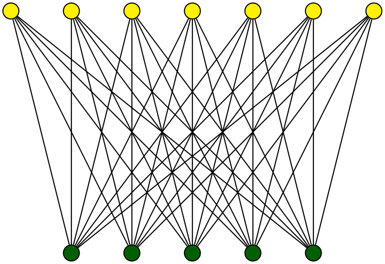
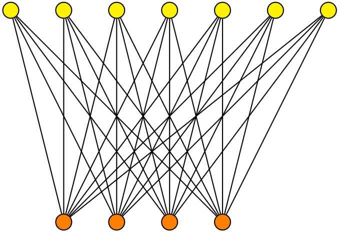
New lower bounds on crossing numbers of $K_{m,n}$

Sven Polak
CWI



Joint with Daniel Brosch (Universität Klagenfurt)
arXiv:2206.02755

Crossing numbers of $K_{m,n}$



The graph $K_{7,4}$

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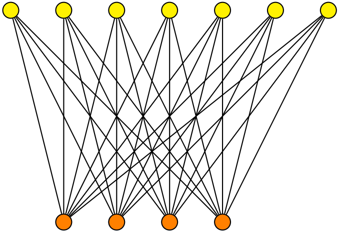

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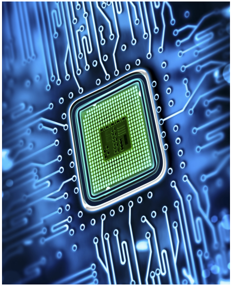

Computer chip

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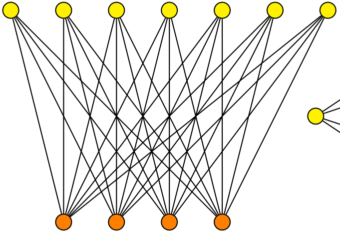

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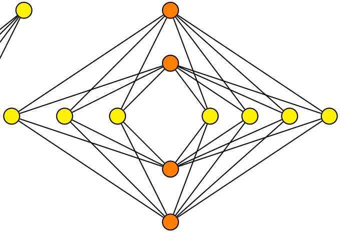
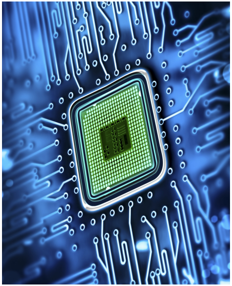


Optimal drawing



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- Minimum number of crossings? (Turán, 1940s)
- $\text{cr}(K_{m,n}) = \lfloor\frac{(m-1)^2}{4}\rfloor\lfloor\frac{(n-1)^2}{4}\rfloor$? (Zarankiewicz's conjecture, 1956)

Overview

Main goal

Compute lower bounds on $\mathrm{cr}(K_{m,n})$.

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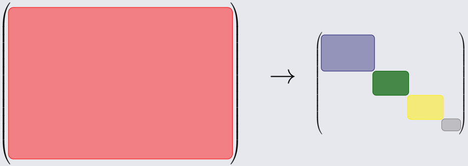

n	best lower bound previously known	new lower bound	$Z(n,n)$
10	384	**388**	400
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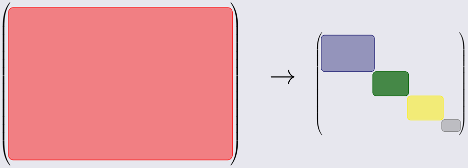

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New lower bounds on $\mathrm{cr}(K_{n,n})$

- We also give a lower bound β_m on α_m and compute it for $m \leq 13$.

Cyclic orders

Let $\{1, \ldots, m\}$ and $\{b_1, \ldots, b_n\}$ be the sides of the bipartition of $K_{m,n}$. Given a drawing of $K_{m,n}$, define for each vertex b_i:

$$\gamma(b_i) := \text{ the cyclic order in which edges from } b_i \text{ go to } \{1, \ldots, m\}$$

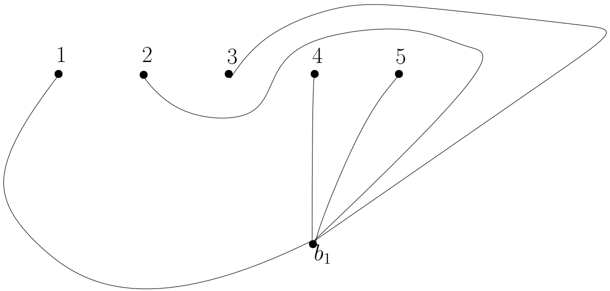

Figure: $\gamma(b_1) = (14523)$.

Cyclic orders

Lemma (Kleitman; 1970, Woodall; 1993)

#crossings of edges incident with b_i and $b_j \geq$
min. #swaps of adjacent elements of $\gamma(b_i)$ to change $\gamma(b_i)$ into $\gamma(b_j)^{-1}$.

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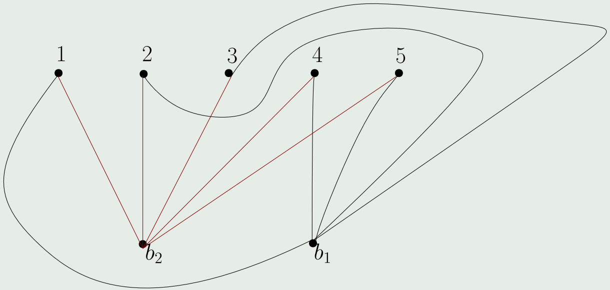

Figure: $\gamma(b_2) = (12345)$ and $\gamma(b_1)^{-1} = (14523)^{-1} = (13254)$.
Min. #swaps of adjacent elements of $\gamma(b_2)$ to change $\gamma(b_2)$ into $\gamma(b_1)^{-1}$ is 2.

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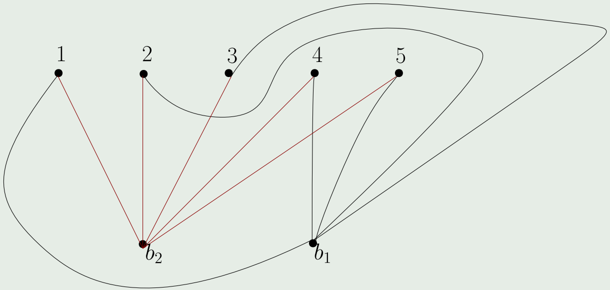

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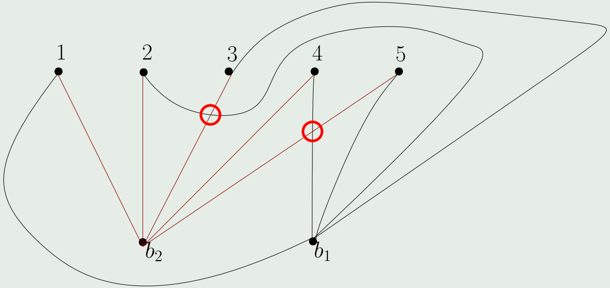

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Lower bound on crossing number

Let Z_m be the set of m-cycles, and let $Q \in \mathbb{R}^{Z_m \times Z_m}$ with

$$Q_{\sigma,\tau} := \text{min. \#swaps of adjacent elements of } \sigma \text{ to change } \sigma \text{ into } \tau^{-1}.$$

Lower bound: $\mathrm{cr}(K_{m,n}) \geq \sum_{i<j} Q_{\gamma(b_i),\gamma(b_j)}$ for γ from optimal drawing.

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Define $q_m := \min \left\{ x^\mathsf{T} Q x \mid x \in \mathbb{R}^{Z_m}_{\geq 0},\, \mathbf{1}^\mathsf{T} x = 1 \right\}$. Then

$$\mathrm{cr}(K_{m,n}) \geq \tfrac{1}{2} n^2 q_m - \tfrac{1}{2} n \lfloor \tfrac{1}{4}(m-1)^2 \rfloor \text{ for all } m, n.$$

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Semidefinite relaxation α_m of q_m

$q_m \geq \alpha_m := \min\{\langle X, Q\rangle \mid X \in \mathbb{R}^{Z_m \times Z_m}_{\geq 0}, \langle X, J\rangle = 1, X \text{ positive semidefinite}\}.$

Exploiting symmetry for computing α_m

Let $G_m := S_m \times \{-1, +1\}$. Then G_m acts on Z_m via

$$(\pi, \varepsilon) \cdot \sigma = \pi \sigma^{\varepsilon} \pi^{-1}, \qquad \text{for } (\pi, \varepsilon) \in G_m \text{ and } \sigma \in Z_m.$$

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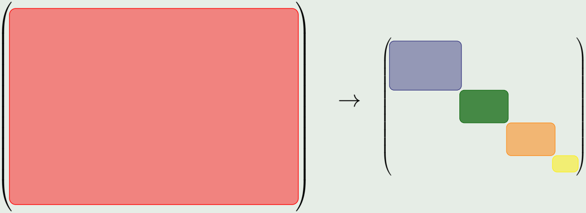
Crucial fact

Restriction of search space: find X in algebra of G_m-invariant matrices.
There exists a *block-diagonalization* of this algebra. (Artin-Wedderburn)



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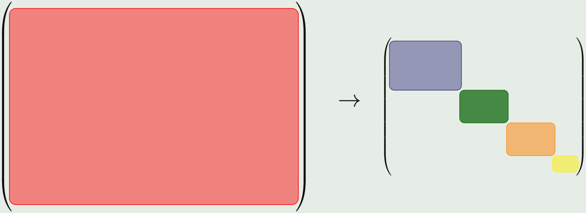

m	$(m-1)!$	$\sum m_i^2$	max block size
7	720	78	3
8	5040	380	7
9	40320	2438	12
10	362880	18744	38

Group invariance and Artin-Wedderburn

Let G be a finite group acting on a finite set Z. Decompose

$$\mathbb{C}^Z = \bigoplus_{i=1}^{k} \bigoplus_{j=1}^{m_i} V_{i,j},$$

for irreducible G-modules $V_{i,j}$ with $V_{i,j} \cong V_{i',j'}$ iff $i = i'$.

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$$\Phi\ :\ \left(\mathbb{C}^{Z\times Z}\right)^G \to \bigoplus_{i=1}^{k} \mathbb{C}^{m_i \times m_i},$$

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Key fact

For all $A \in \left(\mathbb{C}^{Z \times Z}\right)^G$ we have $A \succeq 0 \iff \Phi(A) \succeq 0$.

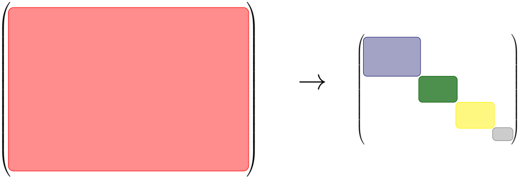
Symmetry reduction for α_m

Surjective S_m-homomorphism

$$f : M^{(1^m)} \to \mathbb{C}^{Z_m},$$

$$\begin{array}{|c|}\hline i_1 \\ \hline i_2 \\ \hline \vdots \\ \hline i_m \\ \hline \end{array} \mapsto \text{indicator vector in } \mathbb{C}^{Z_m} \text{ of } (i_1 i_2 \ldots i_m).$$

1. Decompose permutation module $M^{(1^m)}$ for S_m, results in vectors $u_{i,j}$.



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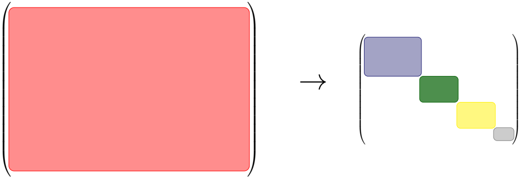
   - See thesis Daniel Brosch: faster decomposition of modules M^μ/F.



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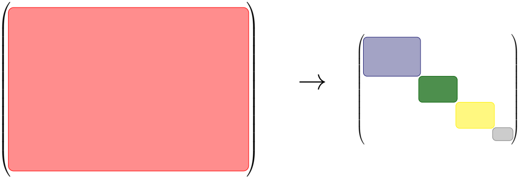
3. Reduce further with inversion ($\{\pm 1\}$)-action, splits each block in 2.



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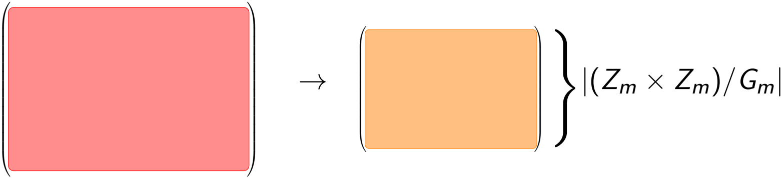

Results

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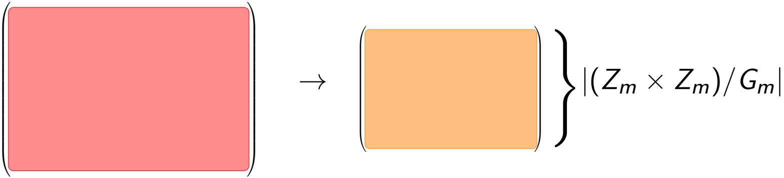

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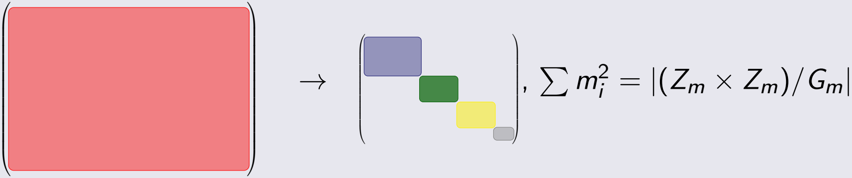

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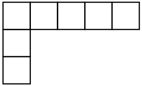
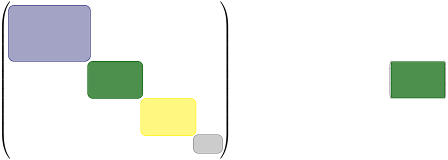
The relaxation β_m



$\lambda = (m-2, 1, 1)$

Observation

Requiring only one specific block of size $\lfloor \frac{m-1}{2} \rfloor$ to be PSD gives a good lower bound on α_m.



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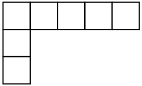

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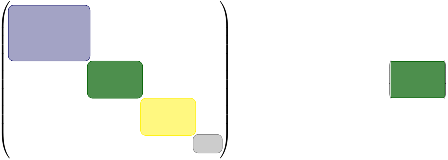
- Dual of β_m: few variables, small matrix block, many linear constraints.
- We solve the program without constraints, and iteratively add the most violated constraint until all constraints are satisfied.



Computational results

Theorem (Brosch, P.; 2022+)

$$\mathrm{cr}(K_{10,n}) \geq 4.8345n^2 - 10n,$$
$$\mathrm{cr}(K_{11,n}) \geq 5.9088n^2 - 12.2222n,$$
$$\mathrm{cr}(K_{12,n}) \geq 7.0906n^2 - 14.6666n,$$
$$\mathrm{cr}(K_{13,n}) \geq 8.3798n^2 - 17.3333n.$$

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$$\mathrm{cr}(K_{10,n}) \geq 4.8705n^2 - 10n,$$
$$\mathrm{cr}(K_{11,n}) \geq 5.9993n^2 - 12.5n,$$
$$\mathrm{cr}(K_{12,n}) \geq 7.2557n^2 - 15n,$$
$$\mathrm{cr}(K_{13,n}) \geq 8.6567n^2 - 18n.$$

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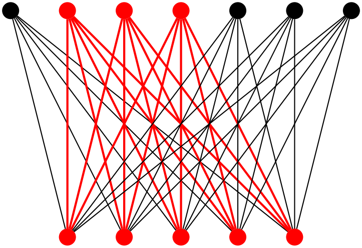


$$\mathrm{cr}(K_{m,n}) \geq \frac{\mathrm{cr}(K_{k,n})\binom{m}{k}}{\binom{m-2}{k-2}}$$

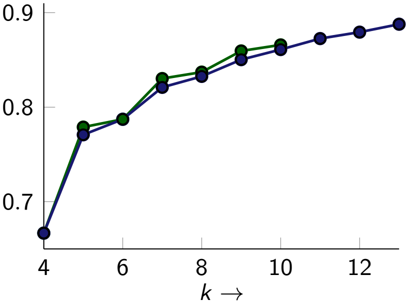
$\rightsquigarrow$ lower bounds on $\mathrm{cr}(K_{m,n})$ for $m \geq 13$.

Asymptotic results

Lemma (de Klerk, Maharry, Pasechnik, Richter, Salazar; 2006)

$$\lim_{n\to\infty} \frac{\mathrm{cr}(K_{n,n})}{Z(n,n)} \geq \frac{8\alpha_k}{k(k-1)}.$$

k	$\frac{8\alpha_k}{k(k-1)}$	$\frac{8\beta_k}{k(k-1)}$
7	0.8303	0.8210
8	0.8371	0.8326
9	0.8595	0.8503
10	**0.8659**	0.8610
11		**0.8726**
12		**0.8794**
13		**0.8878**



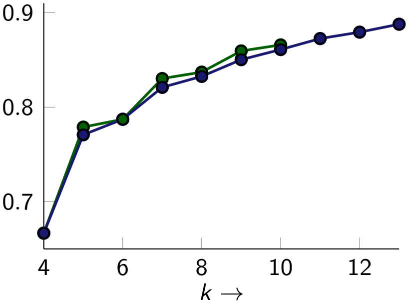
NB: Norin and Zwols claim $\lim_{n\to\infty} \frac{\mathrm{cr}(K_{n,n})}{Z(n,n)} \geq 0.905$. (2013, unpublished)

Asymptotic results

Lemma (de Klerk, Maharry, Pasechnik, Richter, Salazar; 2006)

$$\lim_{n\to\infty} \frac{\mathrm{cr}(K_{n,n})}{Z(n,n)} \geq \frac{8\alpha_k}{k(k-1)}.$$

k	$\frac{8\alpha_k}{k(k-1)}$	$\frac{8\beta_k}{k(k-1)}$
7	0.8303	0.8210
8	0.8371	0.8326
9	0.8595	0.8503
10	**0.8659**	0.8610
11		**0.8726**
12		**0.8794**
13		**0.8878**



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Lemma (Brosch, P.; 2022+)

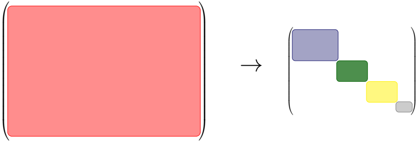
$$\lim_{n\to\infty} \frac{\mathrm{cr}(K_{m,n})}{Z(m,n)} \geq 0.8878 \frac{m}{m-1} \text{ for each } m \geq 13.$$

Summary

- Semidefinite programming gives good lower bounds on $\mathrm{cr}(K_{m,n})$.

n	best lower bound previously known	new lower bound	$Z(n,n)$
10	384	**388**	400
11	581	**589**	625
12	846	**865**	900
13	1192	**1229**	1296

New lower bounds on $\mathrm{cr}(K_{n,n})$



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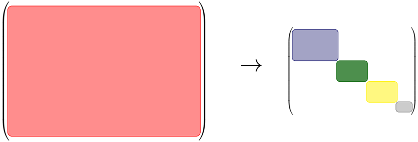

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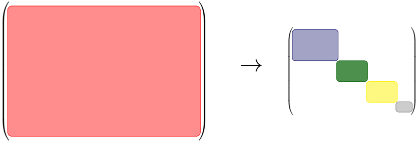

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- Semidefinite programming gives good lower bounds on $\mathrm{cr}(K_{m,n})$.
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- Possible to give optimum solution to β_m analytically?

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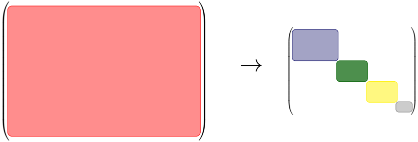

The relaxation β_m: structure in the solutions?

The dual of β_m

$$\beta_m = \max\Big\{t \ : \ Y \in \mathbb{R}^{\lfloor\frac{m-1}{2}\rfloor\times\lfloor\frac{m-1}{2}\rfloor}, \ Y \succeq 0, \\ \forall\omega \in (Z_m \times Z_m)/G_m \ : \langle Y, A_\omega\rangle + |\omega| t \leq |\omega| q_\omega\Big\},$$

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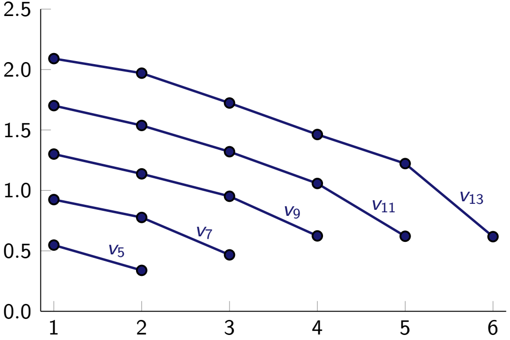
where $q_\omega := Q_{\sigma,\tau}$ for any $(\sigma,\tau) \in \omega$, and $A_\omega := U_\lambda^\mathsf{T} \left(\sum_{(\sigma,\tau)\in\omega} E_{\sigma,\tau}\right) U_\lambda$.



Vectors v_m such that optimal $Y = \frac{1}{(m-1)!} v_m v_m^\mathsf{T}$.

Lower bound on crossing number

Let Z_m be the set of m-cycles, and let $Q \in \mathbb{R}^{Z_m \times Z_m}$ with

$$Q_{\sigma,\tau} := \text{min. \#swaps of adjacent elements to change } \sigma \text{ into } \tau^{-1}.$$

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Define $q_m := \min \left\{ x^\top Q x \mid x \in \mathbb{R}^{Z_m}_{\geq 0}, \mathbf{1}^\top x = 1 \right\}$. Then

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$$= 2 \sum_{i<j} Q_{\gamma(b_i),\gamma(b_j)} + \sum_{i=1}^n Q_{\gamma(b_i),\gamma(b_i)} \leq 2\,\mathrm{cr}(K_{m,n}) + n \lfloor \tfrac{1}{4}(m-1)^2 \rfloor. \quad \square$$