The Lasserre hierarchy for equiangular lines with a fixed angle

David de Laat (TU Delft)

Joint with
Fabrício Machado and
Willem de Muinck Keizer (TU Delft)

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1. The equiangular lines problem

The equiangular lines problem

What is the maximum number $N(n)$ of lines intersecting in a single point in $\mathbb{R}^n$ having pairwise have the same angle?

$N(2) = 3$ (lines through opposite vertices of a regular hexagon)

$N(3) = 6$ (lines through opposite vertices of an icosahedron)

For $n = 18, 19, 20, 42$, and many higher dimensions the determination of $N(n)$ is still open

The complex version of this problem is relevant in quantum information (existence of SIC-POVMs)

The equiangular lines problem with a fixed angle

What is the maximum number $N_\alpha(n)$ of lines intersecting in a single point in $\mathbb{R}^n$ having pairwise the same angle $\arccos \alpha$?

For each $n \geq 15$ we can compute $N(n)$ by computing $N_\alpha(n)$ for a given finite list of inner products of the form $\alpha = 1/a$ with $a \geq 3$ odd

Determination of $N_{1/a}(n)$ is open for many values of a and n

More generally, we can consider the the spherical finite distance problem, which asks for the largest set of points on a unit sphere having pairwise inner products from a finite prescribed set

Recent breakthrough results

Bukh, *Bounds on equiangular lines and on related spherical codes*, SIAM J. Discrete Math. 30 (2016)

For every α there is a constant c_α such that $N_\alpha(n) \leq c_\alpha n$ for all n

Balla, Dräxler, Keevash, and Sudakov, *Equiangular lines and spherical codes in Euclidean space*, Invent. Math. 211 (2018)

For $\alpha = 1/a$ with $a > 3$, $\sup_{n\to\infty} N_\alpha(n)/n$ is at most 1.93

Jiang, Tidor, Yao, Zhang, and Zhao, *Equiangular lines with a fixed angle*, Ann. of Math. (2) 194 (2021)

For every integer $k \geq 2$ and all sufficiently large n we have

$$N_{1/(2k-1)}(n) = \lfloor k(n-1)/(k-1) \rfloor$$

2. LP and SDP bounds

The Delsarte, Goethals, Seidel LP bound

Adaptation of the Delsarte LP bound for binary codes to spherical codes

Let

$$f(u) = 1 + \sum_{k=1}^{d} f_k P_k^n(u),$$

where P_k^n is the ultraspherical polynomial for S^{n-1} of degree d

The problem

$$\begin{aligned} \text{minimize} \quad & f(1) \\ \text{subject to} \quad & f(u) \leq 0, \qquad u \in D, \\ & f_1, \ldots, f_d \geq 0 \end{aligned}$$

gives an upper bound on the largest size of a set in S^{n-1} with pairwise inner products in D

The Lovász theta prime number

The equiangular lines problem with fixed angle $\arccos \alpha$ is an independent set problem for the graph $G = (S^{n-1}, \sim)$, where $x \sim y$ if $x \cdot y = \pm\alpha$

Can compute an upper bound using the Lovász ϑ'-number:

$$
\begin{aligned}
&\text{minimize} && M \\
&\text{subject to} && K(x,x) \leq M - 1, && x \in S^{n-1}, \\
& && K(x,y) \leq -1, && x \neq y \text{ and } x \not\sim y, \\
& && K \in \mathcal{C}(S^{n-1} \times S^{n-1})_{\succeq 0}
\end{aligned}
$$

Here $\mathcal{C}(S^{n-1} \times S^{n-1})_{\succeq 0}$ is the cone of positive kernels

The Lovász theta prime number

Using Schoenberg's characterization we get a reduction to the linear programming bound [Bachoc-Nebe-Oliveira-Vallentin 2009]

Schoenberg's characterization

A positive, $O(n)$-invariant kernel $K\colon S^{n-1} \times S^{n-1} \to \mathbb{R}$ can be written as

$$K(x,y) = \sum_{k=0}^{\infty} c_k P_k^n(x \cdot y)$$

with $c_0, c_1, \cdots \geq 0$, where convergence is uniform absolute

k-point bounds for packing and energy minimization

[Bachoc-Vallentin, 2008] Three-point bound for spherical codes
Adaptation of the three-point bound for binary codes by [Schrijver, 2005]

[Cohn-Woo 2012] Three-point bound for energy minimization

[Barg-Yu 2013] Application of the Bachoc-Vallentin bound to equiangular lines with a fixed angle

[Bilyk-Ferizović-Glazyrin-Matzke-Park-Vlasiuk 2021+] Energy minimization with multivariate kernels

[Cohn-L.-Salmon 2022+] Three-point bounds for sphere packing

[L.-Machado-Oliveira-Vallentin 2021] k-point bounds for equiangular lines with a fixed angle

[Yu 2022+] Analysis for the case $k = 4$

The Lasserre hierarchy for packing and energy minimization

[L.-Vallentin 2015] Generalization of the Lasserre hierarchy for the independent set problem to topological packing graphs

[L. 2020] Computation of the second level of the above hierarchy for an energy minimization problem in dimension 3

[Cohn-Salmon 2022$+$] Noncompact adaptation

Convergence proof: in principle we can solve any equiangular lines problem by computing a high enough level of the Lasserre hierarchy

In general it is difficult to compute higher levels of the hierarchy

3. The Lasserre hierarchy for equiangular lines

Lasserre hierarchy for packing problems

Lasserre hierarchy for packing problems [L.-Vallentin 2015]

$$\begin{aligned} \text{minimize} \quad & K(\emptyset, \emptyset) \\ \text{subject to} \quad & A_t K(S) \leq -1_{I_{=1}}(S), \qquad S \in I_{2t} \setminus \{\emptyset\}, \\ & K \in \mathcal{C}(I_t \times I_t)_{\succeq 0} \end{aligned}$$

- I_t is the set of independent sets of size at most t
- $\mathcal{C}(I_t \times I_t)_{\succeq 0}$ is the cone of positive kernels on I_t
- $1_{I_{=1}}$ is the indicator function of $I_{=1}$, the set of singletons
- A_t is the operator

$$A_t K(S) = \sum_{\substack{J, J' \in I_t \\ J \cup J' = S}} K(J, J')$$

(This is a continuous version of the adjoint of the operator that sends a vector to its moment matrix)

Lasserre hierarchy for packing problems

Lemma

The size of an independent set is at most the optimal value of:

$$\begin{aligned} \text{minimize} \quad & K(\emptyset, \emptyset) \\ \text{subject to} \quad & A_t K(S) \leq -1_{I_{=1}}(S), \qquad S \in I_{2t} \setminus \{\emptyset\}, \\ & K \in \mathcal{C}(I_t \times I_t)_{\succeq 0} \end{aligned}$$

Proof: If C is an independent set and K a feasible kernel, then

$$0 \leq \sum_{\substack{J,J' \in I_t \\ J,J' \subseteq C}} K(J, J') = \sum_{\substack{S \in I_{2t} \\ S \subseteq C}} A_t K(S) \leq K(\emptyset, \emptyset) - |C|,$$

which shows any feasible solution gives an upper bound □

Symmetry

- The action of $O(n)$ on S^{n-1} extends to an action on I_t
- We may assume the kernels $K\colon I_t \times I_t \to \mathbb{R}$ are $O(n)$ invariant:

$$K(\gamma x, \gamma y) = K(x, y) \text{ for } \gamma \in O(n)$$

Constructing positive, $O(n)$-invariant kernels $I_t \times I_t \to \mathbb{R}$

- The space I_t decomposes as a disjoint union of finitely many orbits
- Fix an orbit representative R_i for each orbit X_i
- Let H_i be the stabilizer subgroup of $O(n)$ with respect to R_i
- Let s_i be a function $X_i \to O(n)$ such that $s_i(J)R_i = J$ for $J \in X_i$
- Let $\pi\colon O(n) \to \mathrm{GL}(V)$ be a unitary, irreducible representation
- Let $e_{\pi,i,1}, \ldots, e_{\pi,i,d_{\pi,i}}$ be a basis of the space of invariants

$$V^{H_i} = \{v \mid \pi(h)v = v \text{ for all } h \in H_i\}$$

For F a positive semidefinite matrix, the kernel K defined by

$$K(J_1, J_2) = \sum_{j_1,j_2} F_{(i_1,j_1),(i_2,j_2)} \langle \pi(s_{i_1}(J_1)) e_{\pi,i_1,j_1}, \pi(s_{i_2}(J_2)) e_{\pi,i_2,j_2} \rangle$$

for $J_1 \in X_{i_1}$ and $J_2 \in X_{i_2}$, is positive and $O(n)$ invariant

(Uniform absolute convergence follows from extending [Bochner 1941])

Intuition behind the construction

Let H_k be the space of spherical harmonics of degree k (homogeneous polynomials that vanish under the Laplacian)

Let $e_{k,1}, \ldots, e_{k,d_k}$ be a (real) orthonormal basis of H_k

Can construct a positive, $O(n)$-invariant kernel $S^{n-1} \times S^{n-1} \to \mathbb{R}$ as

$$K(x, y) = \sum_{j=1}^{d_k} e_{k,j}(x) e_{k,j}(y)$$

(by the addition theorem this is the ultraspherical polynomial $P_k^n(x \cdot y)$)

Alternatively, let p be the spherical harmonic spanning $(H_k)^{O(n-1)}$

Then we can construct K as

$$K(x, y) = \int_{S^{n-1}} \pi(s(x)) p(z) \pi(s(y)) p(z) \, dz$$

Stiefel harmonics

I_t is homeomorphic to a union of quotients of Stiefel manifolds (sets of all $n \times k$ matrices with orthonormal columns)

Let $\pi\colon O(n) \to \mathrm{GL}(V)$ be an irreducible, unitary representation

> We need to describe V^{H_i}, where H_i is the stabilizer subgroup of an orbit representative R_i of the action of $O(n)$ on I_t

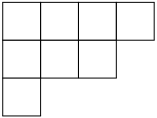
The group H_i is isomorphic to $G \times O(n - t_i)$, where $t_i = |R_i|$

By Frobenius reciprocity, $\dim V^{O(n-k)}$ is the multiplicity of V in the decomposition of $\mathcal{C}(O(n)/O(n-k))$ into $O(n)$ irreducibles

$O(n)/O(n-k)$ is a Stiefel manifold

Representations of $\mathrm{GL}_t\mathbb{C}$

Index the irreps of $\mathrm{GL}_t\mathbb{C}$ by Young diagrams with t rows



In a Young diagram the rows are left-aligned and the length of the rows cannot increase

Let λ be a vector of the row lengths; in the above example $\lambda = (4, 3, 1)$

If $\rho\colon \mathrm{GL}_t\mathbb{C} \to \mathrm{GL}(W)$ is the irrep indexed by λ, then we can give a basis of W indexed by the semistandard tableaux on the Young diagram

A tableaux is a labeling of the boxes using the integers $1, \ldots, t$

A tableaux is semistandard if the entries in each row are nondecreasing and the entries in each column are strictly increasing

For $A \in \mathrm{GL}_t\mathbb{C}$ we can explicitly compute the matrix coefficients of $\rho(A)$

Gelbart's discovery

Let $\pi\colon O(n) \to \mathrm{GL}(V)$ and $\rho\colon \mathrm{GL}_{\mathbb{C}}(k) \to \mathrm{GL}(W)$ be irreducible representations indexed by the same tuple λ of nonincreasing integers

In 1974 Gelbart discovered that

$$\dim V^{O(n-k)} = \dim W$$

He called this "an act of providence"

The Gross-Kunze construction

Let $\omega = \begin{pmatrix} I_t & iI_t & 0 \end{pmatrix}$ and $\epsilon = \begin{pmatrix} I_t \\ 0 \end{pmatrix}$

For each $w \in W$, define the function $\phi(w) \colon O(n, \mathbb{C}) \to W$ by

$$\phi(w)(x) = \rho(\omega x \epsilon) w$$

Define π by $\pi(\xi)\phi(w)(x) = \phi(w)(x\xi)$

Then $V = \pi(O(n, \mathbb{C}))\phi(W)$ is an irreducible representation, and all irreducible representations of $O(n, \mathbb{C})$ can be obtained in this way

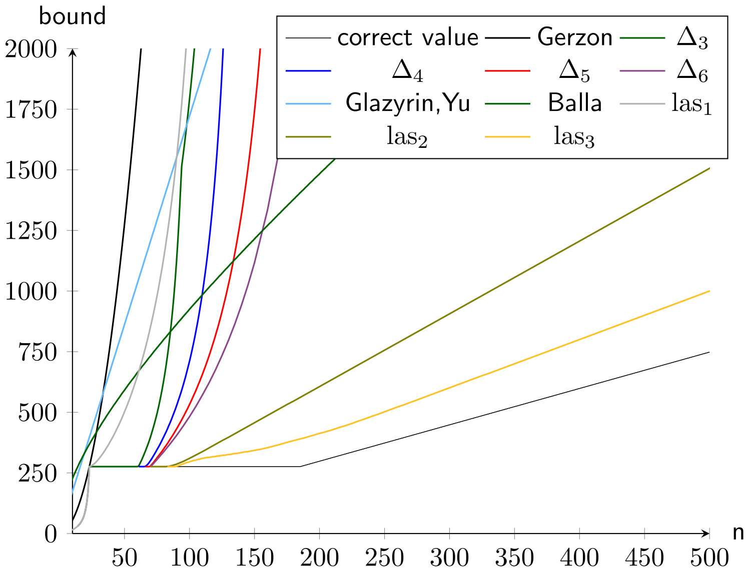
Because $h\epsilon = \epsilon$ for $h \in O(n-t, \mathbb{C})$, it follows that

$$V^{O(n-t)} = \phi(W)$$

For $1 \leq k \leq t$ we have $V^{O(n-k)} = \phi(W_k)$, where W_k is the span of the elements indexed by semistandard tableaux with entries $1, \ldots, k$

4. New bounds for equiangular lines with a fixed angle

$\alpha = 1/5$



bound
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correct value
Gerzon
Δ_3
Δ_4
Δ_5
Δ_6
Glazyrin, Yu
Balla
las_1
las_2
las_3
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1000
1250
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1750
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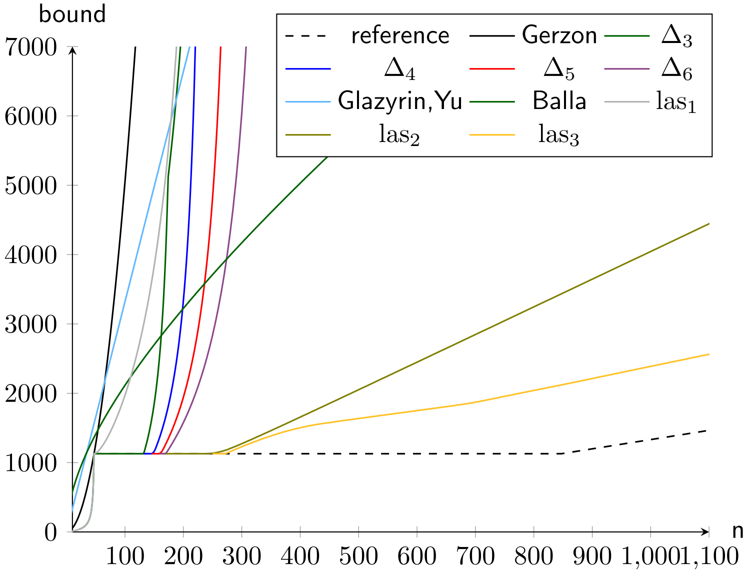
100
150
200
250
300
350
400
450
500

$\alpha = 1/7$



bound
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Δ_3
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Glazyrin, Yu
Balla
las_1
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5. Asymptotics as $n \to \infty$

Asymptotics as $n \to \infty$

Conjecture

Fix $-1 < \alpha < 1$. For each n, the optimal solution of the second level of the Lasserre hierarchy for bounding $N_\alpha(n)$ is obtained using polynomial representations with labels $\lambda = (0), (2), (3,1), (4)$. These bounds are asymptotically linear in n with slope $(1/\alpha + 1)/2$.

For $\alpha = 1/5$ the numerical results indicate that

$$\mathrm{las}_2(n) = 3n + 6 + \frac{120}{n} + \frac{5530}{n^2} + \frac{1449485}{3n^3} + O\left(\frac{1}{n^4}\right).$$

For each rational α, the constant term in the expansion seems to lie in a quadratic number field

Asymptotics as $n \to \infty$

Conjecture

Fix $-1 < \alpha < 1$. There exist matrix-valued functions $F^\lambda(n)$ with entries

$$F^\lambda(n)_{(i_1,j_1),(i_2,j_2)} = \sum_{k=0}^{\infty} A^{\lambda,k}_{(i_1,j_1),(i_2,j_2)} n^{1+\lambda_1+2\lambda_2-t_{i_1}-t_{i_2}-k}$$

such that $\{F^\lambda(n)\}_\lambda$ becomes feasible and optimal as $n \to \infty$ for the problem of bounding $N_\alpha(n)$ with the second level of the Lasserre hierarchy.

For each rational α we seem to be able to compute $A^{\lambda,0}$ as a rational, positive semidefinite matrix

Asymptotics as $n \to \infty$

Let $\rho\colon \mathrm{GL}_t\mathbb{C} \to \mathrm{GL}(W)$ be an irreducible representation

On W we define two inner products. One in which the basis elements given by the semistandard tableaux are orthogonal. Another where the restriction to $U(t) \subseteq \mathrm{GL}_t\mathbb{C}$ becomes a unitary representation.

Conjecture

There is a constant $c > 0$ such that

$$n^d \langle \phi(e_{T_i}), \phi(e_{T_j}) \rangle = c \langle e_{T_i}, e_{T_j} \rangle_{U(t)} + O\left(\frac{1}{n}\right),$$

for all i, j, where $d = \sum_i \lambda_i$ and $\{e_{T_i}\}_i$ is the semistandard tableaux basis

Thank you!

L., Willem de Muinck Keizer, Fabrício Machado, The Lasserre hierarchy for equiangular lines with a fixed angle, In preparation