} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$

SDP: $\Gamma \succeq 0$

subject to: $\mathcal{S}_0 = \mathcal{S}_0^*$, $\mathcal{S}_1 = \mathcal{S}_1^*$, ...



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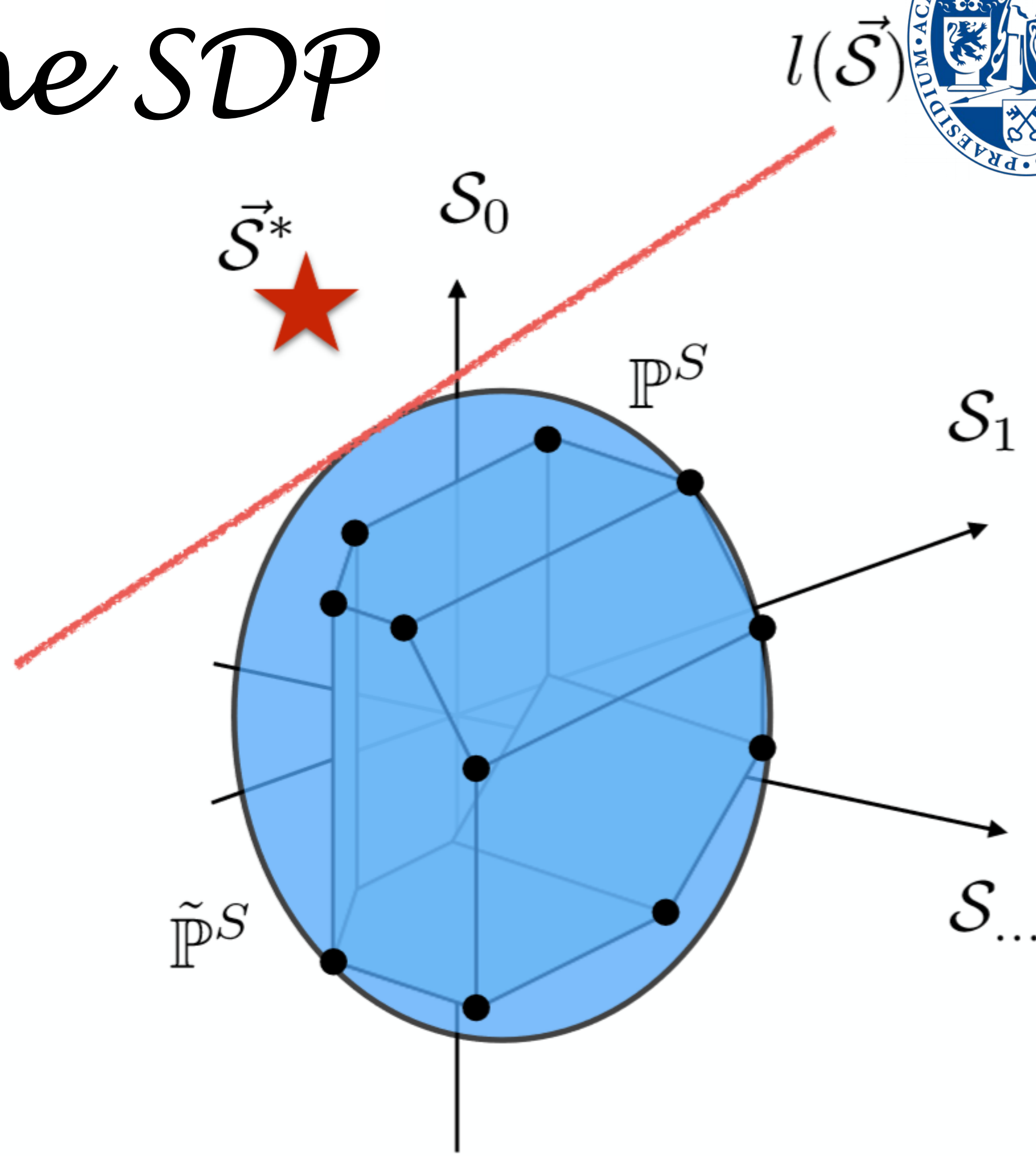

(Experimental data point)

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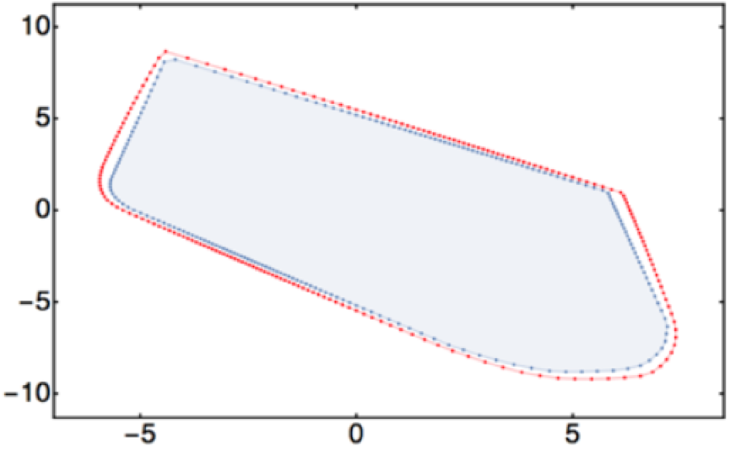
Approximation for N= 10



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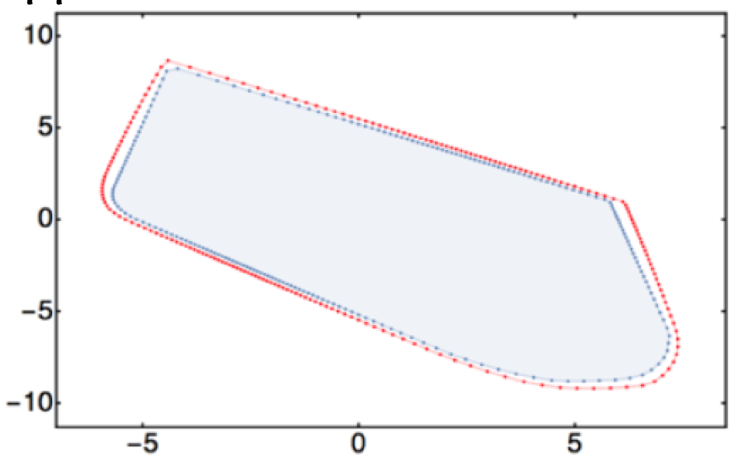
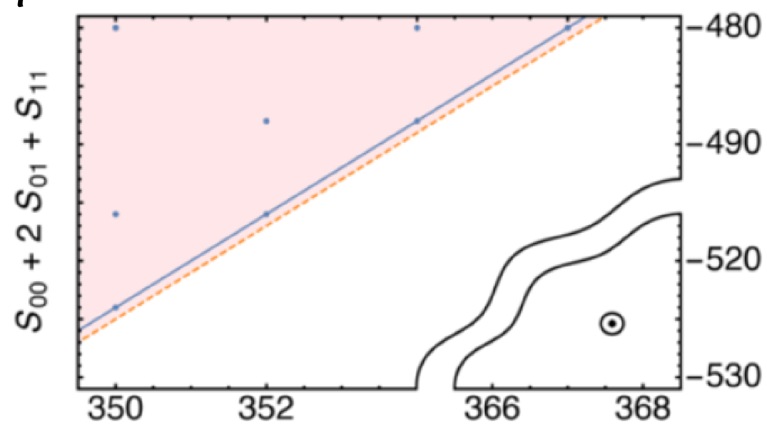


Approximation with actual experimental data and N=476



Generalization

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- More outcomes, higher-order correlators…

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Spin-nematic squeezing

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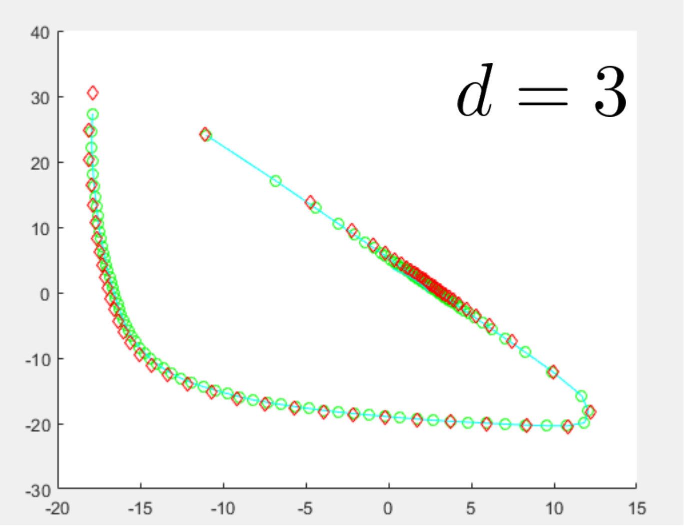

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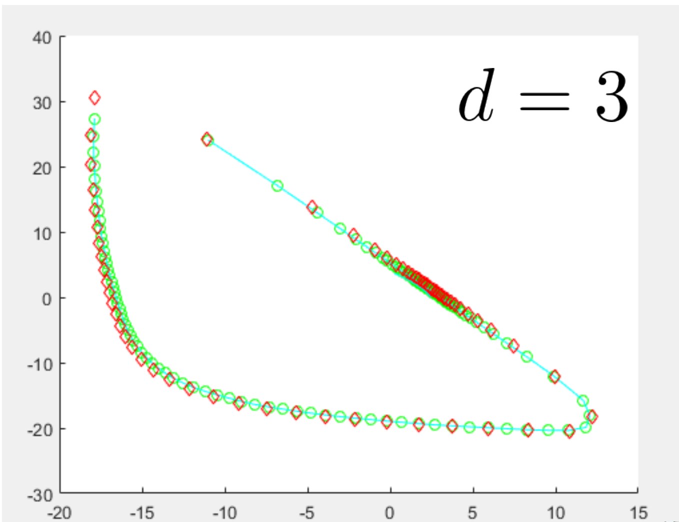

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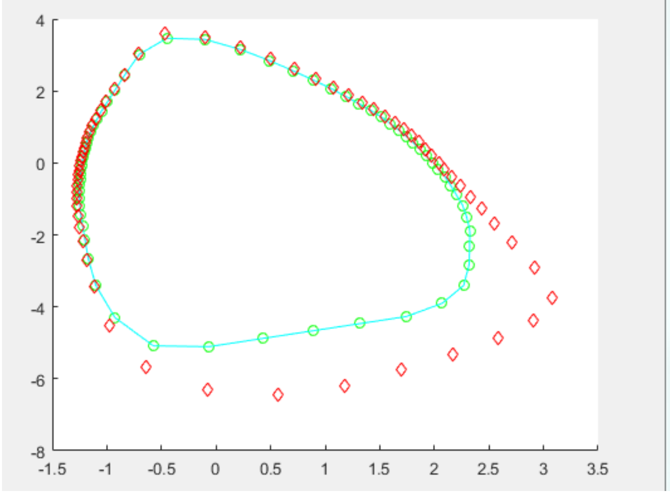

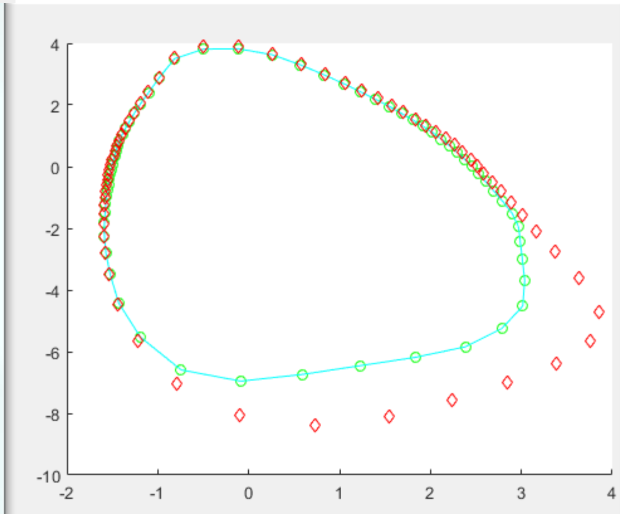

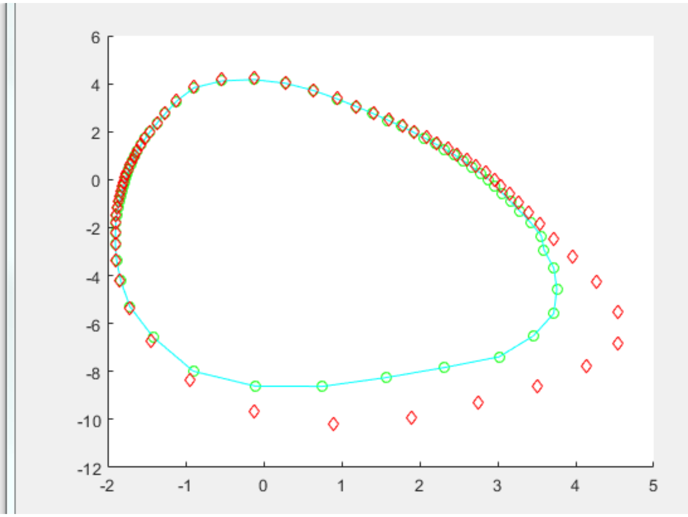

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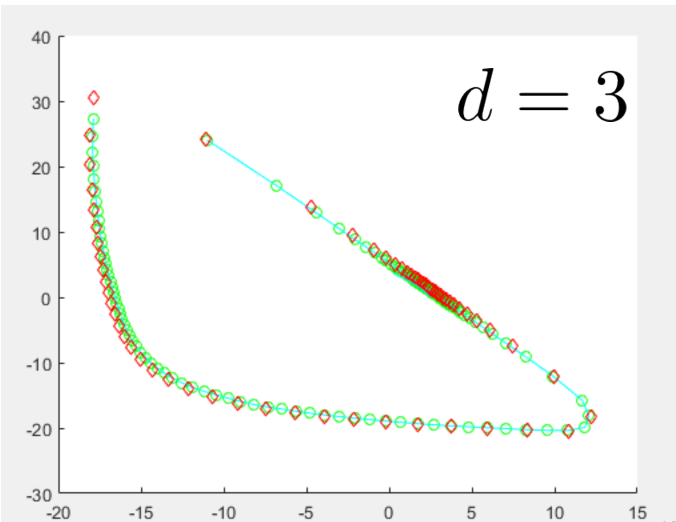

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1	x_1	
2	$x_{01}x_{10} - x_{11}$	$-$
3	$x_{001}x_{010}x_{100} - (x_{011}x_{100} + x_{010}x_{101} + x_{001}x_{110}) + 2x_{111}$	$- \quad +2$
4	$x_{0001}x_{0010}x_{0100}x_{1000} + (x_{0110}x_{1001} + x_{0101}x_{1010} + x_{0011}x_{1100})$ $-(x_{0011}x_{0100}x_{1000} + x_{0010}x_{0101}x_{1000} + x_{0001}x_{0110}x_{1000} + x_{0010}x_{0100}x_{1001} + x_{0001}x_{0100}x_{1010} + x_{0001}x_{0010}x_{1100})$ $+2(x_{0100}x_{1011} + x_{0010}x_{1101} + x_{0001}x_{1110} + x_{0111}x_{1000}) - 6x_{1111}$	$+ \quad - \quad +2 \quad -6$

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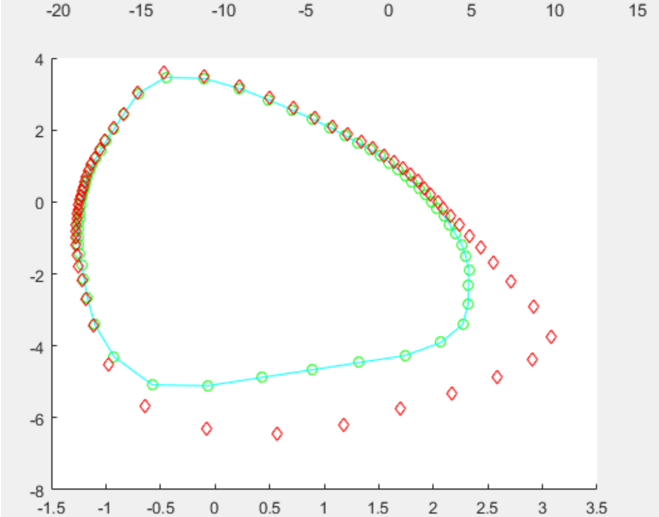

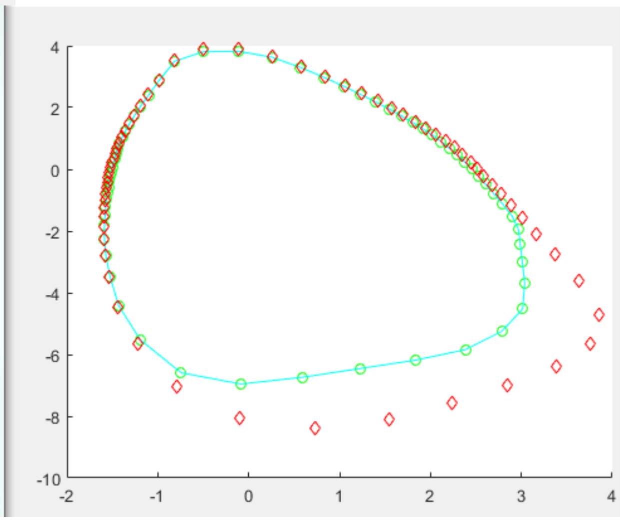

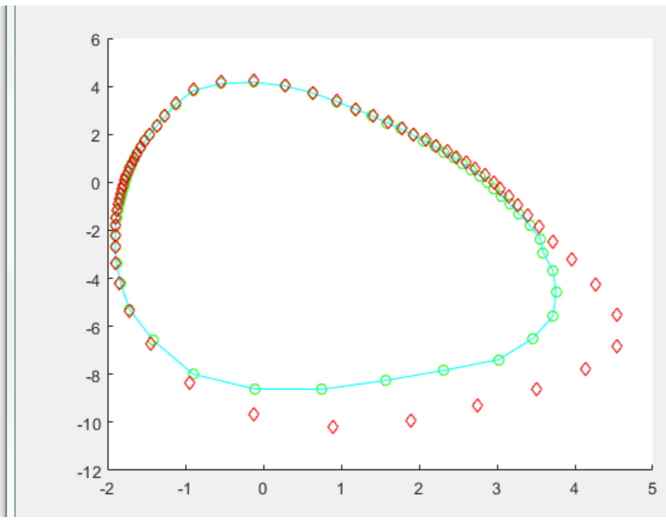

Outline

- A quantum information scientist meets non-negative polynomials
- Sum-of-squares representations
- Characterizing many-body correlations
- An application of sum-of-quares in quantum information: self-testing

Self-testing quantum devices

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⟨aQa^L⟩

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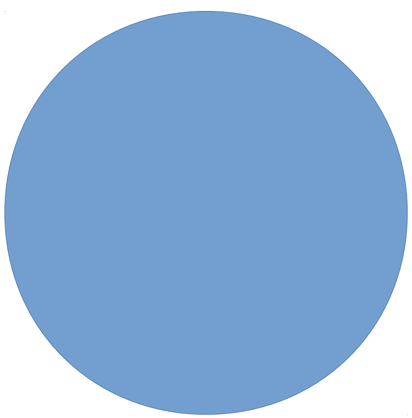

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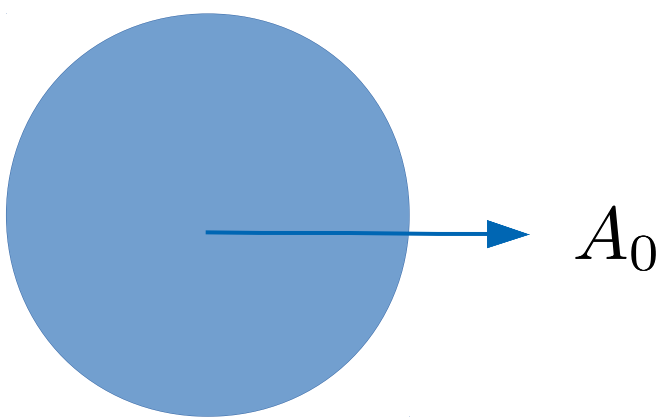

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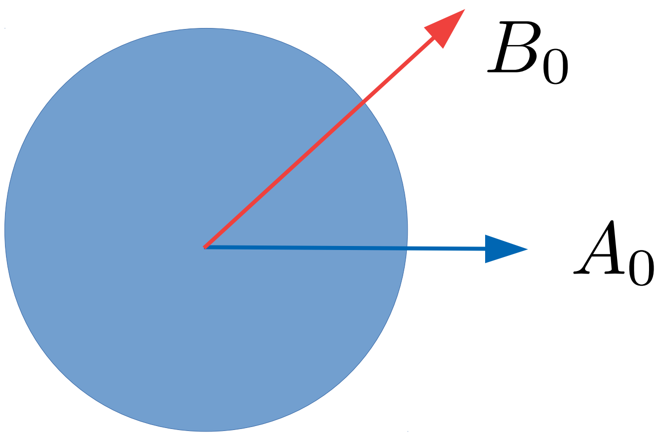

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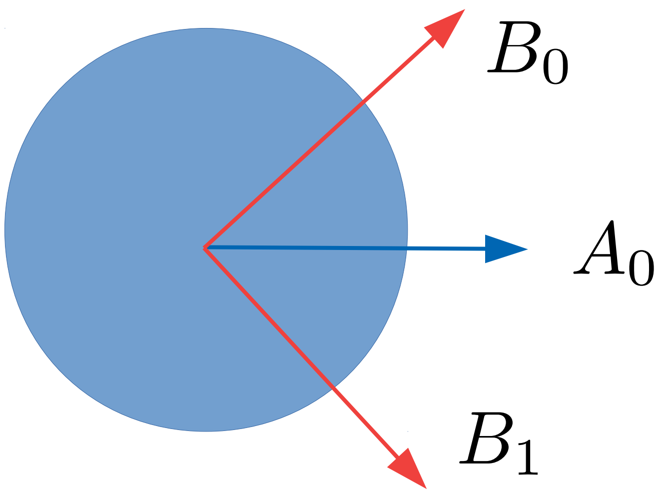

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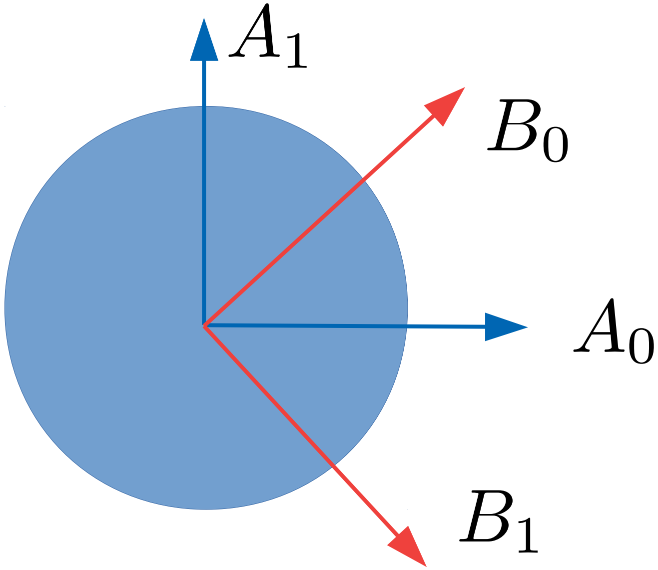

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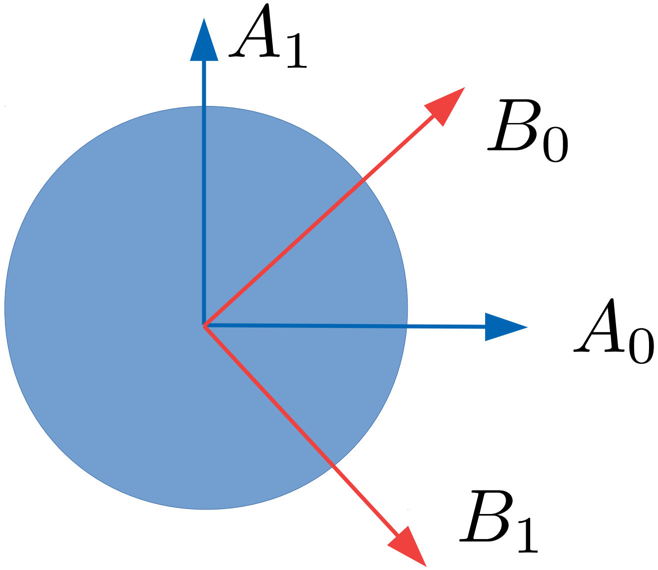

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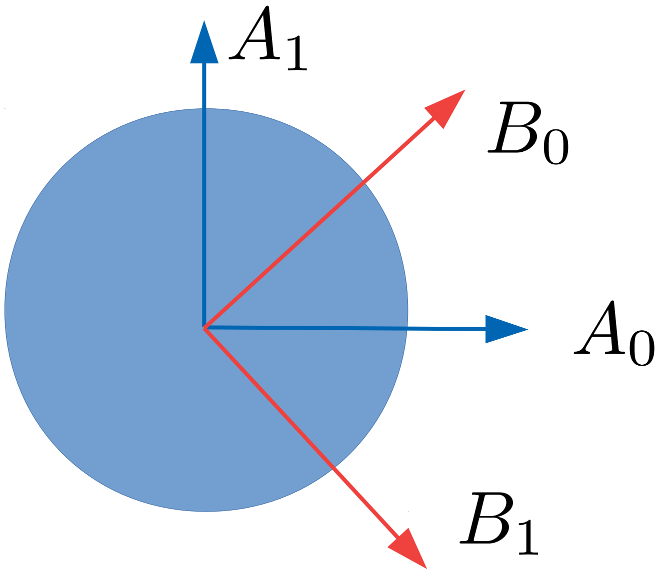

Self-testing quantum devices

- Maximally entangled states

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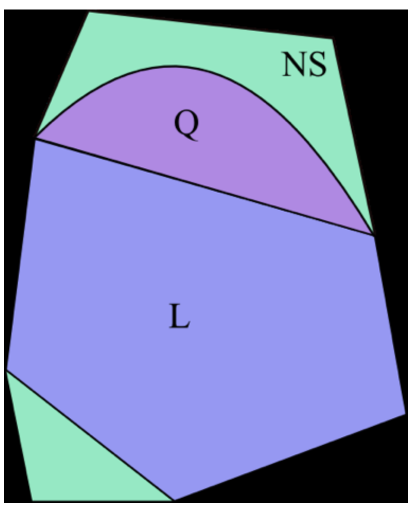

"SATWAP inequality", PRL 119, 040402 (2017)

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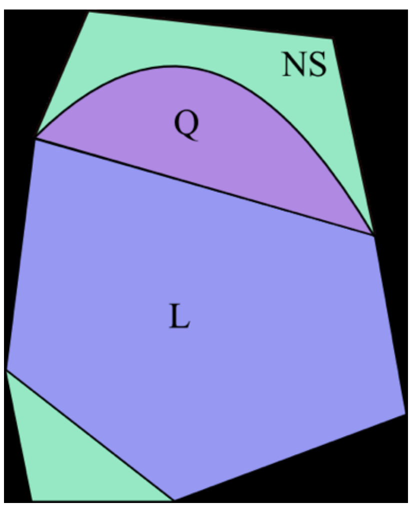

Goal: Self-test the MaxEnt state

$$|\psi\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |ii\rangle$$

"SATWAP inequality", PRL 119, 040402 (2017)

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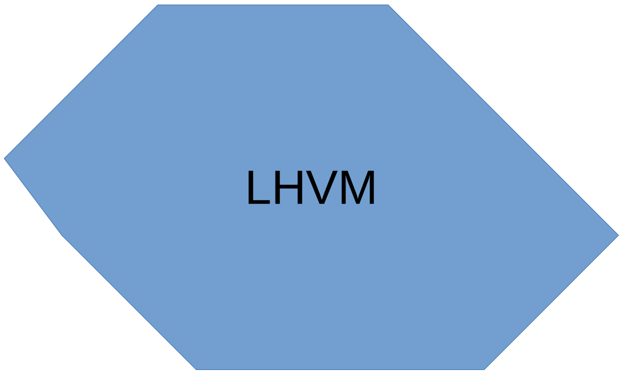

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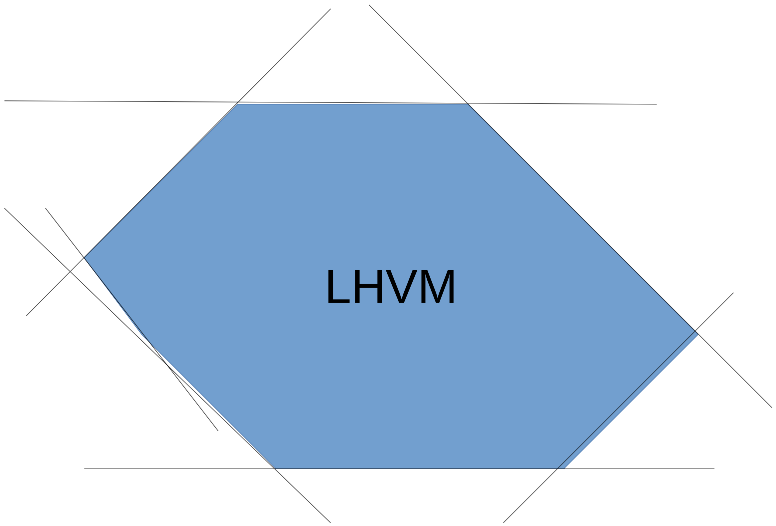

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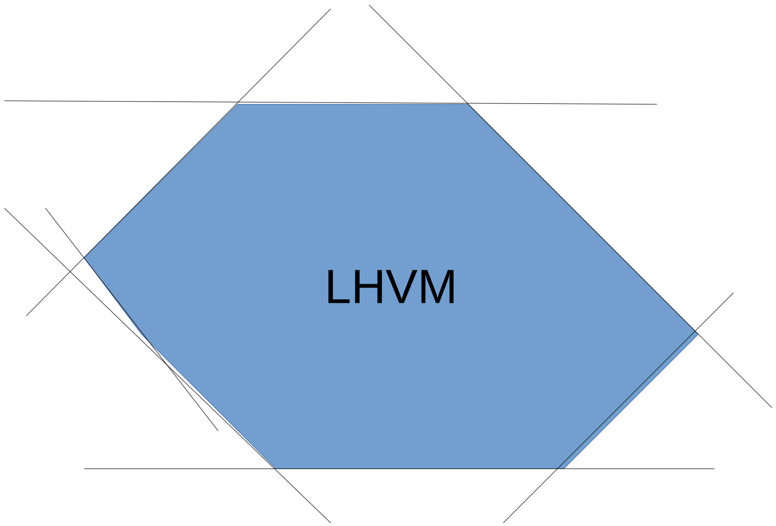

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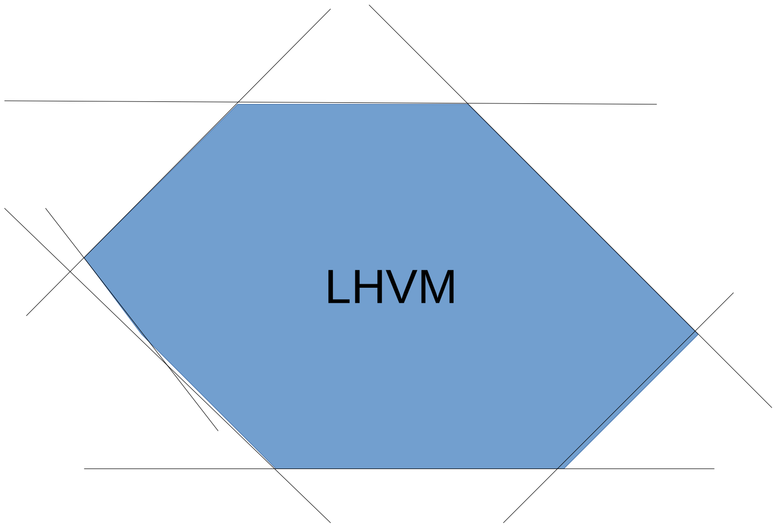

Minimal number
Optimal in what sense?

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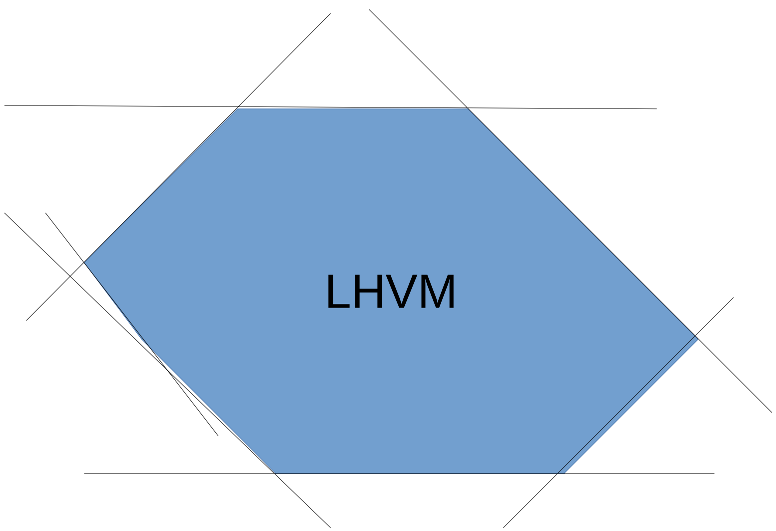

Collins-Gisin-Linden-Massar-Popescu (CGLMP) inequality

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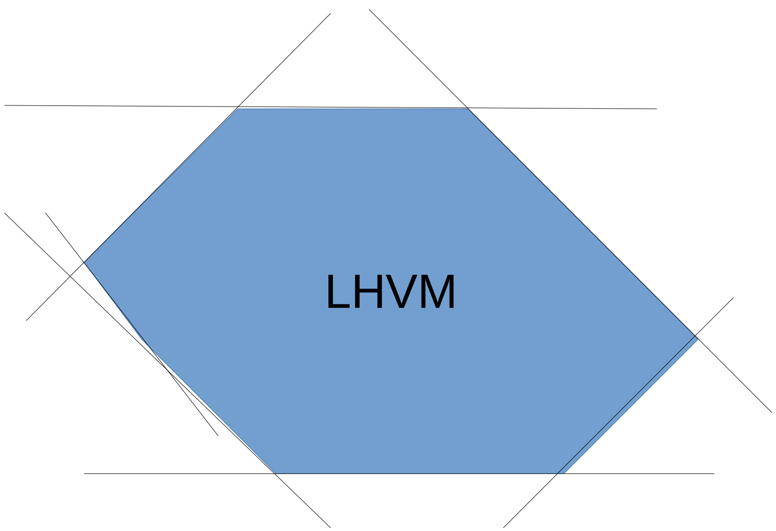

Collins-Gisin-Linden-Massar-Popescu (CGLMP) inequality

$$\mathcal{B}^d_{\text{CGLMP}} = \sum_{k=0}^{\lfloor d/2 \rfloor} \left(1 - \frac{2k}{d-1}\right) ([P(A_1 = B_1 + k) + P(B_1 = A_2 + k + 1) + P(A_2 = B_2 + k) + P(B_2 = A_1 + k)] \\ - [P(A_1 = B_1 - k - 1) + P(B_1 = A_2 - k) + P(A_2 = B_2 - k - 1) + P(B_2 = A_1 - k - 1)]) \leq 2$$

Self-testing quantum devices

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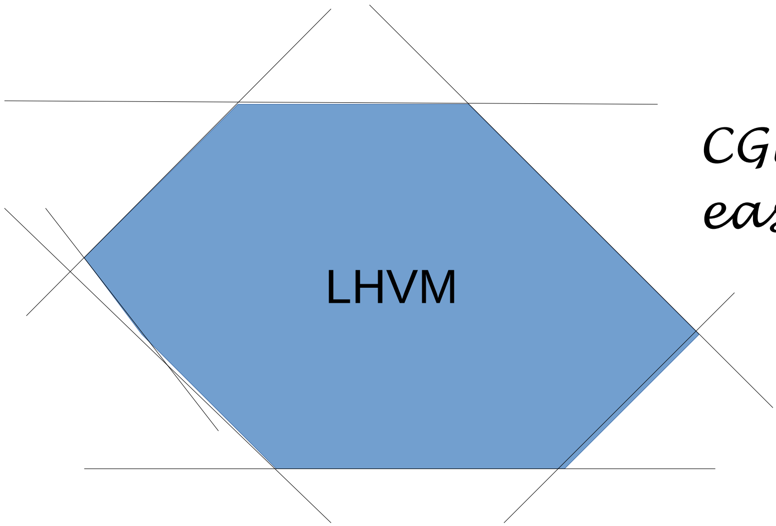

CGLMP is tight, resistant to noise, analytically easy... but has an important "anomaly"

$$|\psi^{\text{CGLMP}}\rangle = \frac{|00\rangle + \gamma|11\rangle + |22\rangle}{\sqrt{2+\gamma^2}}$$

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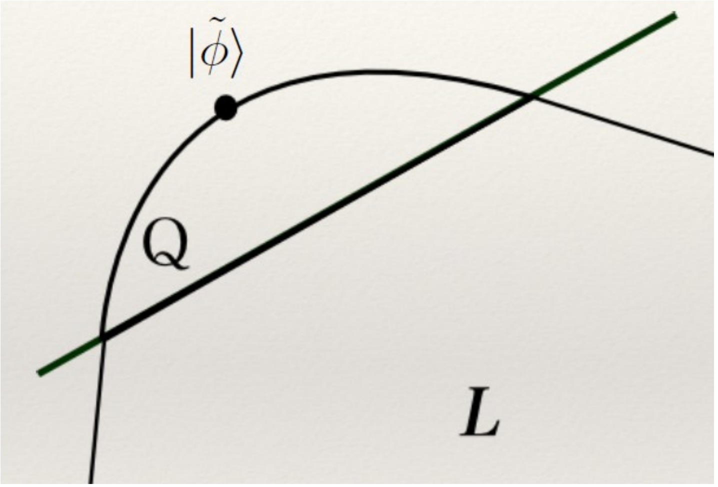

Self-testing quantum devices

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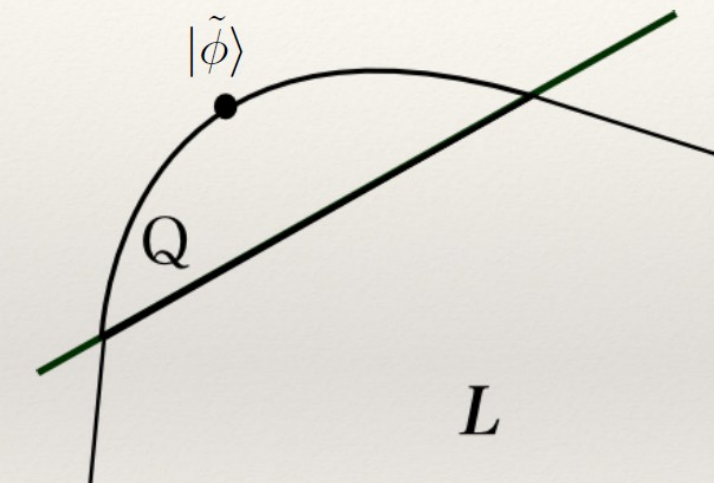

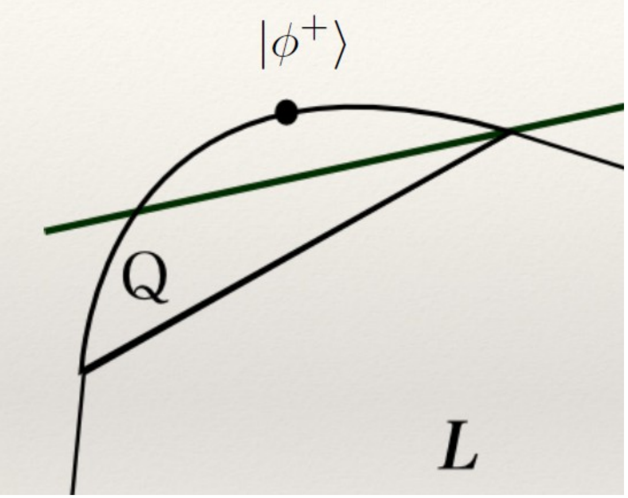

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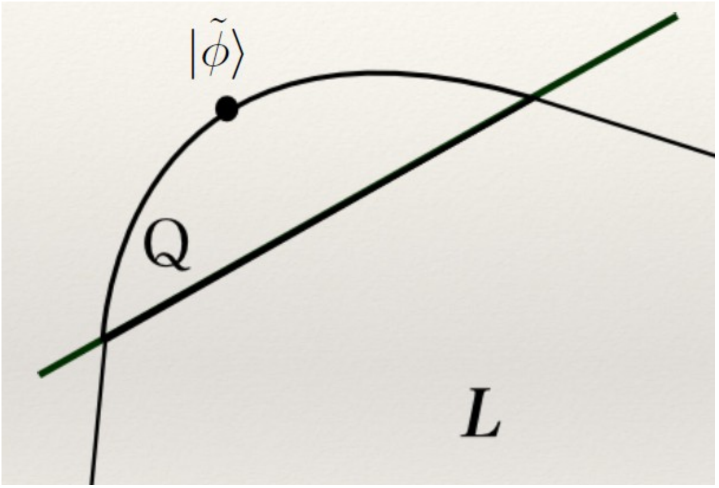
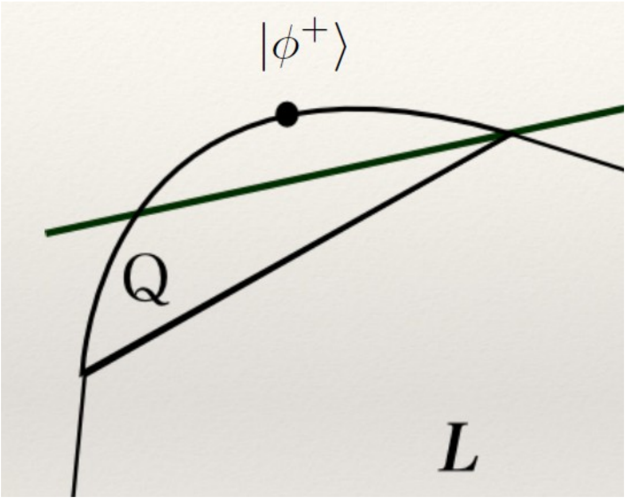


Tight inequality derived from LHVM models



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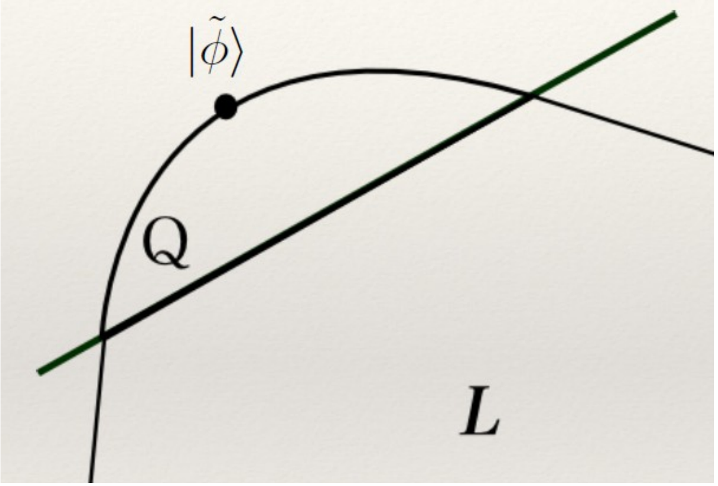

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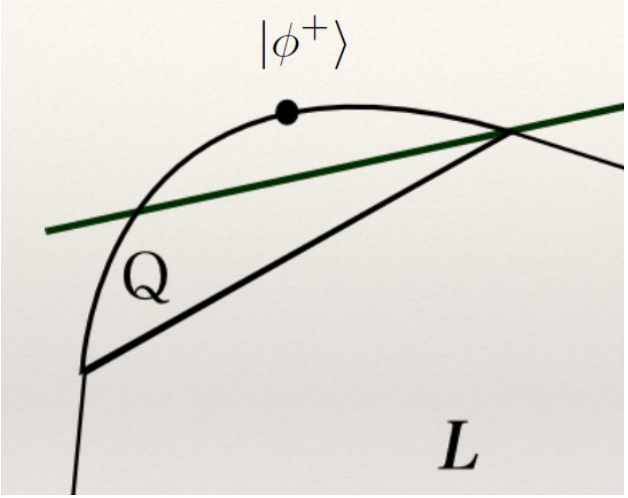

Bell inequality tailored to the maximally entangled state

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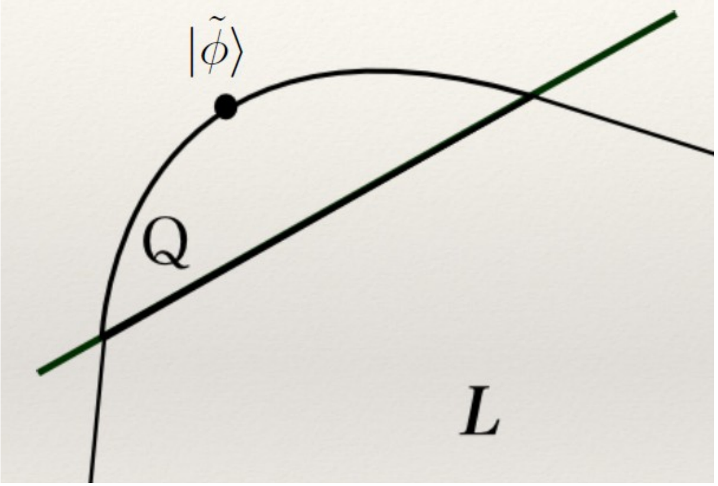
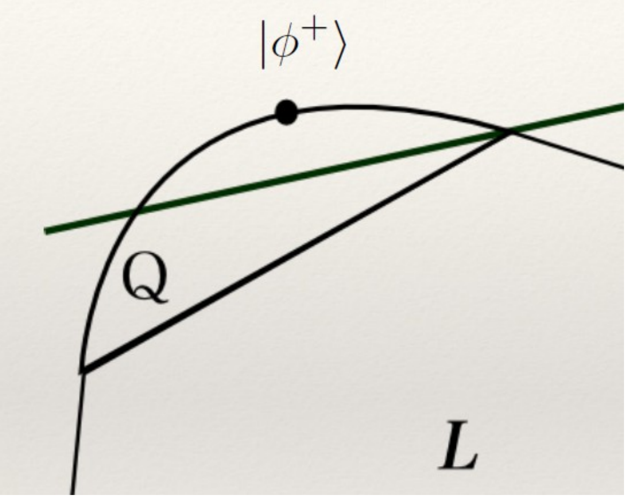


Tight inequality derived from LHVM models

Complicated Operator-sum-of-squares



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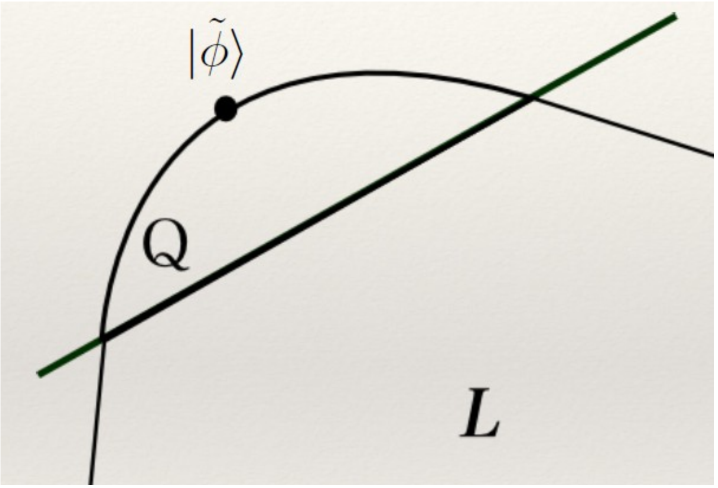

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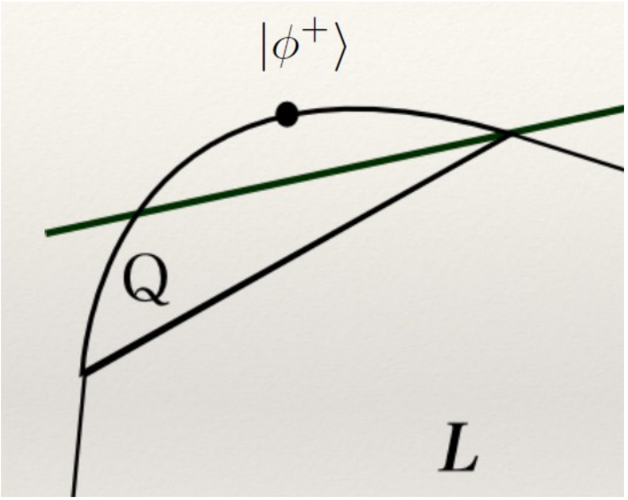


Bell inequality tailored to the maximally entangled state

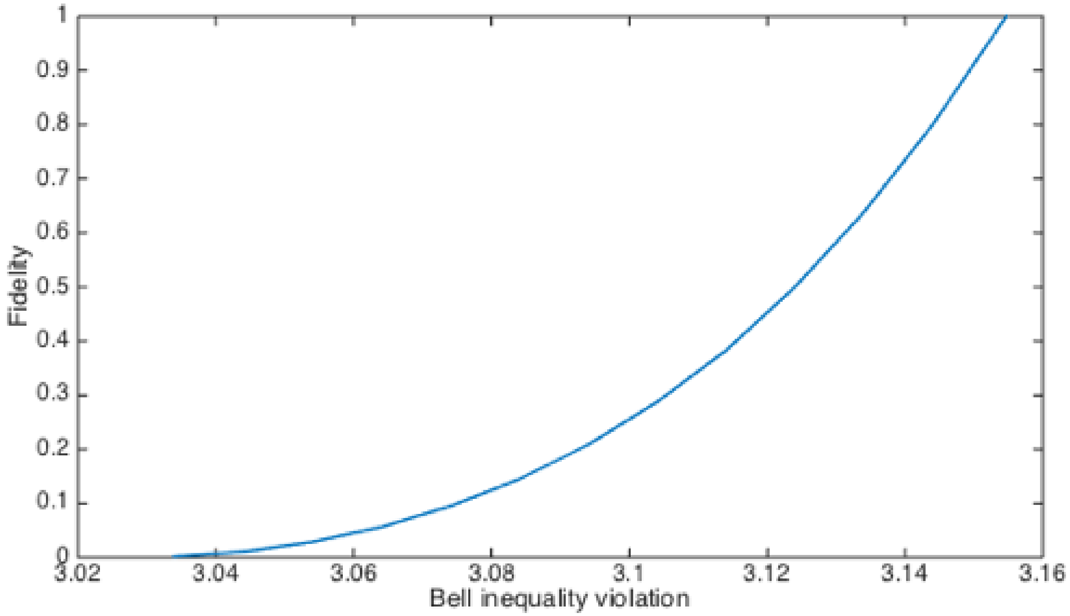
Simple, first-degree operator-sum-of-squares

Self-testing quantum devices

- Fidelity with maximally entangled qutrit pair

Self-testing quantum devices

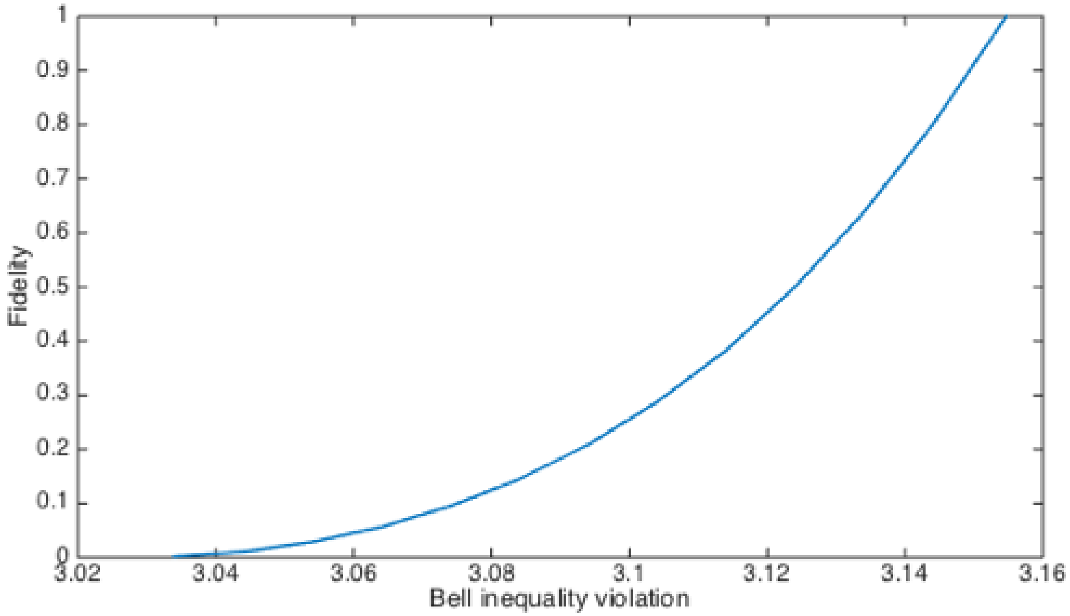
- Fidelity with maximally entangled qutrit pair



Self-testing quantum devices

- Fidelity with maximally entangled qutrit pair

$$|\psi\rangle = \frac{|00\rangle + |11\rangle + |22\rangle}{\sqrt{3}}$$



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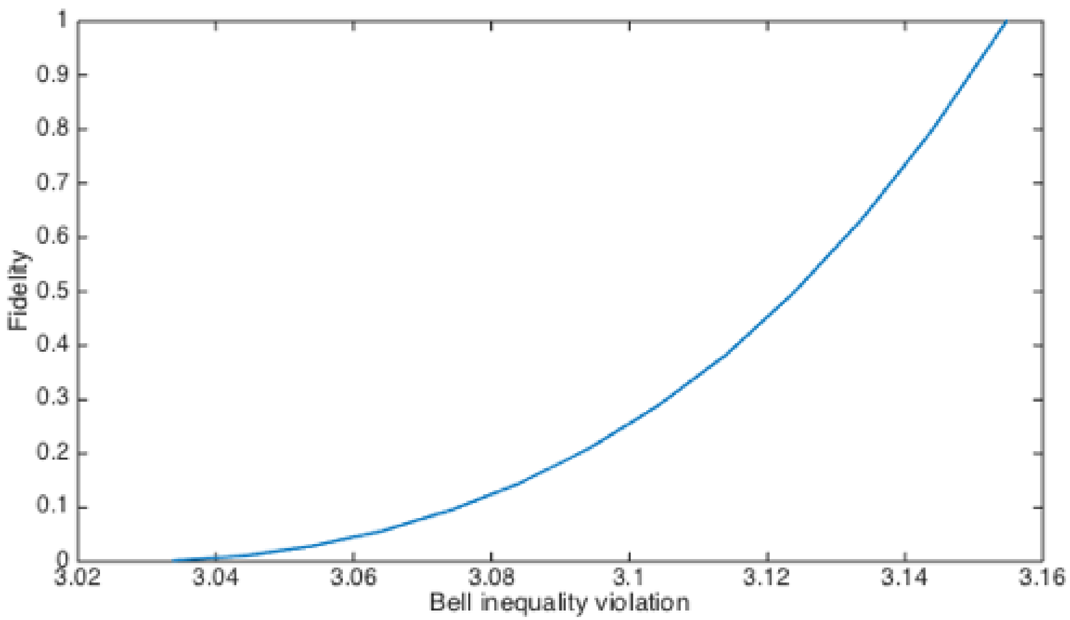

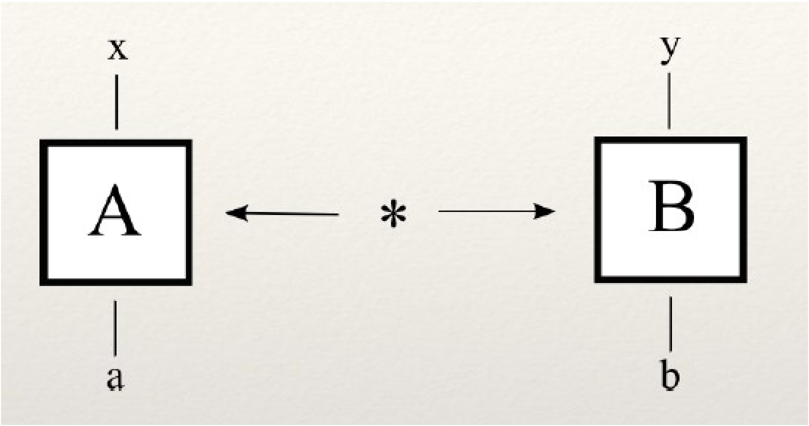

Self-testing quantum devices

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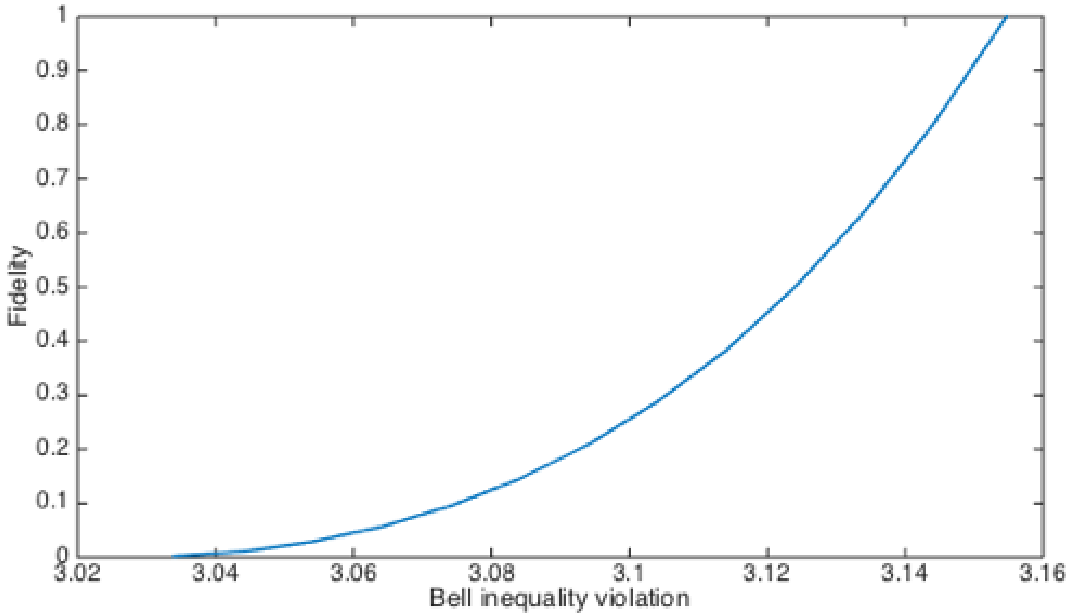
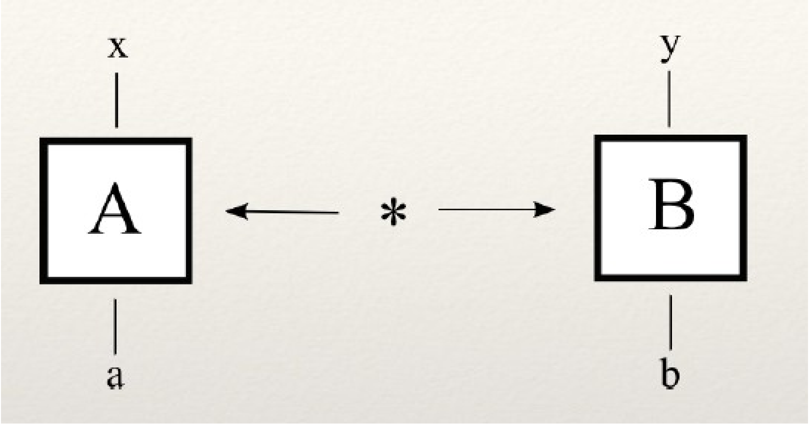
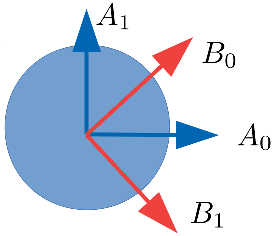




Sum-of-squares naturally generalizes



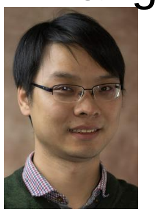
To higher dimensions and number of measurements

Self-testing quantum devices

- Experiment in Bristol

S. Paesani

J. Wang



R. Santagati

J. O'Brien

A. Laing

M. Thompson

P. Skrzypczyk

L. Mancinska

University of BRISTOL

Y. Ding

D. Bacco

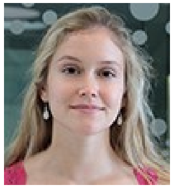
K. Rottwitt

L. K. Oxenløwe

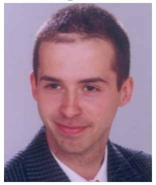
DTU

DTU Fotonik
Department of Photonics Engineering

A. Salavrakos



R. Augusiak



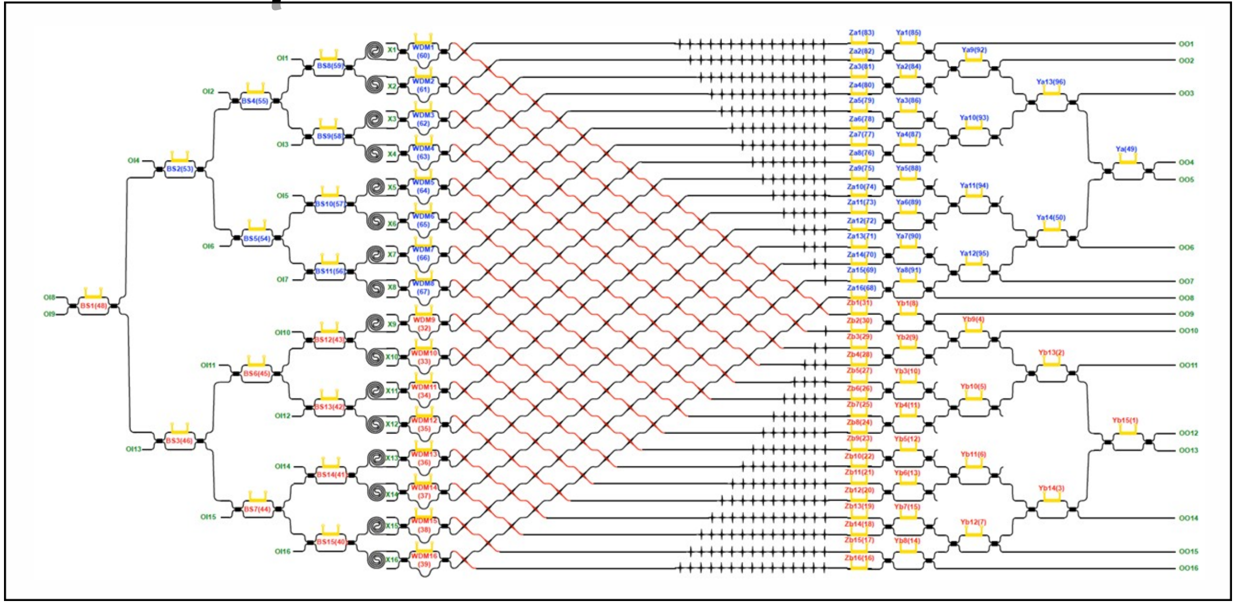
T. Acin

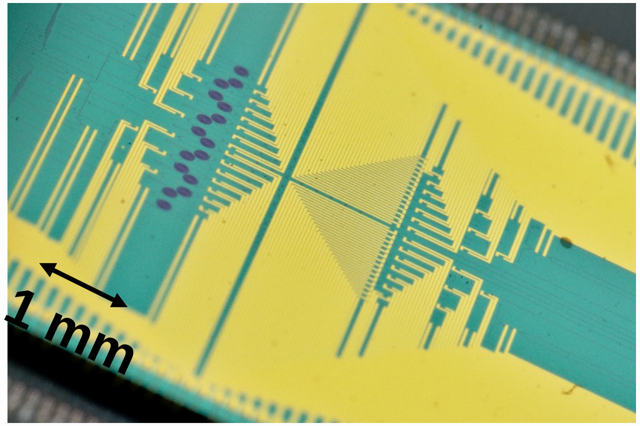
ICFO
The Institute of Photonic Sciences

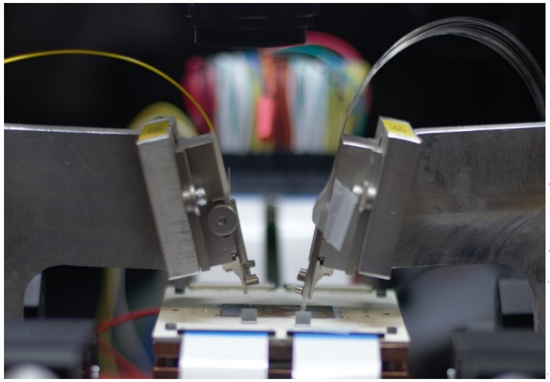
Science **360**, 285 (2018)

Self-testing quantum devices

- *Experiment in Bristol*





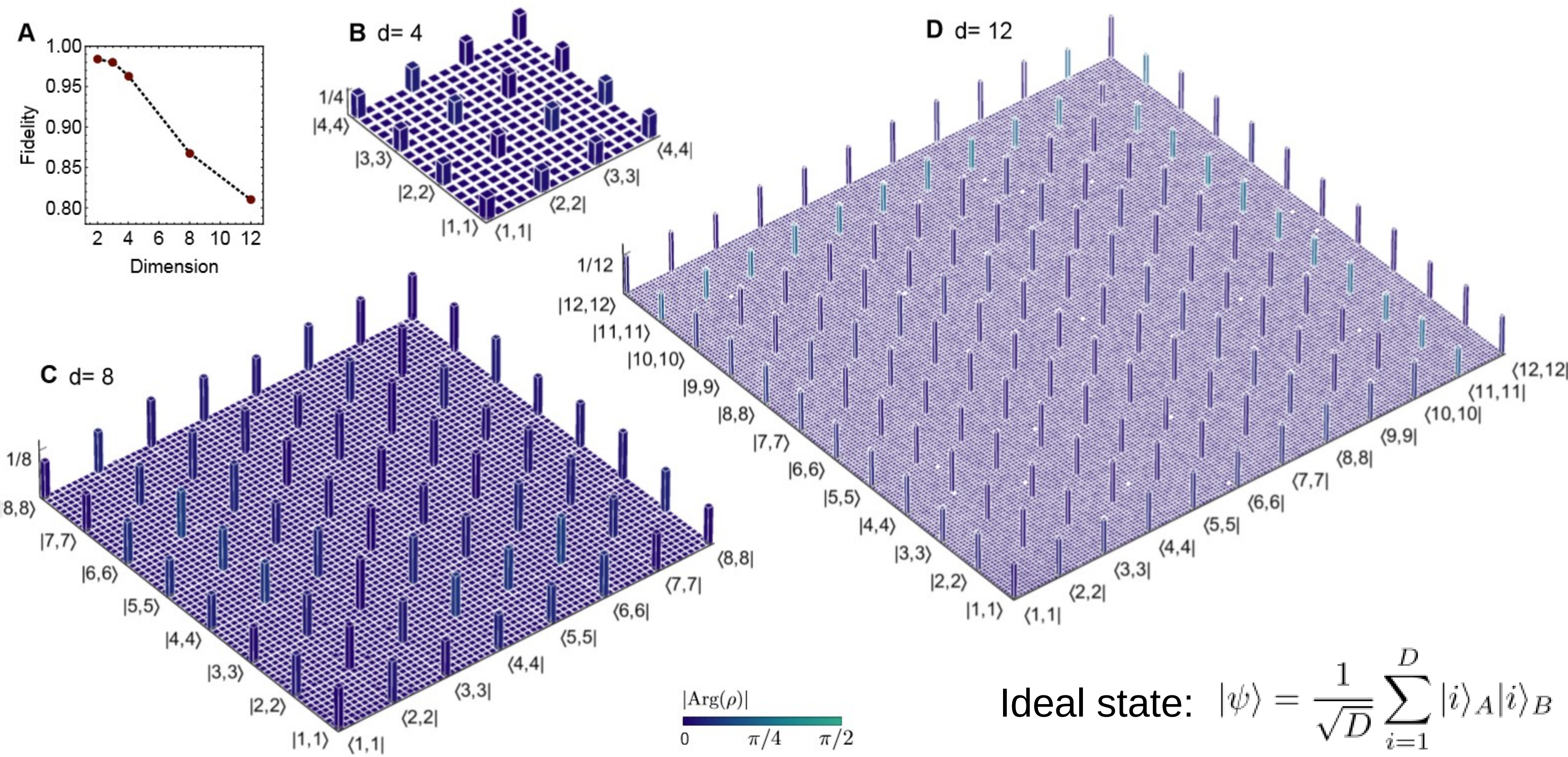


Integration of more than 500 optical components, including:

- 93 active components (phase shifters).
- 16 SFWM single photon sources.

Self-testing quantum devices

- State tomography



Ideal state: $|\psi\rangle = \frac{1}{\sqrt{D}} \sum_{i=1}^{D} |i\rangle_A |i\rangle_B$

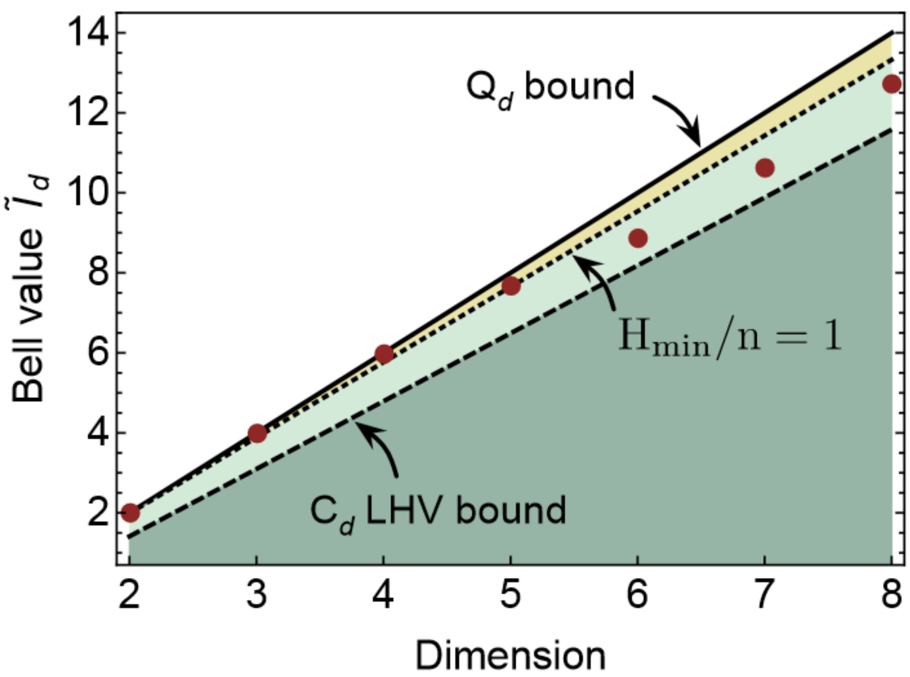
Self-testing quantum devices

Self-testing quantum devices

SATWAP Bell inequalities:

$$\tilde{I}_d = \sum_{i=1}^{2} \sum_{l=1}^{d-1} \langle A_i^l \bar{B}_i^l \rangle$$

A. Salavrakos et al, PRL **119**, 040402 (2017).



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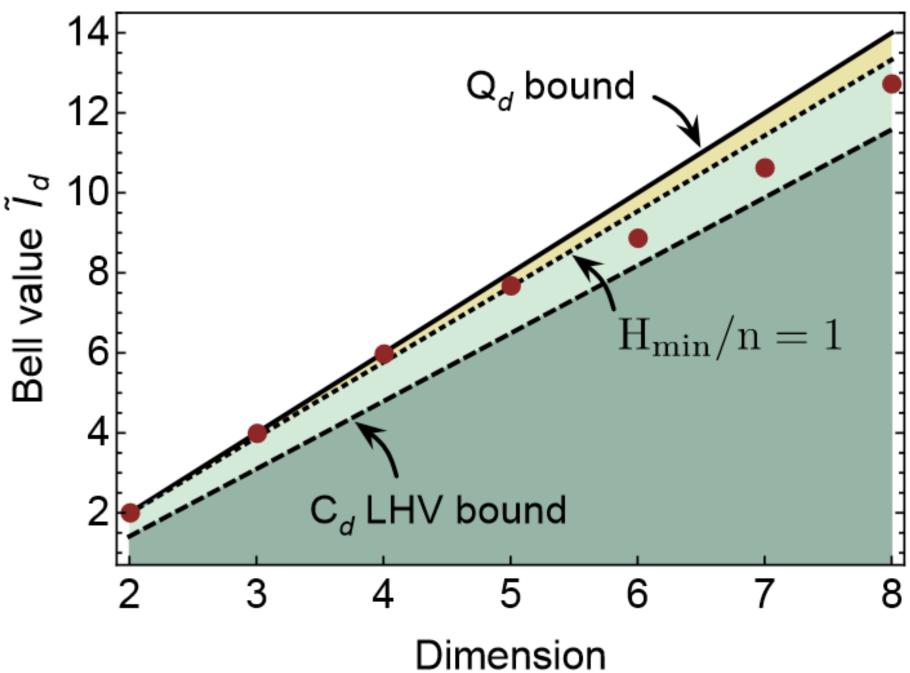
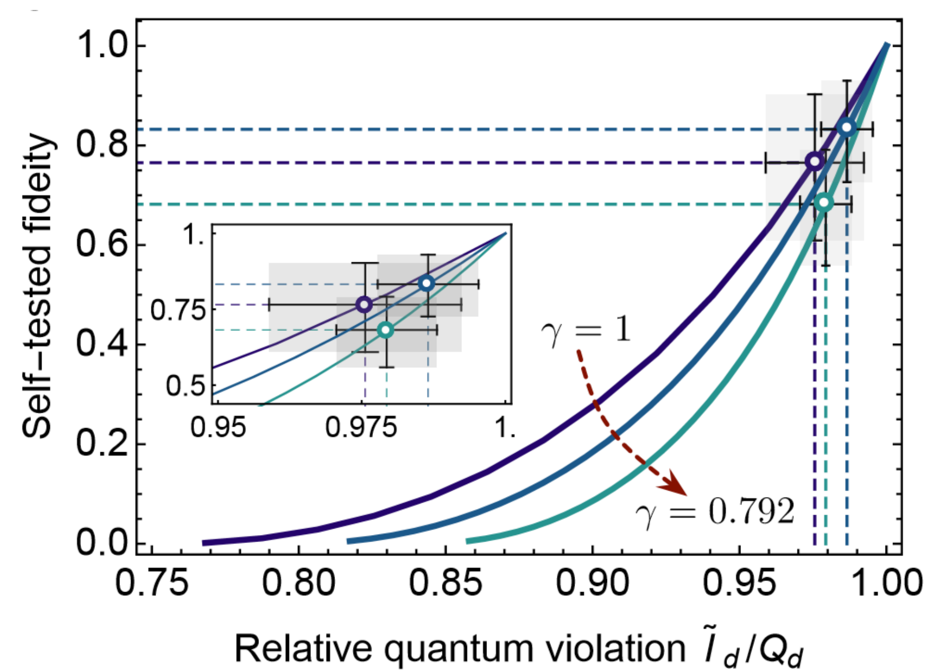


Self-testing:

Device-independent characterisation of quantum devices from nonlocal correlations

Self-testing states: $|\psi_\gamma\rangle = \frac{1}{\sqrt{2+\gamma^2}}(|00\rangle + \gamma|11\rangle + |22\rangle)$



Conclusions

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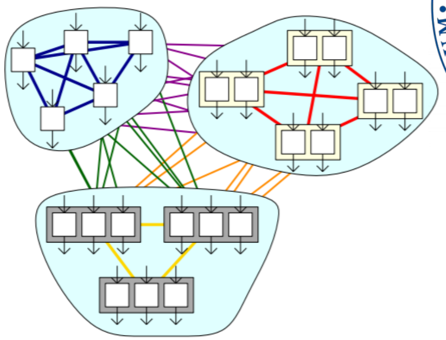
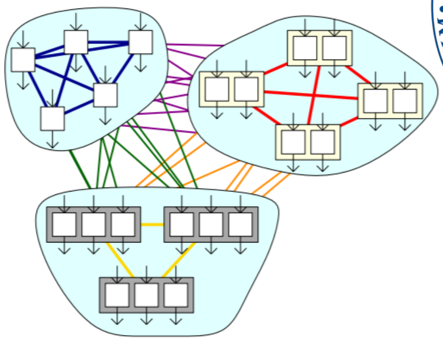
- SDPs and polynomial optimization are **ubiquitous** in quantum information
- Here we showed how they yield accurate **outer approximations** to the LHVM set via **CH of semialgebraic sets**
- Approach is **scalable**, independent of the system size
- SOS certificates allow to **«peek inside»** the black box

Outlook

Outlook



Outlook



- Convergence analysis
- Highly scalable approach
- Self-testing of spin-squeezed states
- Detection of nonlocality in non-Gaussian states
- Incorporate symmetries to the non-commutative case → Quantum upper bounds
- Nonlocality depth quantification

Thanks for your attention!