Shannon Capacity via Real Algebraic Geometry and Strassen's Positivstellensatz

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Strassen, in his seminal 1969 paper

"Gaussian Elimination is Not Optimal"

sent a clear message to the scientific community:

Natural, obvious and centuries-old methods for solving important computational problems may be far from the fastest.

"Gaussian elimination is not optimal"

- multiplying $n \times n$ matrices
- inverting $n \times n$ matrices
- solving a system of n linear equations in n unknowns
- computing the determinant of an $n \times n$ matrix

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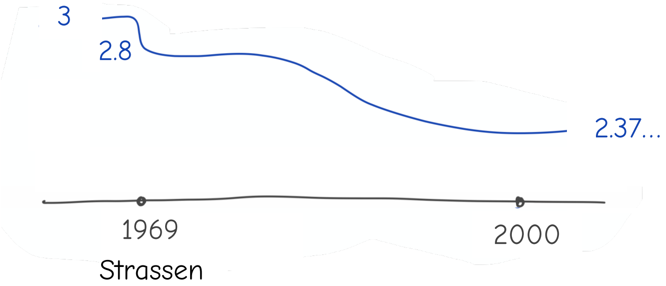
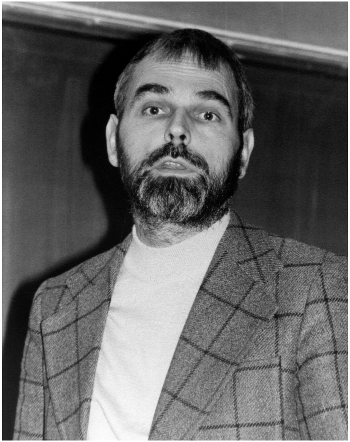
Strassen proved that the obvious $\mathcal{O}(n^3)$ algorithm for these (equivalent) problems is far from optimal

by designing a new one which takes only $\mathcal{O}(n^{2.8})$ operations

The possibility of obtaining even faster algorithms for these central problems set Strassen and many other computer scientists on a quest to obtain them, with the current record below $\mathcal{O}(n^{2.4})$



The quest to understand the matrix multiplication exponent ω is still raging on.

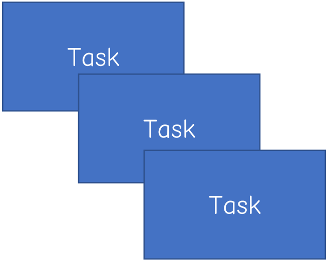


Decades later (1986–1991) Strassen developed his

asymptotic spectrum
Positivstellensatz / duality

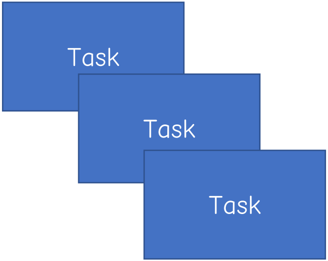
While motivated by trying to understand the complexity of matrix multiplication, this theory is far more general

leading to a broader framework that suits other problems and settings.



Central in this theory of asymptotic spectra:

What is the cost of a task if we have to perform it many times?



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What is the cost of a task if we have to perform it many times?

Arises in numerous parts of mathematics, physics, economics and computer science

- matrix multiplication
- circuit complexity (with Robert Robere)
- direct-sum problems
- Shannon capacity

1. Shannon capacity
2. The asymptotic spectrum of graphs
3. The asymptotic spectrum duality theorem
4. Consequences and new directions

1. Shannon capacity

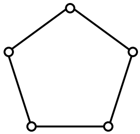
Measures amount of information that can be transmitted over a communication channel.

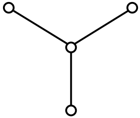
Understanding it has been an open problem in information theory and graph theory since its introduction by Shannon in 1956.

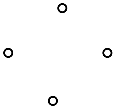
Translates to graph theoretical problem:

channel	graph
protocol	independent set
repeating	strong product

Graph







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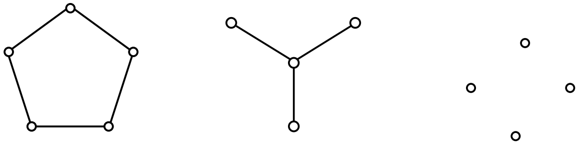
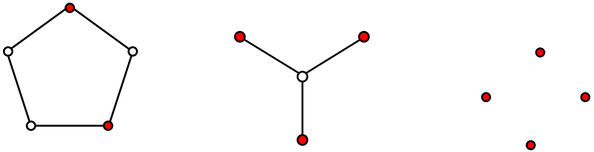


Independent set



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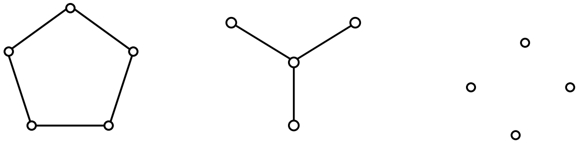

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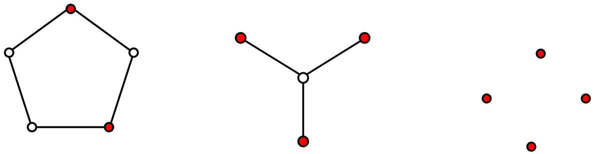

Independence number

$$\alpha(C_5) = 2 \qquad \alpha(S_3) = 3 \qquad \alpha(E_4) = 4$$

Strong product

$G \boxtimes H$

$V(G \boxtimes H) = V(G) \times V(H)$

Adjacency matrix formulation:

The adjacency matrix of $G \boxtimes H$ is the tensor product of those of G and H

Independence number is super-multiplicative

$$\alpha(G \boxtimes H) \geq \alpha(G)\alpha(H)$$

Example

$$\alpha(C_5) = 2$$

$$\alpha(C_5^{\boxtimes 2}) = 5$$

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Shannon capacity

$$\Theta(G) = \sup_n \alpha(G^{\boxtimes n})^{1/n}$$

Example

$$\Theta(C_5) = \sqrt{5} \quad \text{(Lovász)}$$

$$3.2578 \leq \Theta(C_7) \leq 3.3177 \quad \text{(Schrijver–Polak)}$$

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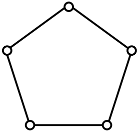
How to upper bound α (and Θ)?

Matrix rank (Haemers bound)



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1 on the diagonal

0 on the non-edges

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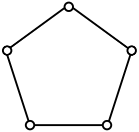

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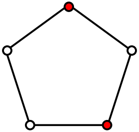
Every independent set gives an identity sub-matrix



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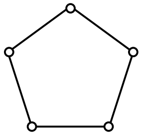

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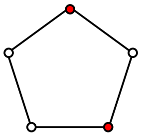

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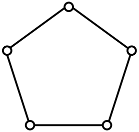
Independence number α is at most rank of any such matrix (and Θ too)

Largest eigenvalue (Lovász theta function)



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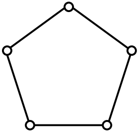

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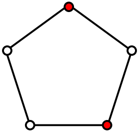

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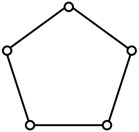

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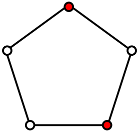

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Independence number is at most largest eigenvalue of such matrix (and Θ too)

Q: How good are the Haemers and Lovász bounds?

2. The asymptotic spectrum of graphs

Models graphs as points in real space

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Defined as the set X of all maps $F : \{\text{graphs}\} \to \mathbb{R}$ that are

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2. multiplicative under $\boxtimes$
3. monotone under cohomomorphism
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Graphs as real points: $G \mapsto (F(G))_{F \in X}$

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Examples of elements of X

- Lovász theta function ϑ
- fractional Haemers bound (Bukh–Cox)
- fractional clique cover number

3. Duality theorem

Recall that

- Shannon capacity is a maximization: $\Theta(G) = \sup_n \alpha(G^{\boxtimes n})^{1/n}$
- Lovász theta gives upper bound: $\Theta(G) \leq \vartheta(G)$

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Lemma
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Q: Are the upper bounds from $F \in X$ powerful enough to reach Θ?

Duality Theorem ("yes", Zuiddam)
Shannon capacity is a minimization: $\Theta(G) = \min_{F \in X} F(G)$

Q: Is the duality theorem non-trivial?

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Theorem (Haemers)
There are G, H for which $\Theta(G \boxtimes H) > \Theta(G)\Theta(H)$

Theorem (Alon)
There are G, H for which $\Theta(G \sqcup H) > \Theta(G) + \Theta(H)$

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Corollary $\Theta \notin X$

Q: How is the duality theorem proven?

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Definition: write

- $G \leq H$ if there is a cohomomorphism $G \to H$
- $G \lesssim H$ when $G^{\boxtimes n} \leq H^{\boxtimes (n+o(n))}$

More General Duality Theorem

$G \lesssim H$ iff $F(G) \leq F(H)$ for all $F \in X$

Ideas:

- Real geometry: inequalities $G \lesssim H$ “cut out” X
- Kadison–Dubois representation theorem, Positivstellensatz
- Extension of Linear Programming Duality
- Works for any “Archimedean” preordered semiring

4. Consequences and new directions

Theorem ("Additivity if and only if multiplicativity", Holzman)
For any graphs G, H the following are equivalent:

(i) $\Theta(G \sqcup H) = \Theta(G) + \Theta(H)$

(ii) $\Theta(G \boxtimes H) = \Theta(G)\Theta(H)$

(iii) There is $F \in X$ such that $F(G) = \Theta(G)$ and $F(H) = \Theta(H)$

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(iii) $\Rightarrow$ (i)

$$\begin{aligned}\Theta(G) + \Theta(H) &\leq \Theta(G \sqcup H) \\ &\leq F(G \sqcup H) = F(G) + F(H) = \Theta(H) + \Theta(H)\end{aligned}$$ $\square$

Example ("Theorems of Haemers and Alon are equivalent")

$$\Theta(G \boxtimes H) > \Theta(G)\Theta(H) \quad \text{iff} \quad \Theta(G \sqcup H) > \Theta(G) + \Theta(H)$$

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Example ("Shannon capacity is not attained at a finite power")

- $C_5 \boxtimes E_1 = C_5$
- $\Theta(C_5 \boxtimes E_1) = \Theta(C_5) = \Theta(C_5)\Theta(E_1)$
- $\Theta(C_5 \sqcup E_1) = \Theta(C_5) + \Theta(E_1) = \sqrt{5} + 1 \neq a^{1/n}$ for $a, n \in \mathbb{N}$

More general theorem

Let $G_1, \ldots, G_n$ be graphs. The following are equivalent:

(i) For every polynomial p we have $\Theta(p(G_1, \ldots, G_n)) = p(\Theta(G_1), \ldots, \Theta(G_n))$

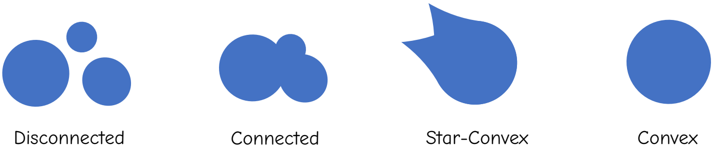
(ii) There exists a polynomial p (depending on all variables) such that $\Theta(p(G_1, \ldots, G_n)) = p(\Theta(G_1), \ldots, \Theta(G_n))$

(iii) There exists $F \in X$ such that $F(G_i) = \Theta(G_i)$ for all i

These we can also make quantitative, relating non-additivity and non-multiplicativity

New directions

- Topological structure of asymptotic spectra



Stronger topological structure $\Rightarrow$ new algorithmic methods
(with Avi Wigderson)

- New notion of graph limits

(with David de Boer and Pjotr Buys)

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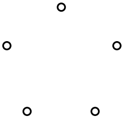
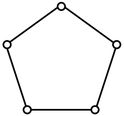
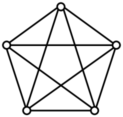
Theorem (de Boer, Buys, Zuiddam)

The space of graphs is not complete: there is a Cauchy sequence that does not converge.

Definition (Fraction graphs)

For $p, q \in \mathbb{N}$ let $E_{p/q}$ be the graph with vertex set $[p]$ and edges between vertices with distance strictly less than q mod p.

Examples

$E_{5/1} =$  $E_{5/2} =$  $E_{5/3} =$ 

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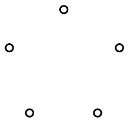
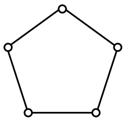
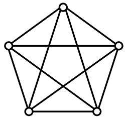
$E_{5/1} =$  $E_{5/2} =$  $E_{5/3} =$ 

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Theorem ("Fraction graphs are ordered as the rationals")

$p/q \leq p'/q'$ iff $E_{p/q} \leq E_{p'/q'}$

Tools

Graphs G, H are equivalent if $G \leq H$ and $H \leq G$

Theorem (DB, B, Z)
Removing any subset of vertices from any fraction graph gives a graph that is equivalent to some fraction graph

Theorem (DB, B, Z)
If G is vertex-transitive and $S \subseteq V(G)$, then for every $F \in X$ we have

$$F(G[S]) \leq F(G) \leq \frac{|V(G)|}{|S|} F(G[S])$$

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Proof ideas:

- Continued fraction expansion of r

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- enough to find that the E_{a_n/b_n} form a Cauchy sequence
- the sequence is not convergent because the fractional clique cover number of any graph is rational, and the fractional clique cover number of this sequence converges to the irrational r

Let $F \in X$ or $F = \Theta$

Theorem (Schrijver, Polak)
$\mathbb{Q}_{\geq 2} \to \mathbb{R} : p/q \mapsto F(Ep/q)$ is left-continuous at p/q when p/q is integer.

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Proof: uses the methods we developed for our non-completeness proof

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We have used the ideas developed here to get new bounds on the Shannon capacity of certain odd cycle graphs.

Problems

- What are the elements of the asymptotic spectrum of graphs?
- What is the structure of X?
- What bounds can we obtain on the Shannon capacity via graph limits?
- What other problems in math, CS and physics have asymptotic spectrum duality?
- Lovász theta function for hypergraphs?

Strassen's asymptotic spectra duality

S semiring	S semiring of graphs
$1 \in S$	$E_1 \in S$
$\leq$ semiring preorder	$\leq$ cohomomorphism
bounded, preserves $\mathbb{N}$	bounded, preserves $\mathbb{N} = \{E_1, E_2, \ldots\}$
subrank	independence number
$\mathrm{Q}(s) = \max\{n \in \mathbb{N} : n \leq s\}$	$\mathrm{Q}(G) = \alpha(G)$
asymptotic subrank	Shannon capacity
$\widetilde{\mathrm{Q}}(s) = \sup_n \mathrm{Q}(s^n)^{1/n}$	$\widetilde{\mathrm{Q}}(G) = \Theta(G)$
duality	duality
$\widetilde{\mathrm{Q}}(s) = \min_{F \in X} F(s)$	$\Theta(G) = \min_{F \in X} F(G)$
$X =$ semiring monotones	$X =$ asymptotic spectrum of graphs