Iterative approximation schemes for SDPs using the factor width cone

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Introduction

An SDP in primal standard form is given by

Primal SDP

$$\begin{aligned} v_P^* = \inf \ & \langle C, X \rangle \\ \text{s.t.} \ & \langle A_i, X \rangle = b_i \ \forall i \in [m] \\ & X \in \mathbb{S}^n_+. \end{aligned}$$

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- Semidefinite programming is more powerful and versatile than LP or SOCP,
- but it is also more expensive (IPM scales as $O(mn^3 + m^2n^2)$).
- We can solve LPs with millions of variables and constraints,
- while for SDPs we are limited to $\approx$ 10.000 constraints and matrices of size $\approx 1000 \times 1000$.

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Solving any instance of

$$\begin{aligned} v_P^{(0)} = \inf \langle C, X \rangle \\ \text{s.t.} \langle A_i, X \rangle = b_i \; \forall i \in [m] \\ X \in \mathcal{K} \subseteq \mathbb{S}^n_+, \end{aligned}$$

yields an upper bound on v_P^*.

What cones to use as $\mathcal{K}$?

Ahmadi and Majumdar[1] considered the cones of diagonally dominant and scaled diagonally dominant matrices.

Definition

A symmetric matrix $S \in \mathbb{S}^n_+$ is said to be *diagonally dominant* if

$$S_{i,i} \geq \sum_{j \neq i} |S_{i,j}| \quad \forall i \in [n].$$

We call $S \in \mathbb{S}^n_+$ *scaled diagonally dominant* if $\exists$ diagonal matrix $D > 0$ such that DSD is diagonally dominant.

[1] [2] A. A. Ahmadi and A. Majumdar, "DSOS and SDSOS optimization: LP and SOCP-based alternatives to sum of squares optimization," 2014 48th Annual Conference on Information Sciences and Systems (CISS), 2014, pp. 1-5, doi: 10.1109/CISS.2014.6814141.

DD_n and SDD_n

We denote

$$DD_n = \{X \in \mathbb{S}^n : X \text{ is diagonally dominant}\} \subseteq \mathbb{S}^n_+$$

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Example

Let $G = (V, E)$ be a random graph with $n = 50$ vertices. Consider the Lovász-ϑ number of G

$$\vartheta(G) = \min \left\{ t : tI + \sum_{\{i,j\} \in E} y_{i,j} E_{i,j} - J \succeq 0, t \in \mathbb{R}, y \in \mathbb{R}^{|E|} \right\} = 7.6311,$$

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Replacing the SDP constraint by a DD_n (resp. SDD_n) constraint lead to an objective value of 33 (resp. 26.298).

Update data matrices

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0th iteration

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1st iteration

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Identity is feasible

Note that $X = I$ is feasible for (2) because

$$\langle A_i^{(1)}, I \rangle = \langle \mathrm{Chol}(\mathrm{X}^*)\, A_i^{(0)}\, \mathrm{Chol}(\mathrm{X}^*), I \rangle = \langle A_i^{(0)}, X^* \rangle = b_i.$$

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Moreover, $\langle C^{(1)}, I\rangle = v_P^{(0)}$. Hence,

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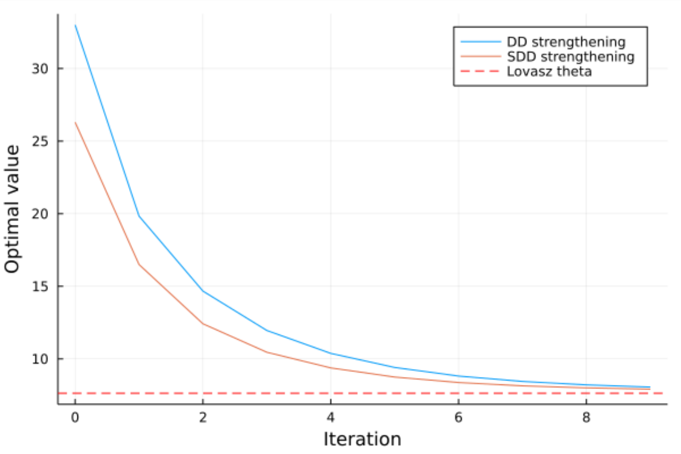
- Reiterating this procedure leads to a non-increasing sequence $\{v_P^{(k)}\}_{k \in \mathbb{N}}$.
- Let us compare the performance of the two cones.

SDD_n vs DD_n

Figure 1: Comparison of performance of the iterative scheme using the DD_n and the SDD_n cone on the problem of computing Lovász-ϑ on a random graph with 50 vertices.



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- a decrease phase, solving the relaxations iteratively and updating the data matrices (the procedure we just learned)
- and a centering phase, producing feasible solutions with the same objective value that are closer to the central path of the SDP.

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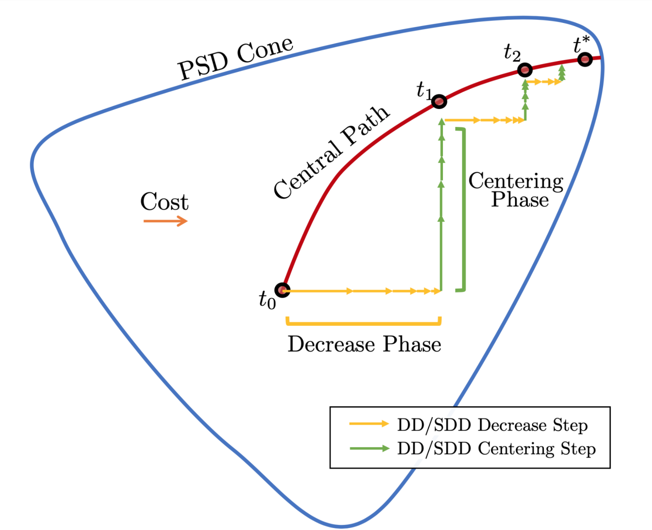
Visualisation

Figure 2: Figure from [5] visualizing the two phases of the algorithm



Generalizing the SDD cone

Lemma

The cone of scaled diagonally dominant matrices can be written as

$$SDD_n = \left\{ \sum_{i \in I} x_i x_i^T : x_i \in \mathbb{R}^n, \mathrm{supp}(x_i) \leq 2 \right\}.$$

This cone is known as the cone of matrices of factor-width 2.

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Factor-width k cone

We define the factor-width k cone as

$$\mathrm{FW}_n(k) := \left\{ \sum_{i \in I} x_i x_i^T : x_i \in \mathbb{R}^n, \mathrm{supp}(x_i) \leq k \right\}.$$

Clearly, $\mathrm{FW}_n(k) \subseteq \mathbb{S}^n_+$.

Optimizing over the $\mathrm{FW}_n(k)$ cone

- Ding and Lim[4] studied this class of cones and their duals and developed self-concordant barrier functions.

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- Algorithm by Sznaier and Roig-Solvas generalizes* to $\mathrm{FW}_n(k)$ for $k \leq n$.

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Discussion

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- Another promising idea by Zheng, Sootla[6] is using block-factor width 2 matrices.

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Block-factor width 2 matrices

The idea is as follows.

- Let $\gamma = \{k_1, \dots, k_p\}$ be a partition of n (i.e. $k_i \in \mathbb{N}$ and $k_1 + \cdots + k_p = n$).
- Let A be a block matrix

$$A = \begin{pmatrix} A_{1,1} & A_{1,2} & \dots & A_{1,p} \\ A_{2,1} & A_{2,2} & \dots & A_{2,p} \\ \vdots & \vdots & \ddots & \vdots \\ A_{p,1} & A_{p,2} & \dots & A_{p,p} \end{pmatrix},$$

  consisting of matrices $A_{i,j} \in \mathbb{R}^{k_i \times k_j}$.
- Define the lift operator $L_{i,j} = \mathbb{R}^{k_i + k_j \times k_i + k_j} \to \mathbb{R}^{n \times n}$

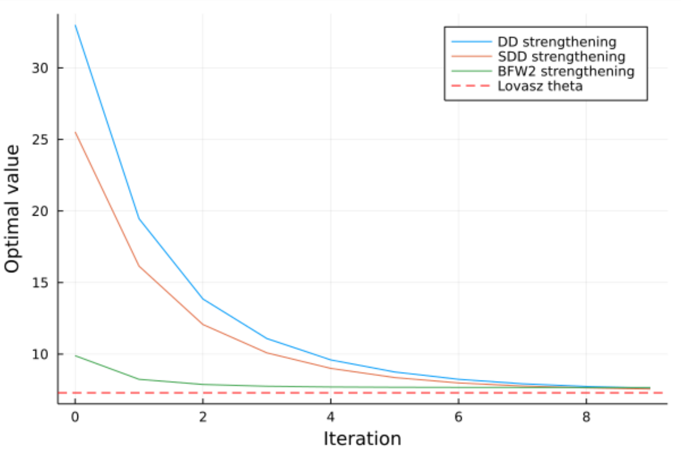
$$\left(L_{i,j} \begin{bmatrix} A_{i,i} & A_{i,j} \\ A_{j,i} & A_{jj} \end{bmatrix} \right)_{k,l} = \begin{cases} A_{i,j} & k \in \{i,j\}, l \in \{i,j\} \\ 0 & \text{otherwise.} \end{cases}$$

Block-factor width 2 matrices (ctd)

- Now replace the psd condition $X \succeq 0$ by
  - $X = \sum_{i<j} L_{i,j} \begin{bmatrix} A_{i,i} & A_{i,j} \\ A_{j,i} & A_{jj} \end{bmatrix}$,
  - such that $\begin{pmatrix} A_{i,i} & A_{i,j} \\ A_{j,i} & A_{j,j} \end{pmatrix} \in \mathbb{S}_+^{k_i+k_j}$.
  - We call X a block-factor width 2 matrix.
- Let $\mathrm{FW}_n(\gamma, 2) = \{X \in \mathbb{S}^n : X \text{ is a block-factor width 2 matrix wrt } \gamma\}$.
- Note that $\mathrm{FW}_n(2) \subseteq \mathrm{FW}_n(\gamma, 2)$ for all partitions γ of n.
- This cone is coarser that $\mathrm{FW}_n(2)$ and scales better than $\mathrm{FW}_n(3)$.

SDD_n vs DD_n vs $\mathrm{FW}_n((10, 10, \ldots, 10), 2)$

Figure 3: Comparison of performance of the iterative scheme using DD_n, SDD_n and $\mathrm{FW}_n((10, 10, \ldots, 10), 2)$ cone on the problem of computing Lovász-ϑ on a random graph with 50 vertices.



References

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Thank you for your attention

Questions?