A 3-POINT BOUND FOR DISTANCE-AVOIDING SETS ON THE SPHERE

Fernando Mário de Oliveira Filho (TU Delft)
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Joint work with:
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THEOREM. $\vartheta'(G) \geq \alpha(G)$.

Binary Codes

- $V = \{0,1\}^n$
- $xy \in E$ if $0 < d(x, y) < d$
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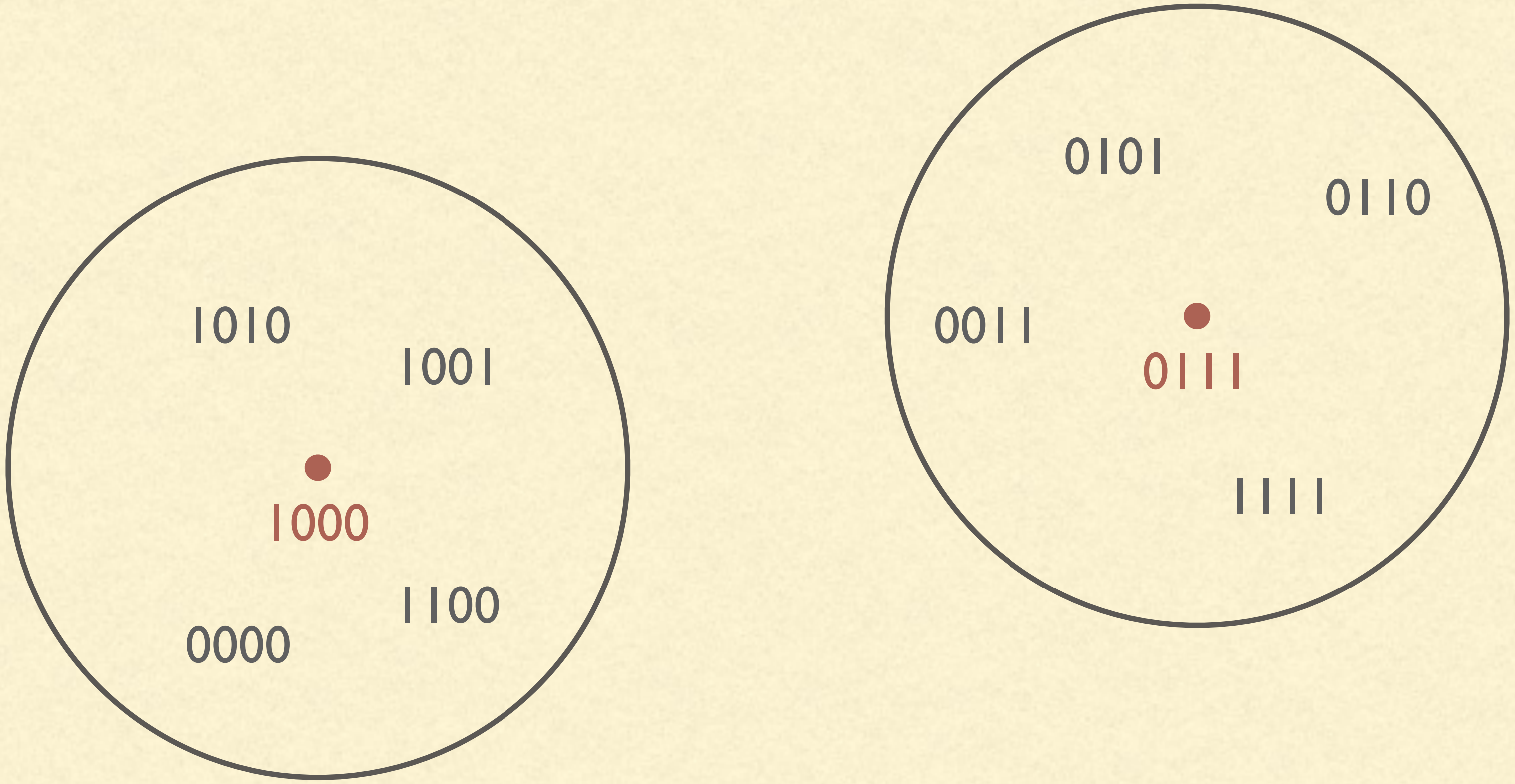

Kissing Number

- $V = S^{n-1} = \{ x \in \mathbb{R}^n : \|x\| = 1 \}$
- $xy \in E$ if $0 < \angle x, y < 60°$
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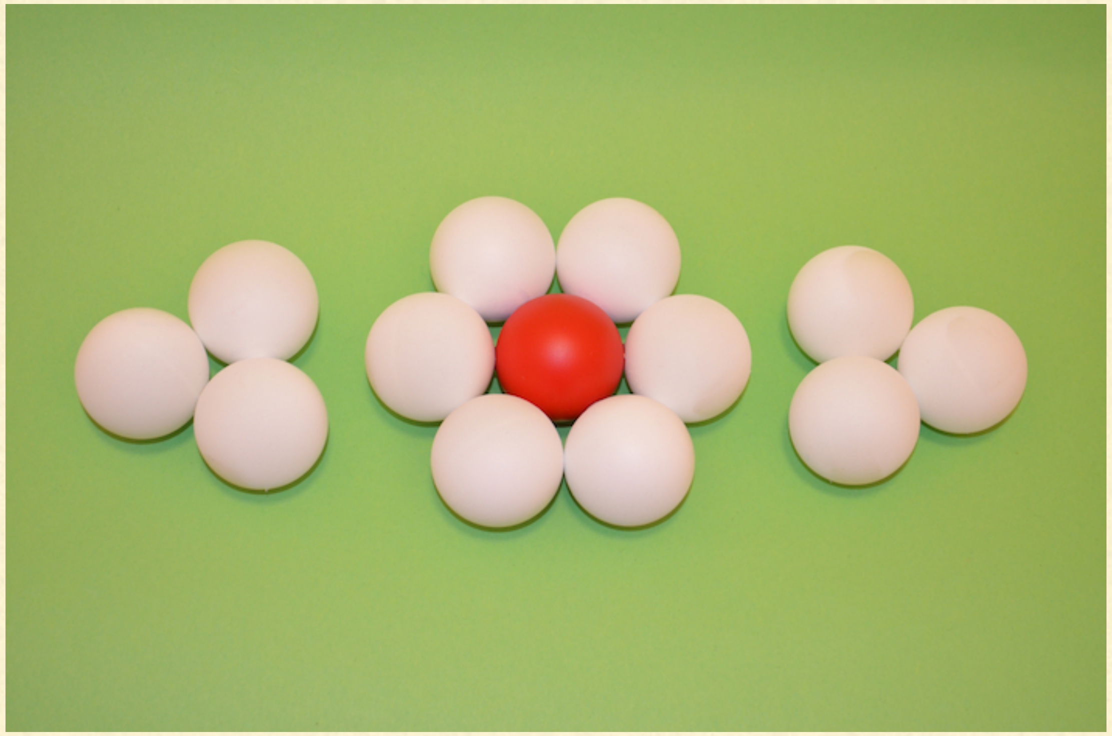

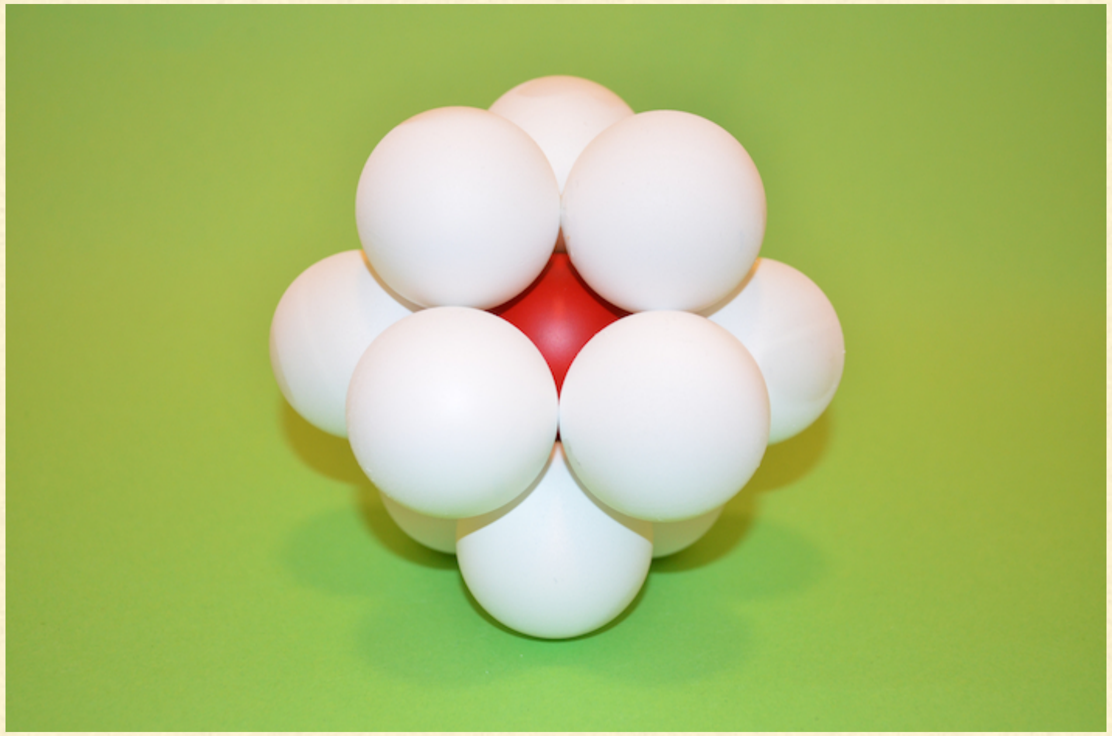

Sphere Packings

- $V = \mathbb{R}^n$
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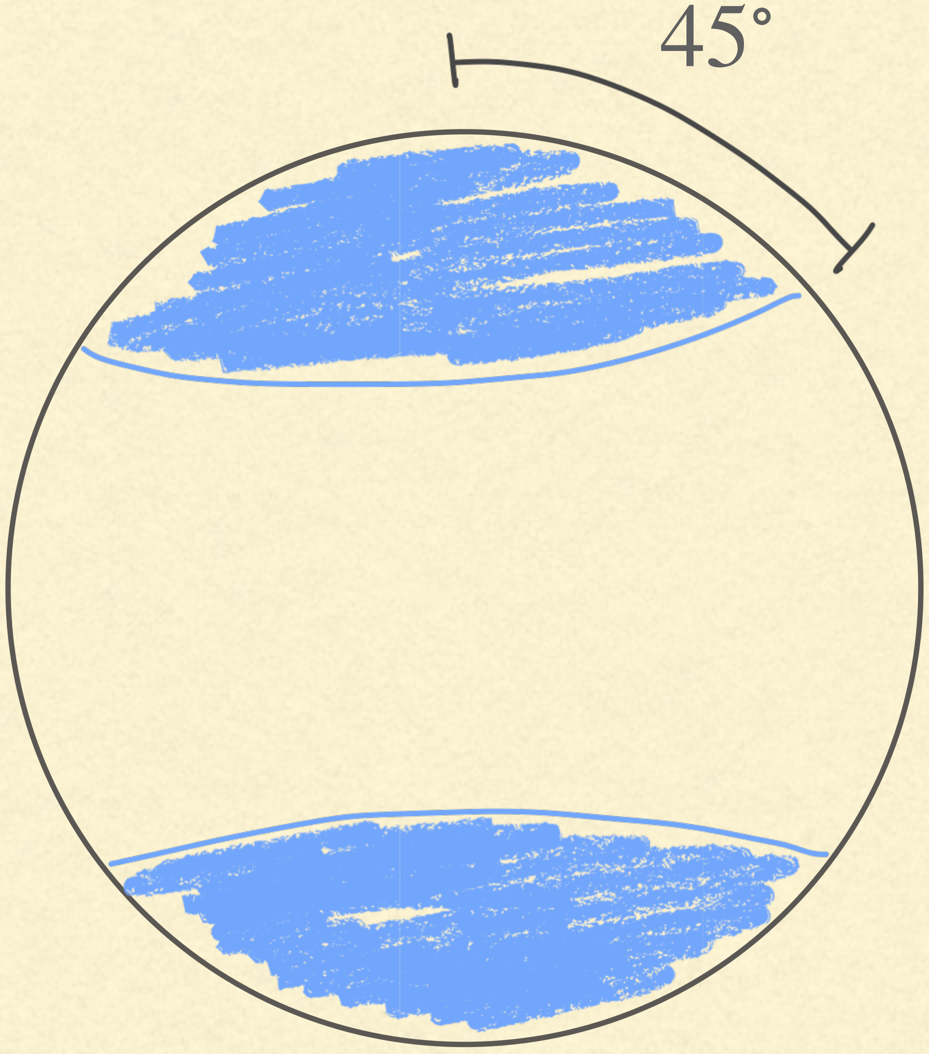
DOUBLE-CAP CONJECTURE (KALAI)



Distance-avoiding Sets

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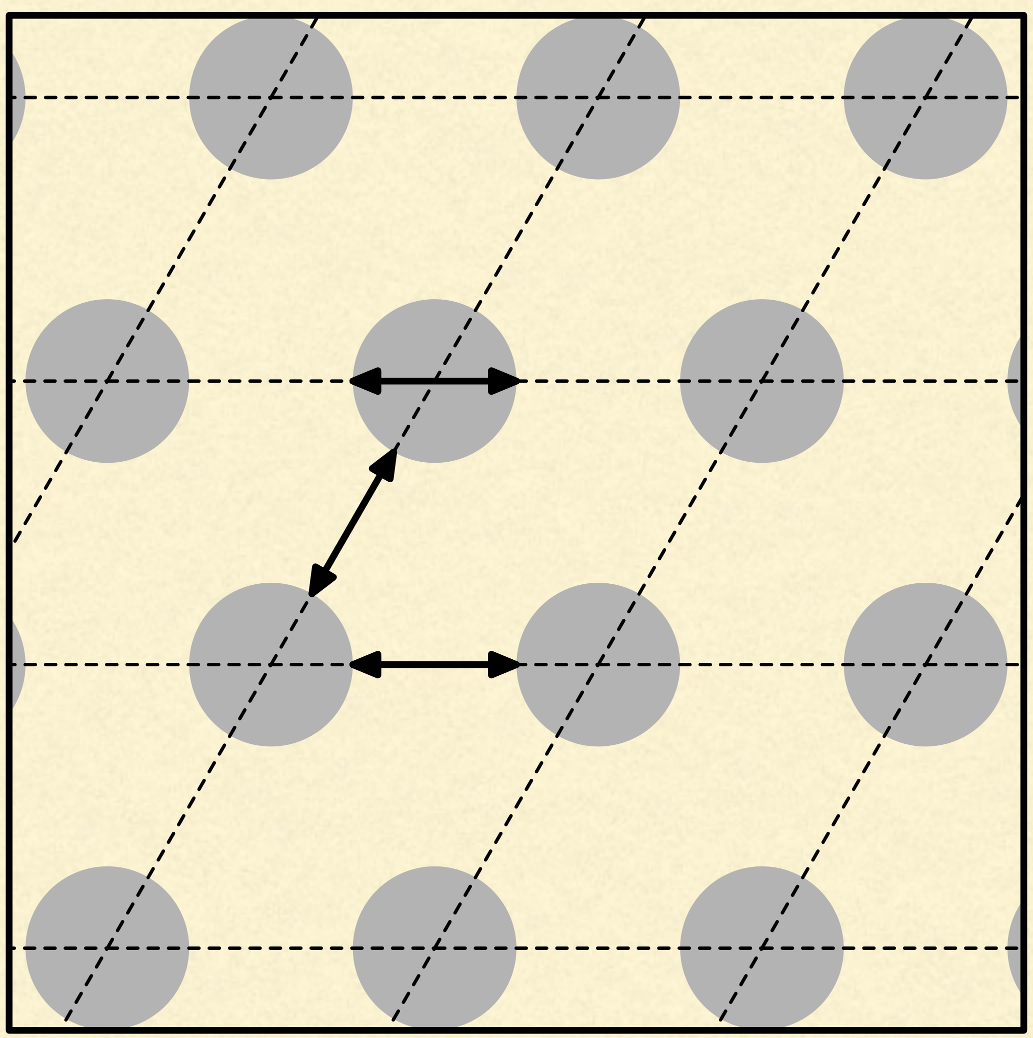

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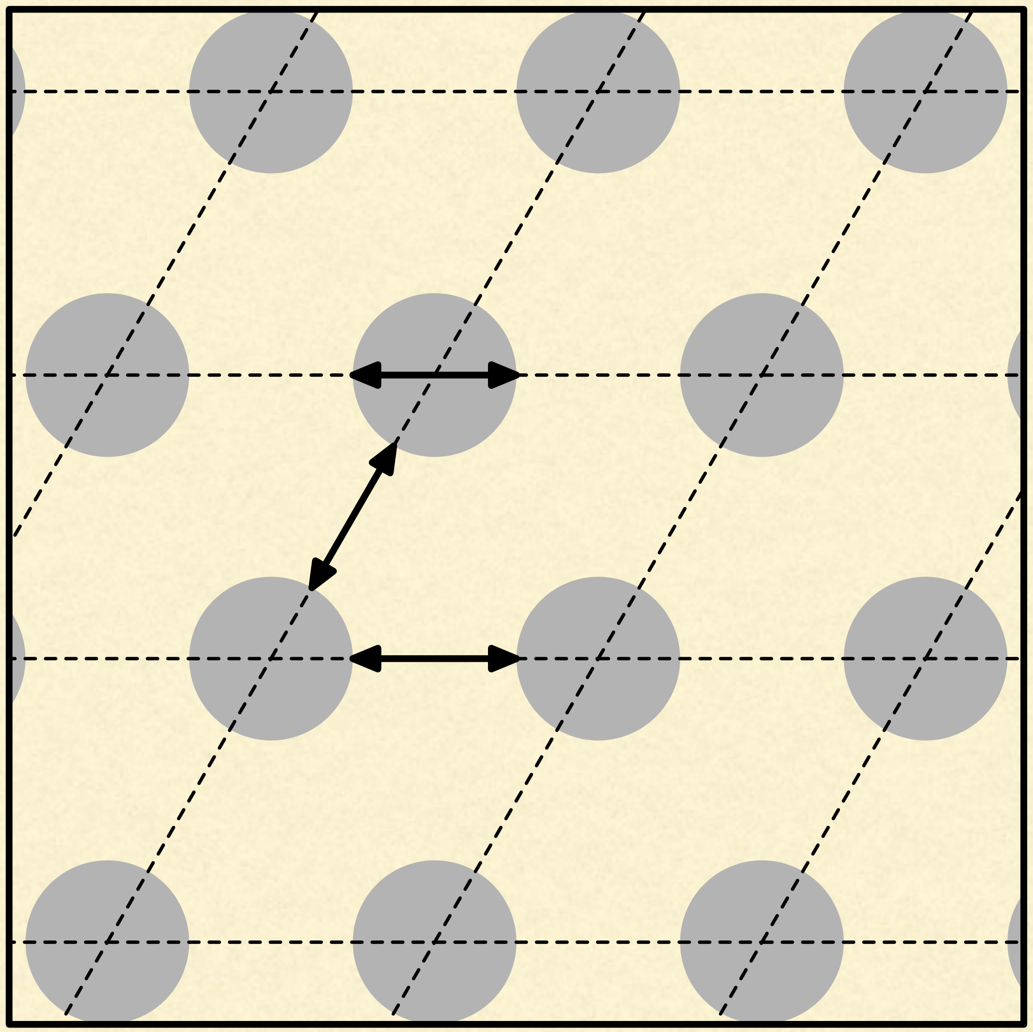

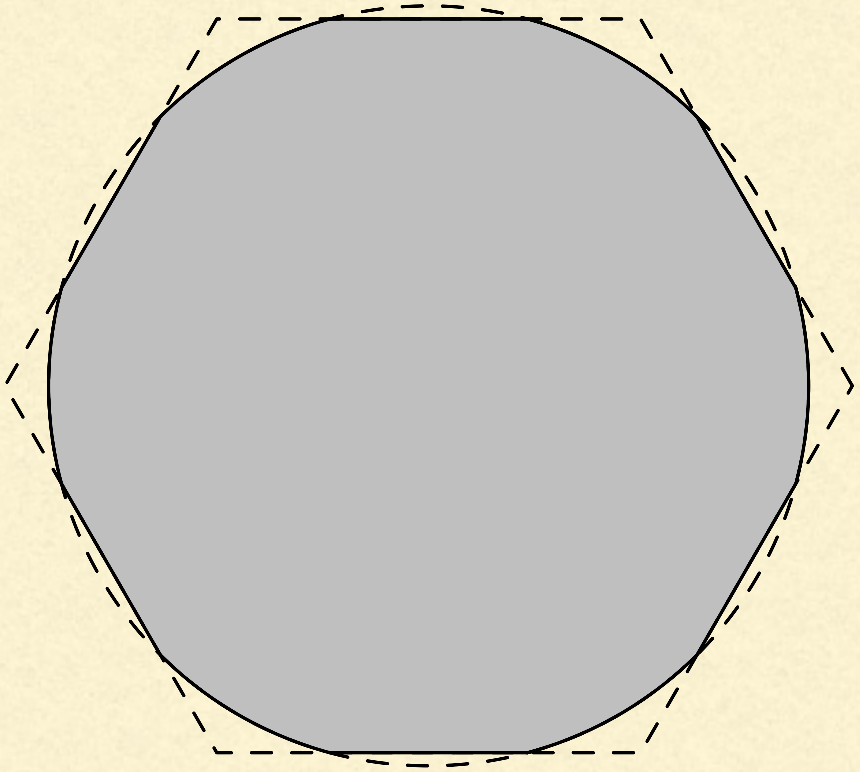

TOPOLOGICAL PACKING GRAPH

- Any finite clique subset of an open clique
- Any point has a complete neighborhood

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Locally Independent Graph

- Any compact independent set subset of an open independent set
- Any point has an independent neighborhood

	Compact	Not Compact
Topological Packing	• Binary codes • Spherical codes	• Sphere packing
Locally Independent	• Binary codes • Witsenhausen's problem	• Distance-avoiding sets in Euclidean space

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\min\ & \lambda \\
& Z(x,x) \leq \lambda - 1 \quad \text{for all } x \in S^{n-1}, \\
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& Z \succeq 0, \\
& Z \in C(S^{n-1} \times S^{n-1})
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- $V = S^{n-1}$
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$$F(x, y, z) = 0 \quad \text{if } \{x, y, z\} \text{ not independent},$$

$$A\colon V^2 \to \mathbb{R},$$

$$F\colon V^3 \to \mathbb{R} \text{ is symmetric and slice-positive}$$

	2-POINT	3-POINT
BINARY CODES	Delsarte (1973)	Schrijver (2005)
SPHERICAL CODES	Delsarte, Goethals, and Seidel (1977)	Bachoc and Vallentin (2007)
WITSENHAUSEN	Bachoc, Nebe, O., Vallentin (2008)	**THIS**
SPHERE PACKINGS	Cohn and Elkies (2003)	Cohn, de Laat, and Salmon (2022)
EUCLIDEAN DISTANCE-AVOIDING	O. and Vallentin (2008)	?

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- Packing graphs: Dobre, Dür, Frerick, and Vallentin (2016)
- Locally independent graphs: DeCorte, O., and Vallentin (2020)

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THEOREM (KURYATNIKOVA, VERA 2019). $\bigcup_{r \geq 0} \mathscr{C}_r(V)$ contains the algebraic interior of $\mathrm{COP}(V) \cap C(V)$.

FINITE SETTING

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- Pólya's Theorem: $p(x) \geq 0$ for all $x \geq 0$ iff there is $r \geq 0$ such that $q(x) = \langle \mathbf{1}, x \rangle^r \langle Ax, x \rangle$ has nonnegative coefficients

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- Now note that $q(x) = \langle \sigma(A \otimes \mathbf{1}^{\otimes r}), x^{\otimes (r+2)} \rangle$
- So the coefficients of q are nonnegative iff $\sigma(A \otimes \mathbf{1}^{\otimes r}) \geq 0$

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THEOREM. $\mathrm{CP}(V) \subseteq \bigcap_{r \geq 0} \mathscr{C}_r(V)^*$ and $\gamma_r(G) \to \alpha(G)$.

Dimension	Lower Bound	2-point bound	3-point bound
3	0.2929	0.33…	0.303292
4	0.1817	0.25	0.201443
5	0.1161	0.2	0.142338
6	0.0756	0.166…	0.106725
7	0.0498	0.1428…	0.084785
8	0.0331	0.125	0.070605