An accelerated first-order method for a class of semidefinite programs

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Workshop on Semidefinite and Polynomial Optimization
August 31, 2022

5-minute flash talk

- Semidefinite program

$$(\textbf{SDP}) = \min_{Y \in \mathbb{S}^{n+k}} \left\{ \langle M_0, Y \rangle : \begin{array}{l} \langle M_i, Y \rangle + d_i = 0, \ \forall i \in [m] \\ Y \succeq 0 \end{array} \right\}$$

Related: Beck [2007], Beck et al. [2012], Burer and Monteiro [2003], Ding et al. [2021], Yurtsever et al. [2021]

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- **Sneak peek**
  - SDPs with “low rank structure” can be reformulated as “easy” QMPs

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- **Sneak peek**
  - SDPs with "low rank structure" can be reformulated as "easy" QMPs
  - First-order method (FOM) for "easy" QMPs

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- **Sneak peek**
  - SDPs with "low rank structure" can be reformulated as "easy" QMPs
  - First-order method (FOM) for "easy" QMPs
  - $\longrightarrow$ Storage-optimal, low complexity FOM for SDPs with "low rank structure"

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  - The Burer–Monteiro method + drawbacks of symmetry

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- **Conclusion**

Quadratic matrix programs

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  where q_i are quadratic functions

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- QMP is “easy” if the SDP relaxation solves it

Related: Beck [2007], Beck et al. [2012]

Two views of the SDP relaxation of a QMP

- Lift and relax:

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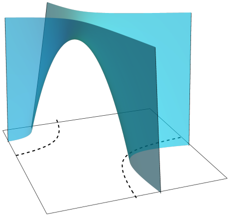
- **Standing assumption**: (**QMP**) feasible and (**SDP**) dual strictly feasible

Two views of the SDP relaxation of a QMP cont.

- Lagrangian aggregation:



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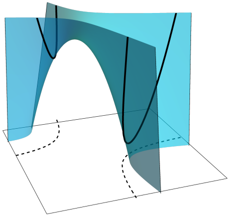

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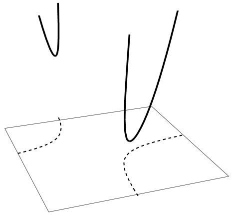

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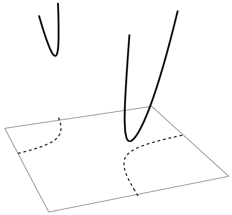
Two views of the SDP relaxation of a QMP cont.

- Lagrangian aggregation: $q_i(\bar{X}) = 0$ and $\gamma \in \mathbb{R}^m$

$$q_0(\bar{X}) = q_0(\bar{X}) + \sum_{i=1}^{m} \gamma_i q_i(\bar{X}) =: q(\gamma, \bar{X})$$



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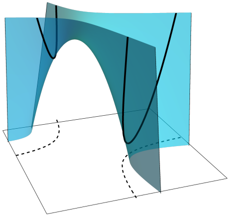

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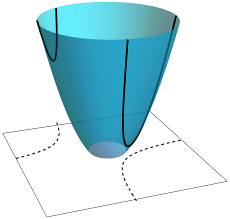

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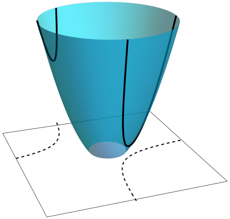

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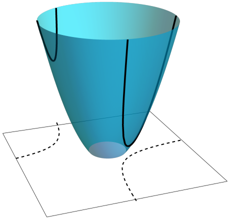

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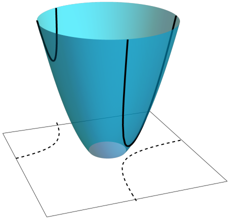

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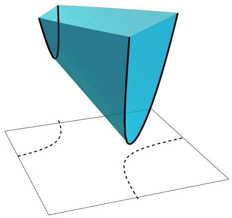

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Two views of the SDP relaxation of a QMP cont.

- Suppose (**SDP**) feasible and (**SDP**) dual strictly feasible. Then,

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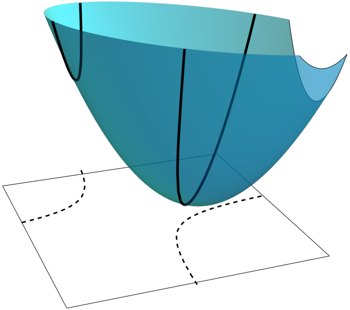
- Then, QMP has unique optimal solution X^* and X^* is unique solution (**Lagr.**)

Easy instances of QMPs cont.

X^* is unique solution (**Lagr.**) $\quad A(\gamma^*) \succ 0$



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Structural assumptions on SDP

$$(\textbf{SDP}) = \min_{Y \in \mathbb{S}^{n+k}} \left\{ \langle M_0, Y \rangle : \begin{array}{l} \langle M_i, Y \rangle + d_i = 0, \ \forall i \in [m] \\ Y \succeq 0 \end{array} \right\}$$
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From k-exact SDP to easy QMP

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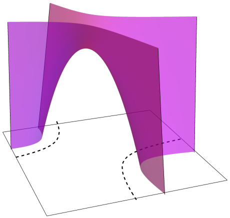

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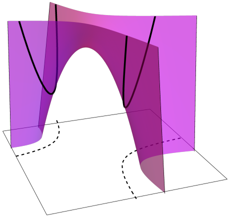

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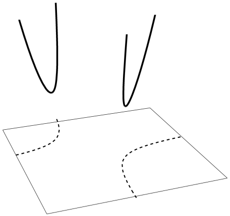

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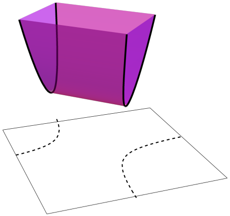

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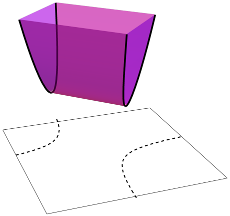

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- Too much symmetry!

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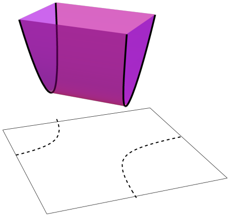

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- Too much symmetry! We set $\tilde{X} = \begin{pmatrix} X \\ I_k \end{pmatrix}$

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Recap

- **Quadratic matrix programs**

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  - Easy if (**Lagr.**) dual has optimal solution γ^* with $A(\gamma^*) \succ 0$

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Recap

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- **Semidefinite programs**
  - k-exact if can be rewritten as easy QMP
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- **Next**: A new FOM for easy QMPs

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- **Alg. plan**: Construct $\mathcal{C}$, solve strongly convex reformulation

CautiousAGD

- How to solve strongly convex quadratic matrix minimax program?

$$(\textbf{QMMP}) = \min_{X \in \mathbb{R}^{n \times k}} \max_{\gamma \in \mathcal{C}} q(\gamma, X)$$

Related: Devolder et al. [2013, 2014], Nesterov [2005, 2018]

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CautiousAGD guarantees

Theorem (CautiousAGD)

CautiousAGD produces iterates X_t such that

$$\max_{\gamma\in\mathcal{C}} q(\gamma, X_t) \leq \min_{X} \max_{\gamma\in\mathcal{C}} q(\gamma, X) + \epsilon$$

after $O\left(\log(\epsilon^{-1})\right)$ iterations, $O(m\epsilon^{-1/2})$ matrix-vector products per iteration

CertSDP

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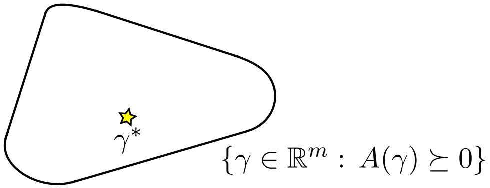

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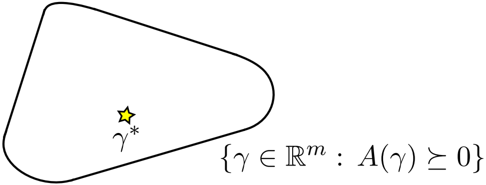

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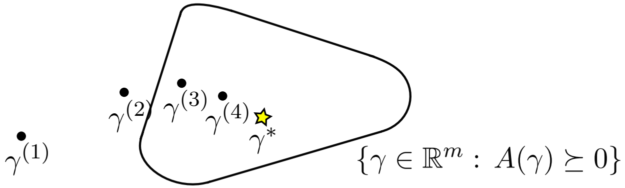

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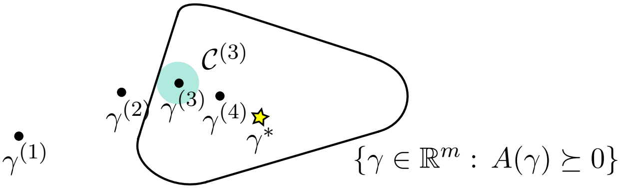

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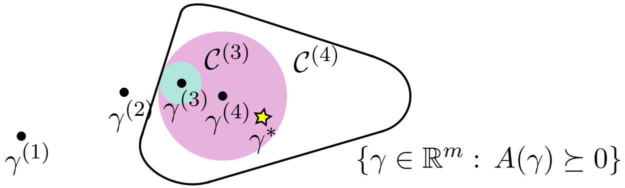

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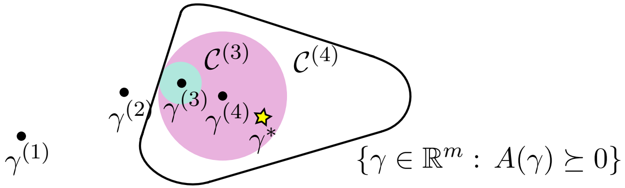

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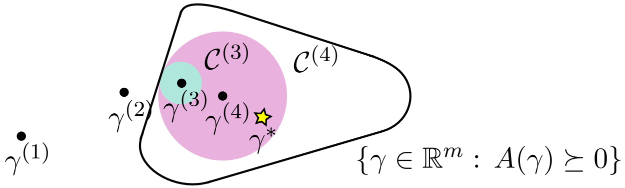

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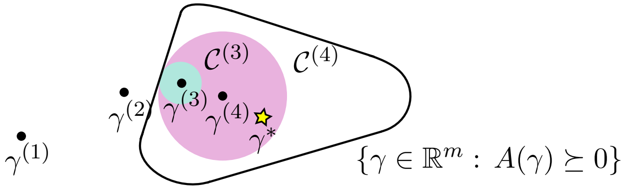

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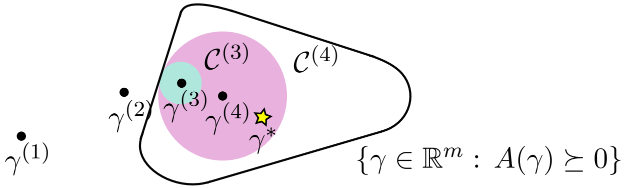

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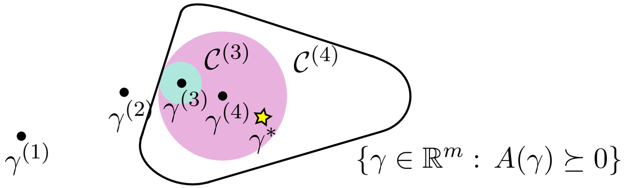
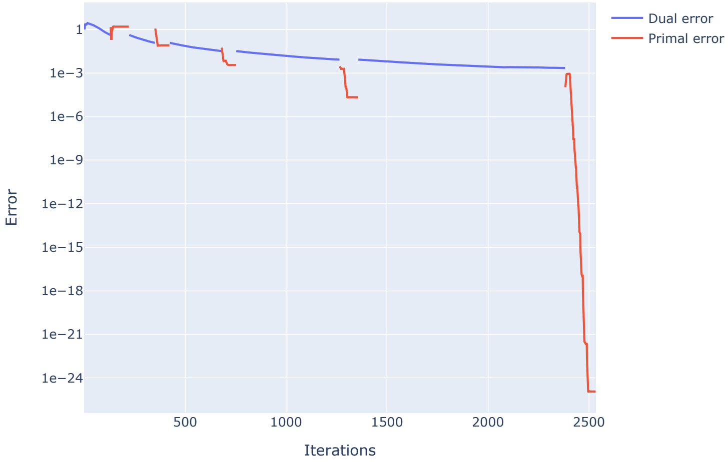


CertSDP convergence behavior



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Numerical results: experimental setup

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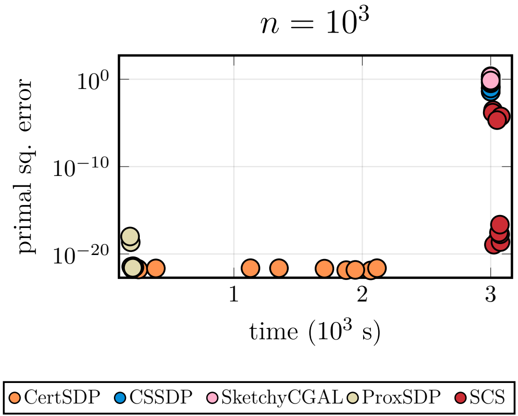

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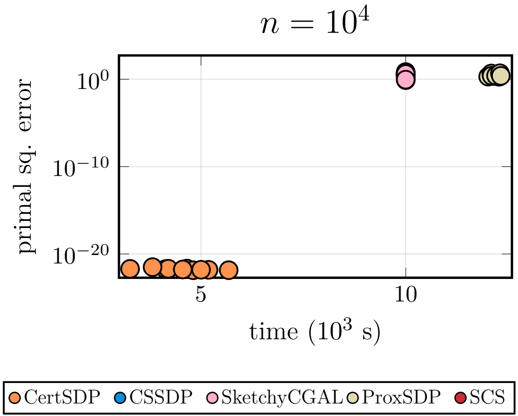

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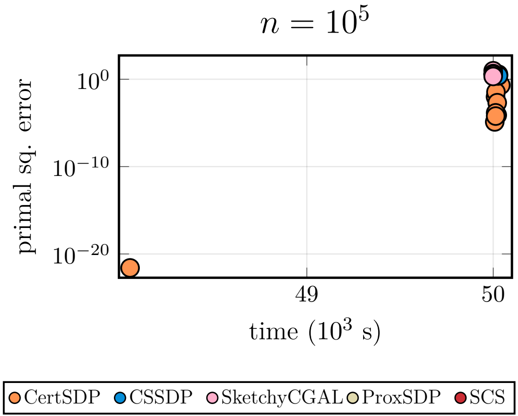
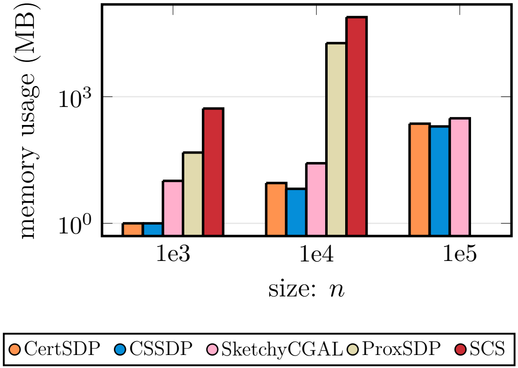


Numerical results: memory usage



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Thank you! Questions?

`https://arxiv.org/abs/2206.00224`

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