A semidefinite program for least distortion embeddings of flat tori into Hilbert spaces

Frank Vallentin

University of Cologne, Germany

based on joint work with
Arne Heimendahl, Moritz Lücke, Marc Christian Zimmermann

CWI Workshop
Semidefinite and Polynomial Optimization
August 29, 2022

The beginning of the story

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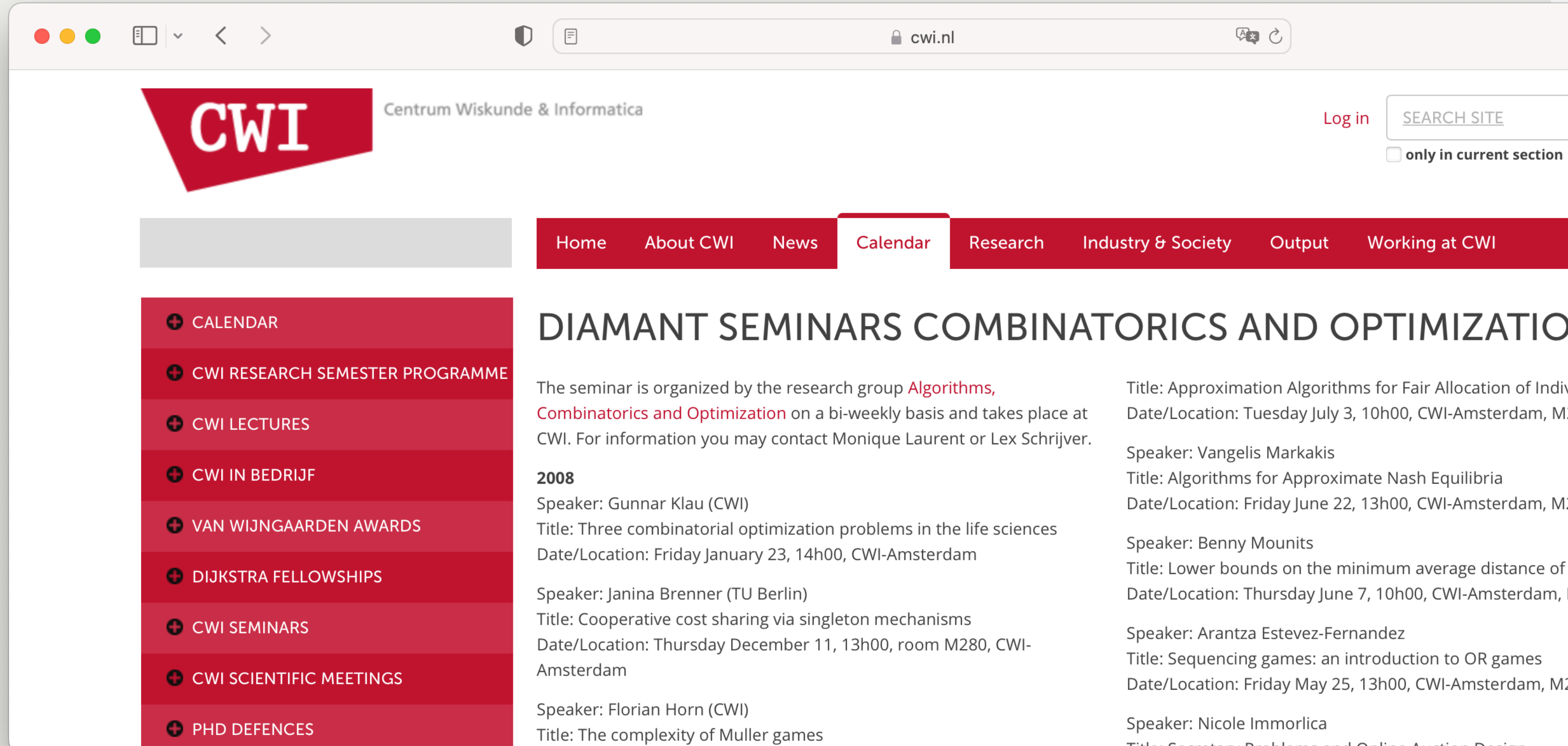

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Title: Realizability of oriented matroids
Date/Location: Friday November 2, 13h00, CWI-Amsterdam, M280.

Speaker: Nicole Immorlica (CWI)
Title: Revenue maximization in digital goods auctions
Date/Location: Friday October 12, 13h00, CWI-Amsterdam, M280.

Speaker: Frank McSherry (Microsoft Research)
Title: Mechanism Design via Differential Privacy
Date/Location: Monday October 1st, 16h00, CWI-Amsterdam, M279.

Speaker: Jarek Byrka and Steven Kelk
Title: Worst-case optimal approximation algorithms for maximizing triplet consistency within phylogenetic networks
Date/Location: Monday October 1st, 10h00, CWI-Amsterdam, M279.

Speaker: Erik Jan van Leeuwen
Title: Domination in Geometric Intersection Graphs
Date/Location: Friday September 7, 10h00, CWI-Amsterdam, M280.

Speaker: Gyula Pap
Title: Packing A-paths subject to node-capacities
Date/Location: Friday August 24, 13h00, CWI-Amsterdam, M279.

Speaker: Gyula Karolyi
Title: The Combinatorial Nullstellensatz and the Polynomial Method
Date/Location: Friday August 24, 10h00, CWI-Amsterdam, M279.

Speaker: Amin Saberi

Title: Fast matrix multiplication (II)
Date/Location: 20-01-2006, 11h00, CWI-Amsterdam, Room M280.

2005

Speaker: Fernando de Oliveira Filho, CWI
Title: Fast matrix multiplication (I)
Date/Location: 16-12-2005, 13h00, CWI-Amsterdam, Room M279.

Speaker: Benny Mounits, Technion, Haifa.
Title: Upper bounds on sizes of codes via association schemes and linear programming Date/Location: 02-12-2005, 10h00, CWI-Amsterdam, Room M279.

Speaker: Erik Jan van Leeuwen, CWI
Title: Better approximation schemes on disk graphs
Date/Location: 04-11-2005, 13h00, CWI-Amsterdam, Room M279.

Speaker: Frank Vallentin, CWI
Title: Computing the minimum distortion for embedding distance-transitive graphs in the Euclidean space
Date/Location: 21-10-2005, 13h00, CWI-Amsterdam, Room H220 (NIKHEF building).

Speaker: Nebojsa Gvozdenovic, CWI
Title: Approximating the chromatic number of a graph by semidefinite programming Date/Location: 07-10-2005, 13h00, CWI-Amsterdam, Room M27.

Notes from October 21, 2005

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OPTIMAL EMBEDDINGS OF DISTANCE TRANSITIVE GRAPHS INTO EUCLIDEAN SPACES

FRANK VALLENTIN

1. Problem, Motivation and some Background

1.1. **Problem.** Let (X, d) be a finite metric space. Find an *isometric embedding* into n-dimensional Euclidean space. This is a map $\varrho : X \to \mathbb{R}^n$ with

$$\forall x, y \in X : d(x, y) = \|\varrho(x) - \varrho(y)\|,$$

where the norm of $(x_1, \ldots, x_n) \in \mathbb{R}^n$ is

$$\|(x_1, \ldots, x_n)\| = \sqrt{x_1^2 + \cdots + x_n^2}.$$

1.2. **Applications.** Visualization of DNA data, Voronoi cells for fingerprint data, Design of approximation algorithms for the sparsest cut problem, ...

1.3. **Bad News.** There is no isometric embedding into any n-dimensional Euclidean space for $X = \{\alpha, \beta, \gamma, \delta\}$ with

$$d(\alpha, \beta) = d(\beta, \gamma) = d(\gamma, \delta) = d(\delta, \alpha) = 1$$

and

$$d(\alpha, \gamma) = d(\beta, \delta) = 2.$$

1.4. **Good News.**

Definition 1.1. Let $\varrho : X \to \mathbb{R}^n$ an embedding. The embedding has *distortion* $D = \frac{\beta}{\alpha}$ if

$$\forall x, y \in X : \alpha d(x, y) \le \|\varrho(x) - \varrho(y)\| \le \beta d(x, y).$$

Denote by $c_2(X, d)$ the minimal possible distortion of (X, d) into any n-dimensional Euclidean space. An embedding of (X, d) attaining distortion $c_2(X, d)$ is called *optimal embedding*.

Example 1.2. The "square embedding" has distortion $\sqrt{2}$ where $\beta = 1$ and $\alpha = \frac{2}{\sqrt{2}}$. We shall

I. Least distortion embeddings of (finite) metric spaces

II. Flat tori

III. SDP perspective

Recap

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given: (X, d) metric space

goal: find Hilbert space H and injective map $\varphi : X \to H$
so that distortion(φ) is minimized.

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$$\text{distortion}(\varphi) = \text{expansion}(\varphi) \cdot \text{contraction}(\varphi)$$

$$\text{expansion}(\varphi) = \sup_{\substack{x,y \in X \\ x \neq y}} \frac{\|\varphi(x) - \varphi(y)\|}{d(x,y)}$$

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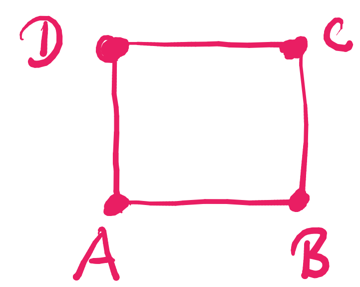
natural embedding of square graph



$d(A,C) = 2$

$\|\varphi(A) - \varphi(C)\| = \sqrt{2}$

expansion $(\varphi) = 1$

contraction $(\varphi) = 2/\sqrt{2} = \sqrt{2}$.

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good news: $c_2(X, d) = O(\log |X|)$ if $|X| < \infty$ (Bourgain, 1985)

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Least distortion is central to the "Ribe program" in functional analysis: Relate Banach space concepts (linear, normed, complete) to metric space concepts (fascinating surveys by Ball, Naor)

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Applications to data science and algorithm design: Data sets come equipped with a natural similarity metric but not with a linear structure. Now embed the data into Banach space with least distortion and exploit the linear structure.

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primal SDP

Scale φ so that contraction$(\varphi) = 1$ and set $\varphi(x) \cdot \varphi(y) = Q_{xy}$

$$c_2(X,d)^2 = \inf\{C \ : C \in \mathbb{R}_+, Q \in \mathcal{S}_+^X,$$
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dual SDP: systematic way to derive lower bounds for $c_2(X, d)$

$$c_2(X,d)^2 = \sup\left\{ \frac{\sum_{i,j=1:Y_{ij}>0}^n Y_{ij} d(x_i,x_j)^2}{-\sum_{i,j=1:Y_{ij}<0}^n Y_{ij} d(x_i,x_j)^2} : Y \in \mathcal{S}_+^n, Y\mathbf{e} = 0 \right\}.$$

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weak duality: $Y_{ij} > 0$ only for most contracted pairs

$Y_{ij} < 0$ only for most expanded pairs

Using the dual

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The positive semidefinite matrix

$$Y = \begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix} = (1,-1,1,-1)^t(1,-1,1,-1)$$

proves that the "square embedding" is optimal. We have $c_2(X,d)^2 \le 2$ due to the existence of the embedding and we have

$$c_2(X,d)^2 \ge \frac{\sum_{i,j=1:Y_{ij}>0}^4 Y_{ij} d(x_i,x_j)^2}{-\sum_{i,j=1:Y_{ij}<0}^4 Y_{ij} d(x_i,x_j)^2} = \frac{4 \cdot 2^2 \cdot 1}{8 \cdot 1^2 \cdot (-(-1))} = 2.$$

Using the dual: References

dual SDP has been used to find least distortion embeddings of several graph classes:

Linial, Magen, 2000: product of cycles and expander graphs (in particular Bourgain's result is tight)

Linial, Magen, Naor, 2002: graphs of high girth

Vallentin, 2008: strongly regular graphs, distance regular graphs
(extended by Kobayashi, Kondo, 2015,
Cioabă, Gupta, Ihringer, Kurihara, 2021)

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$b_1, \ldots, b_n$ basis of $\mathbb{R}^n$

$L = \mathbb{Z}b_1 + \cdots + \mathbb{Z}b_n$ lattice

$T = \mathbb{R}^n/L$ flat torus

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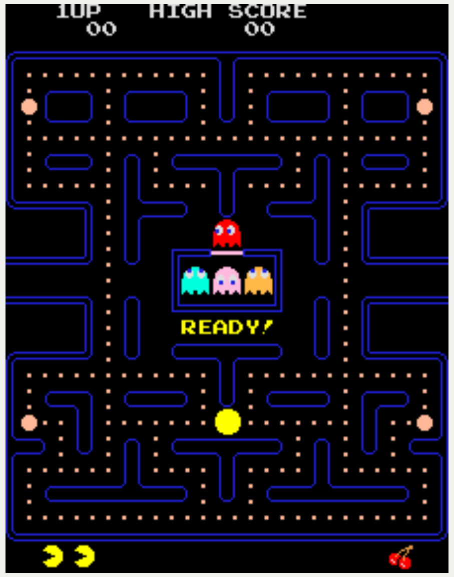

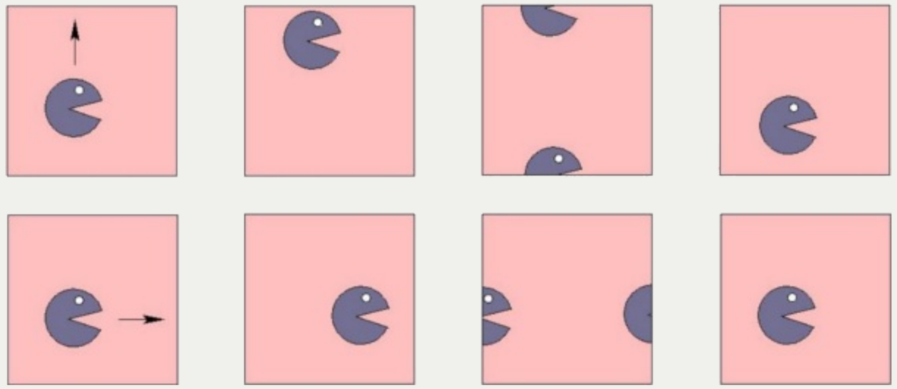

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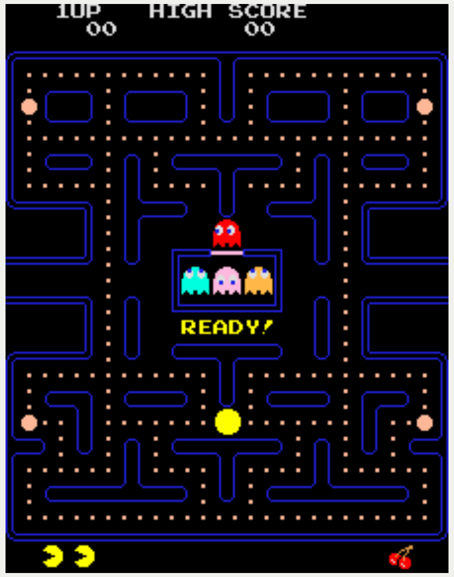

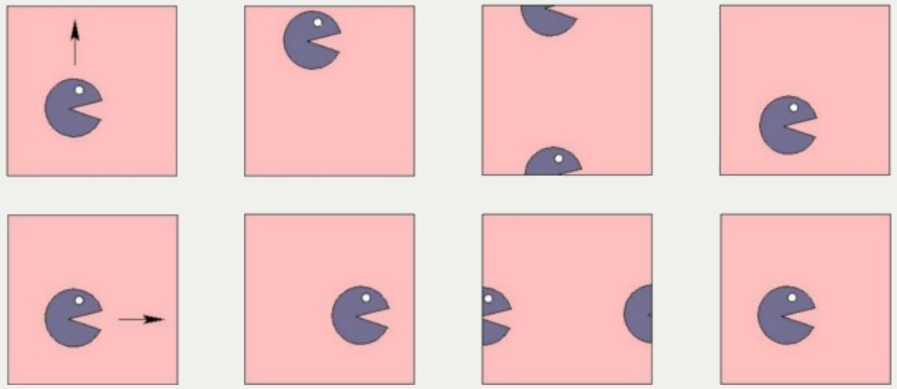

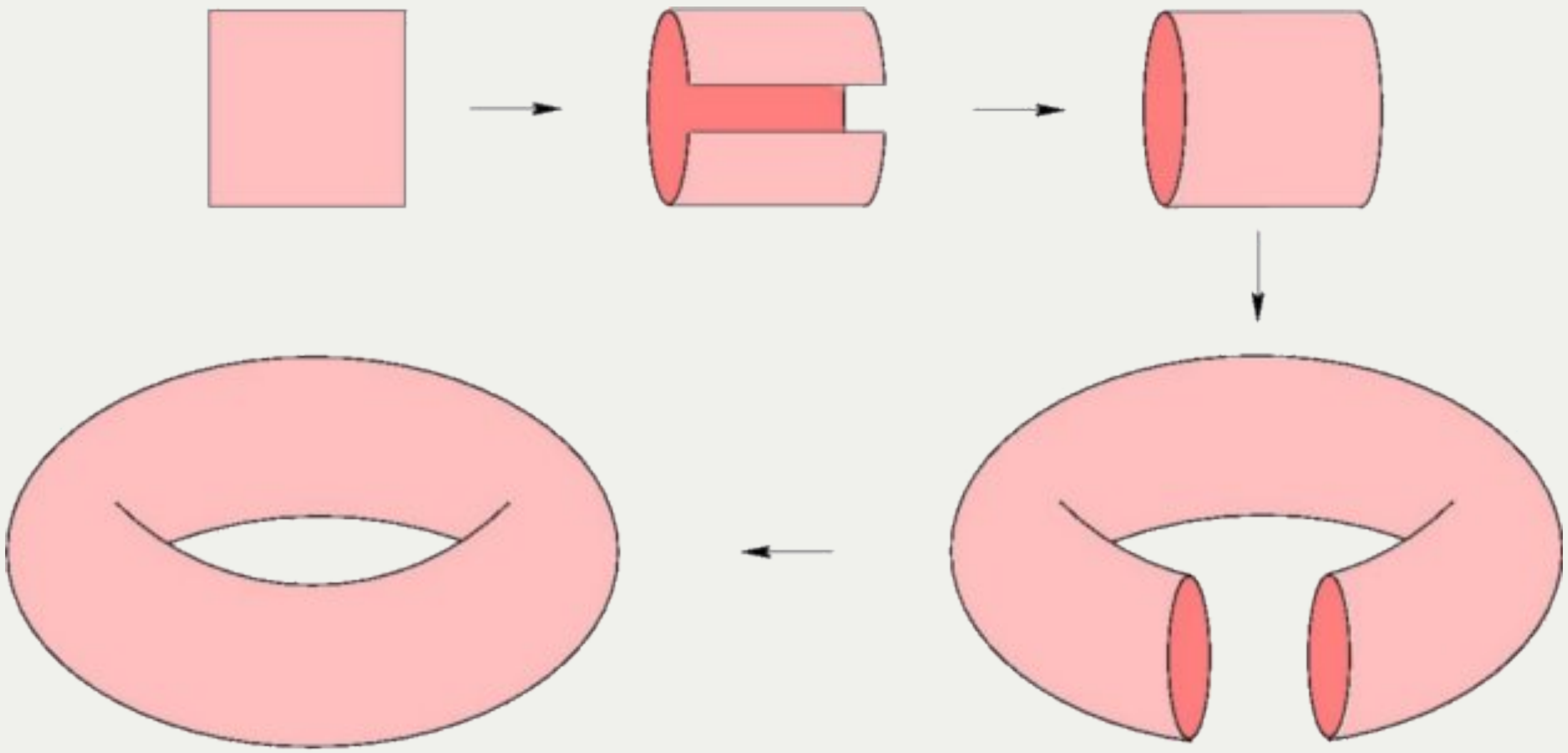

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Flat tori: Euclidean embeddings

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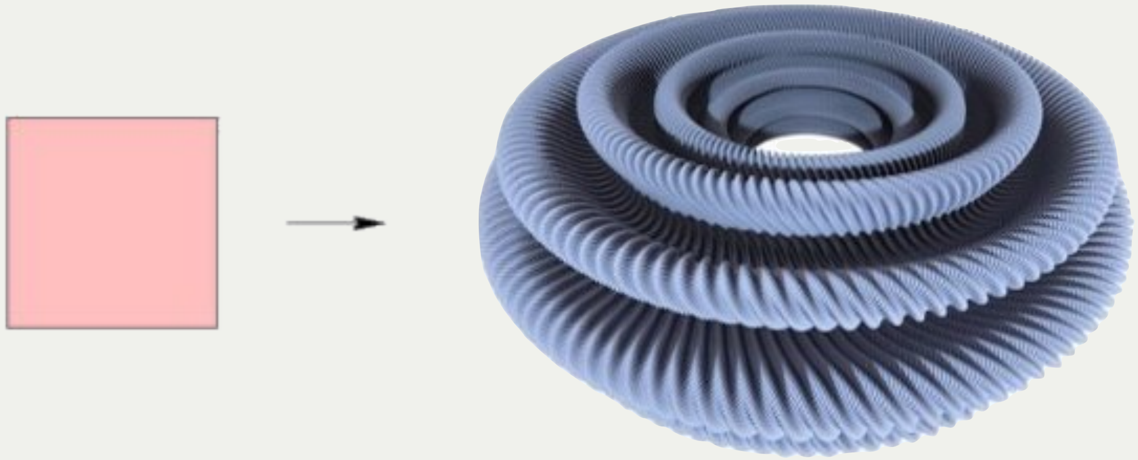

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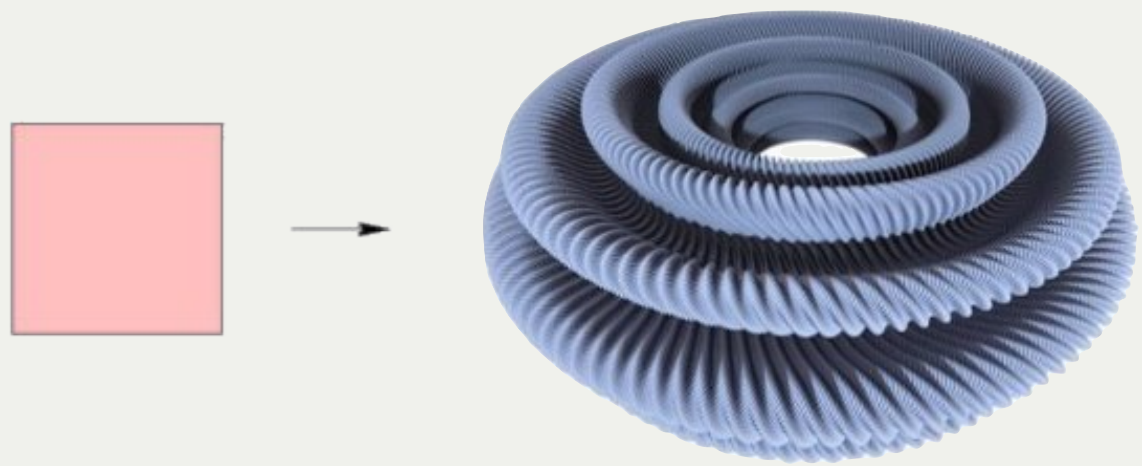

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Potential applications: complexity of lattice problems, like closest vector problem

Standard embedding of standard torus

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$\varphi(x_1, \ldots, x_n) = (\cos 2\pi x_1, \sin 2\pi x_1, \ldots, \cos 2\pi x_n, \sin 2\pi x_n).$

will see: φ is optimal with distortion$(\varphi) = \pi/2$

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$n = 1$

$$\text{contraction}(\varphi) = \frac{1/2}{\|\varphi(0) - \varphi(1/2)\|} = 1/4$$

$$\text{expansion}(\varphi) = \frac{\|\varphi(0) - \varphi(\varepsilon)\|}{\varepsilon} = \frac{\sqrt{2 - 2\cos(2\pi\varepsilon)}}{\varepsilon} \to 2\pi$$

Flat tori can be highly non-Euclidean

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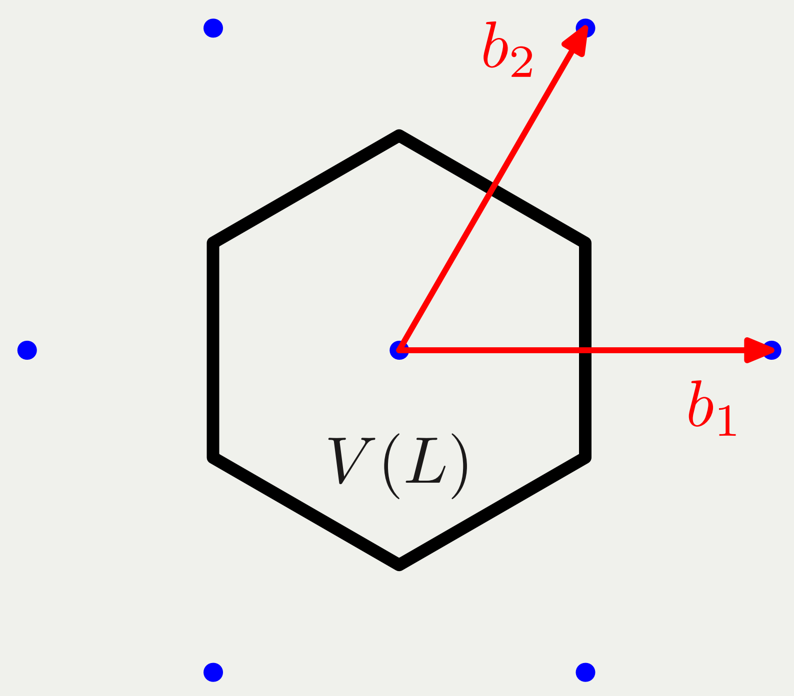

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$V(L)$ = Voronoi cell

$\lambda(L) = 2 \cdot \text{inradius of } V(L)$

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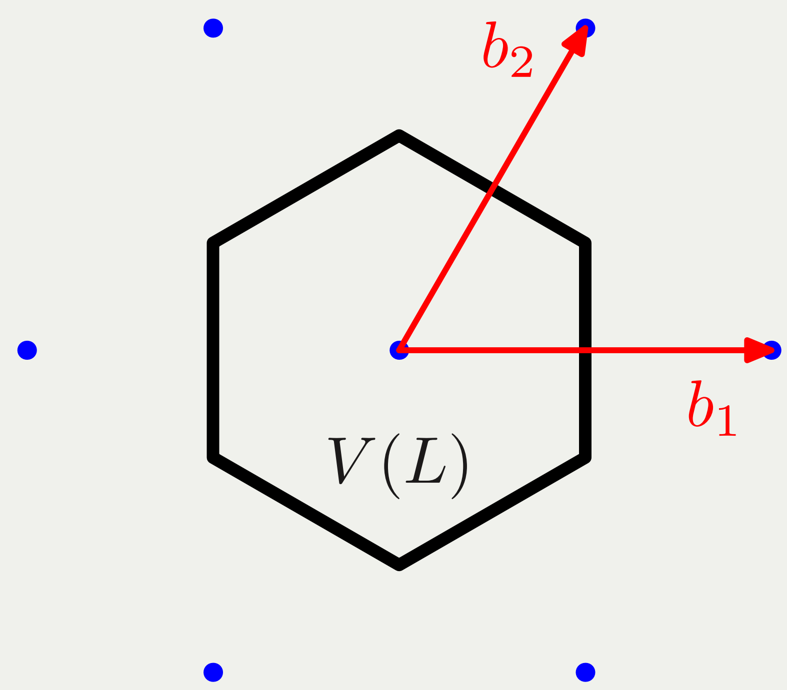

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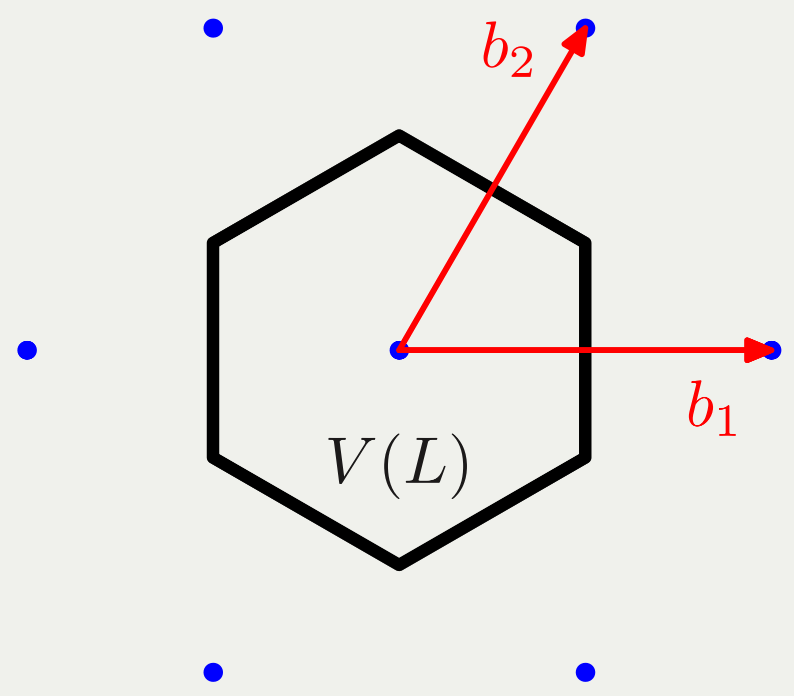

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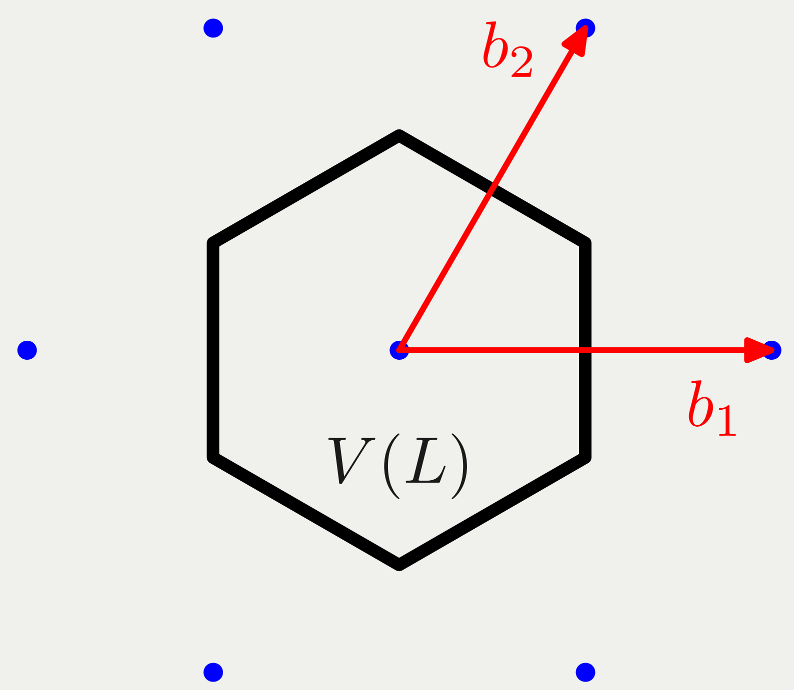

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Corollary. $c_2(\mathbb{R}^n/L_n) = \Omega(\sqrt{n})$

More about Euclidean embeddings of flat tori

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Theorem. Haviv, Regev, 2010 $$c_2(\mathbb{R}^n/L) \geq \frac{\lambda(L^*)\mu(L)}{4\sqrt{n}},$$

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because $\mu(L) = \Omega(\sqrt{n}(\text{vol } L)^{1/n})$

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almost tight lower bound

Theorem. Agarwal, Regev, Tang, 2020 $$c_2(\mathbb{R}^n/L) = O(\sqrt{n \log n})$$

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Want: similar SDP for flat torus

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Moore's theorem (1916): There exist a Hilbert space H and a map $\varphi : \mathbb{R}^n/L \to H$ if and only if there is a positive definite kernel Q so that

$$Q : \mathbb{R}^n/L \times \mathbb{R}^n/L \to \mathbb{C} \text{ so that } Q(x, y) = (\varphi(x), \varphi(y)) \text{ for all } x, y \in \mathbb{R}^n/L.$$

Kernel Q is called positive definite if and only if for all $N \in \mathbb{N}$ and for all $x_1, \ldots, x_N \in \mathbb{R}^n/L$ the matrix $(Q(x_i, x_j))_{1 \leq i,j \leq \mathbb{N}} \in \mathbb{C}^{N \times N}$ is Hermitian and positive semidefinite.

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Moore's theorem (1916): There exist a Hilbert space H and a map $\varphi : \mathbb{R}^n/L \to H$ if and only if there is a positive definite kernel Q so that

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Kernel Q is called positive definite if and only if for all $N \in \mathbb{N}$ and for all $x_1, \ldots, x_N \in \mathbb{R}^n/L$ the matrix $(Q(x_i, x_j))_{1 \le i,j \le \mathbb{N}} \in \mathbb{C}^{N \times N}$ is Hermitian and positive semidefinite.

This gives:

$$\begin{aligned} c_2(\mathbb{R}^n/L)^2 = \inf\{C\ : C \in \mathbb{R}_+, Q \text{ positive definite}, \\ d_{\mathbb{R}^n/L}(x,y)^2 &\le Q(x,x) - 2\Re(Q(x,y)) + Q(y,y) \\ &\le C d_{\mathbb{R}^n/L}(x,y)^2 \text{ for all } x, y \in \mathbb{R}^n/L\} \end{aligned}$$

Our favorite trick: Symmetry reduction

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so: $\exists$ continuous positive type function $f : \mathbb{R}^n/L \to \mathbb{R}$ with $\overline{Q}(x,y) = f(x-y)$

$$c_2(\mathbb{R}^n/L)^2 = \inf\{C \ : C \in \mathbb{R}_+, f\colon \mathbb{R}^n/L \to \mathbb{R} \text{ continuous and of positive type},$$
$$|x|^2 \leq 2(f(0) - f(x)) \leq C|x|^2 \ \text{ for all } x \in V(L)\}.$$

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$f : \mathbb{R}^n/L \to \mathbb{C}$ is of positive type if and only if all its Fourier coefficients

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This gives: infinite-dimesional LP

$$c_2(\mathbb{R}^n/L)^2 = \inf \big\{ C \ : C \in \mathbb{R}_+, z \in \ell^1(L^*), z(u) = z(-u) \geq 0 \text{ for all } u \in L^*,$$
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- understand principal structure of Euclidean embeddings

A feasible solution of the above minimization problem (C, z) determines a Euclidean embedding φ of $\mathbb{R}^n/L$ with distortion $\leq \sqrt{C}$ by

$$\varphi : \mathbb{R}^n/L \to \ell^2(L^*), \quad x \mapsto \left(\sqrt{z(u)}e^{2\pi i u^{\mathsf{T}} x}\right)_{u \in L^*},$$

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- inf is max

because bounded, continuous, positive type functions are weak* compact

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infinite-dimensional linear program

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equivalent to finite SDP condition

$$CI - 4\pi^2 \sum_{u \in L^*} z(u) u u^{\mathsf{T}} \in \mathcal{S}^n_+,$$

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dual SDP

$$c_2(\mathbb{R}^n/L)^2 = \sup \{2\pi^2 \int_{V(L)} |x|^2 \, d\nu(x) \ :$$
$$\nu \in \mathcal{M}_+(V(L)), Y \in \mathcal{S}_+^n, \mathrm{Tr}(Y) = 1,$$
$$\int_{V(L)} (1 - \cos(2\pi u^\mathsf{T} x)) \, d\nu(x) \leq u^\mathsf{T} Y u \text{ for all } u \in L^* \}$$

$\mathcal{M}_+(V(L))$ is the cone of Borel measures on $V(L)$

First application: Constant factor improvement of lower bound

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Theorem. Let L be an n-dimensional lattice, then

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Proof. Let y be a vertex of the Voronoi cell $V(L)$ which realizes the covering radius, that is $|y| = \mu(L)$.

Choose $\nu = \frac{\lambda(L^*)^2}{2n}\delta_y$ to be a point measure supported at y and set $Y = \frac{1}{n}I$.

Then (Y, ν) is feasible for dual.

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Second application: Least distortion embedding of standard torus

$\varphi : \mathbb{R}^n/\mathbb{Z}^n \to \mathbb{R}^{2n}$

$\varphi(x_1, \ldots, x_n) = (\cos 2\pi x_1, \sin 2\pi x_1, \ldots, \cos 2\pi x_n, \sin 2\pi x_n).$

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Theorem. $c_2(\mathbb{R}^n/\mathbb{Z}^n) = \pi/2$

$c_2(\mathbb{R}^n/L) \geq \dfrac{\pi\lambda(L^*)\mu(L)}{\sqrt{n}}$ is tight for $L = \mathbb{Z}^n$

$\lambda(\mathbb{Z}^n) = 1$ and $\mu(\mathbb{Z}^n) = \sqrt{n/4}$

Third application: Least distortion embedding of 2-d flat tori

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Suppose $\exists\ u_1, \ldots, u_k \in L^*$, $z_1, \ldots, z_k \geq 0$ such that $\sum_{i=1}^k z_i u_i u_i^\top = I$.

Then

$$\varphi : \mathbb{R}^n / L \to \mathbb{C}^k, \quad H(x)_r = (2\pi^2 D z_r e^{2i\pi u_r^\top x})$$

with

$$D = \max_{V(L)\setminus\{0\}} \frac{|x|^2}{\sum_{i=1}^k z_i(1 - \cos(2\pi u_i^\top x))}$$

has distortion $\sqrt{2\pi^2 D}$.

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Drawback: somewhat indirect, D seems hard to determine.

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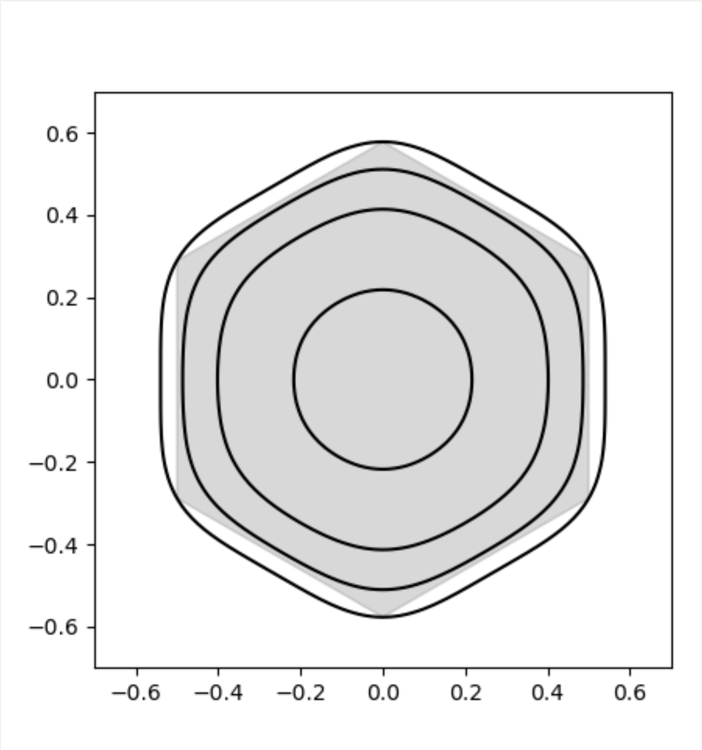
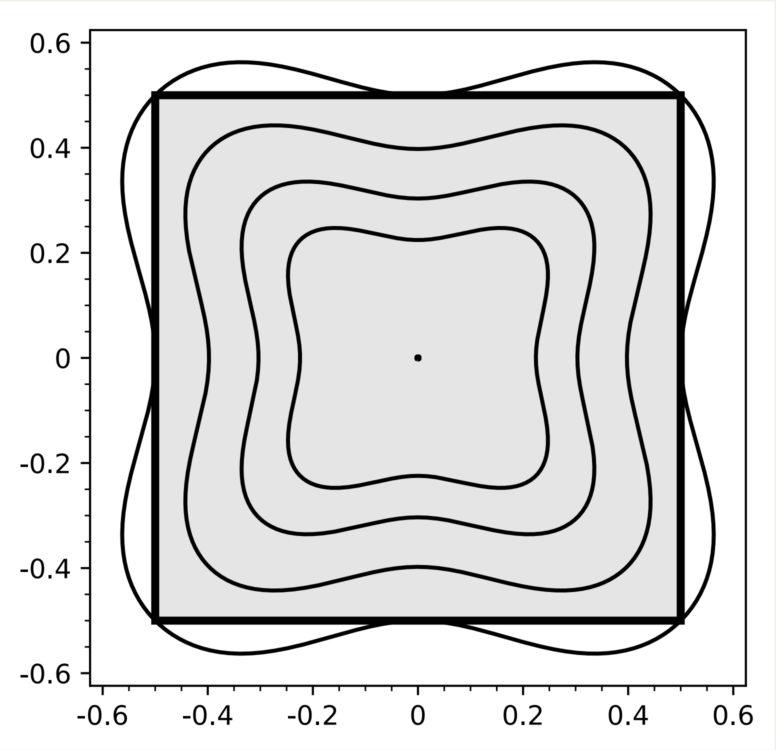
Seems to be $(0, y)$ where y vertex of Voronoi cell





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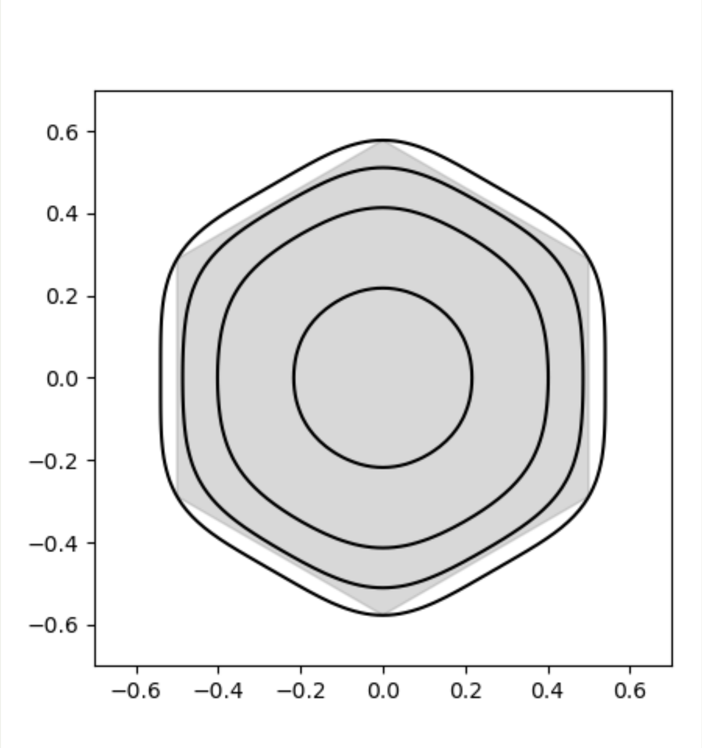

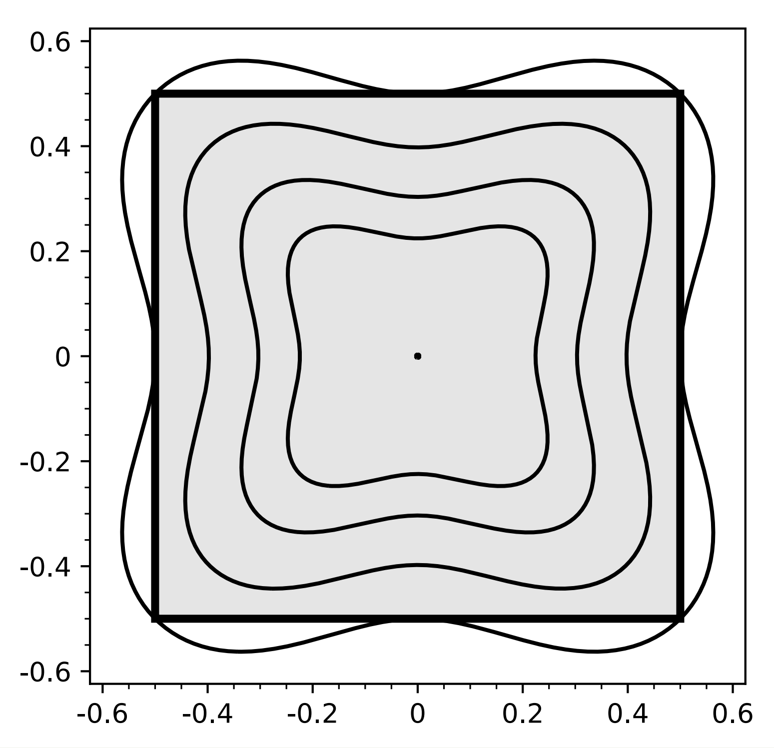

2. We do not understand the optimal Hilbert space

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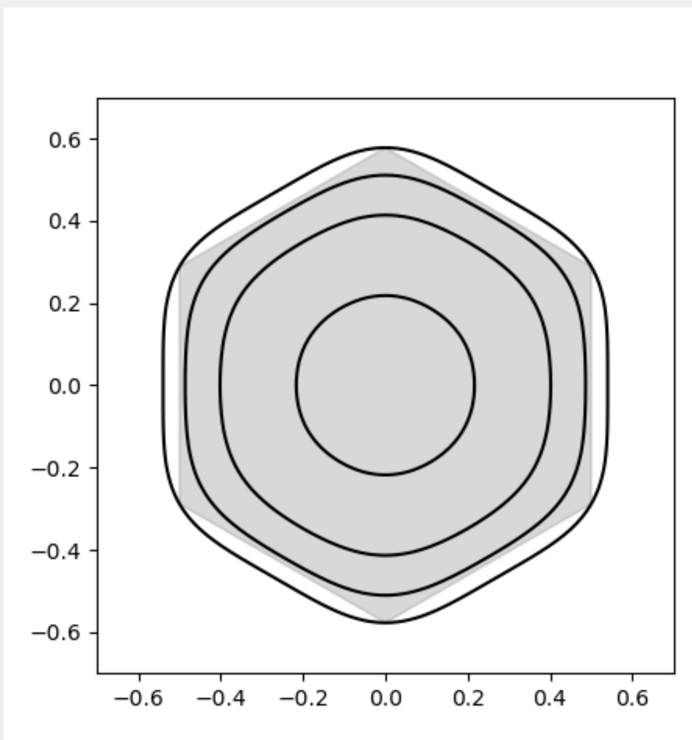

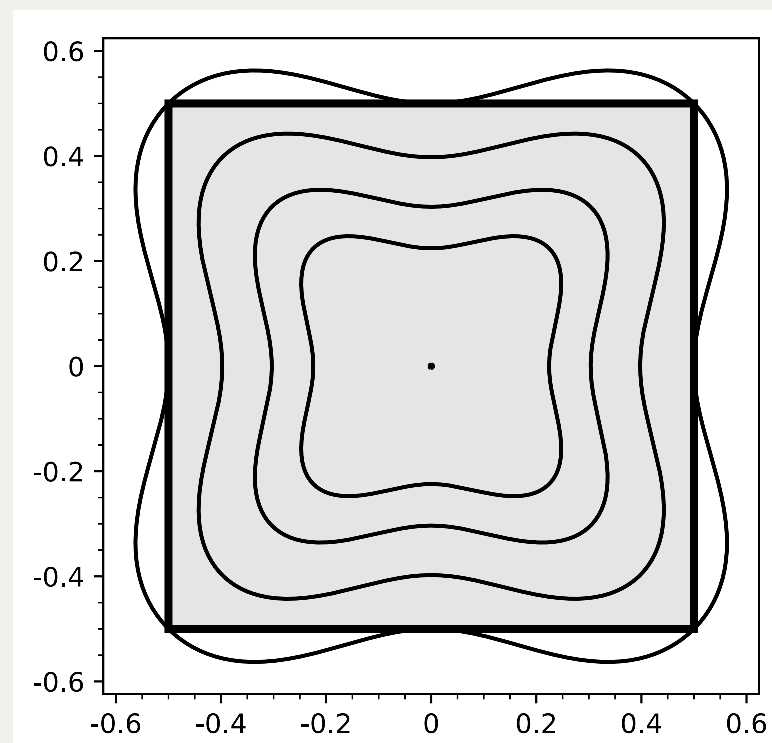

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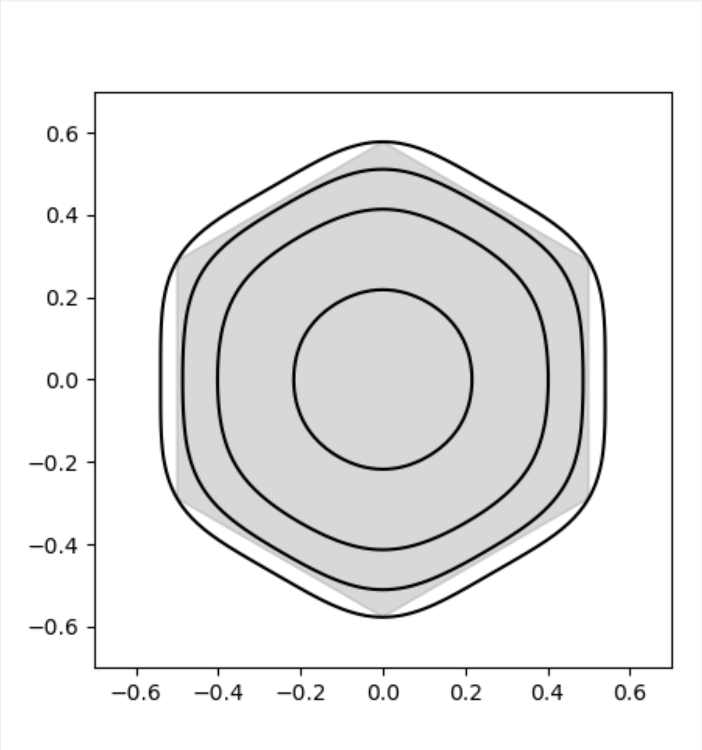

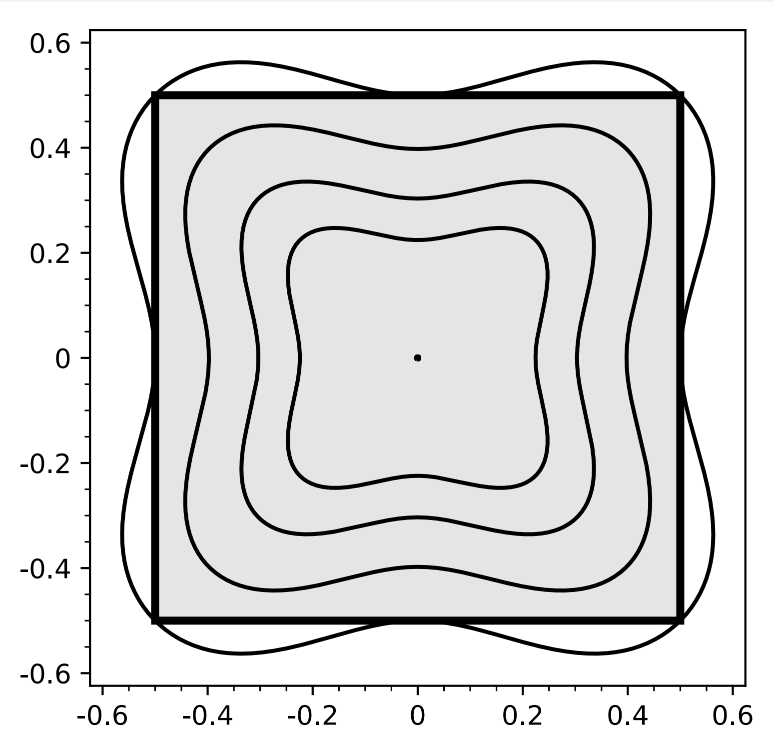

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If both are true: SDP perspective would provide a finite algorithm