A New Result on SOS-certificates for Copositive Matrices

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Workshop on Semidefinite and
Polynomial Optimization

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- Hard problems can be encoded as Opt. over COP_n
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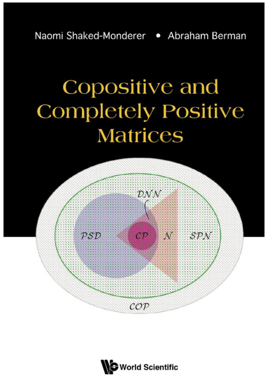
- Every nonconvex quadratic program involving continuous and binary variables can be modelled as a linear program over the Copositive cone [Burer, 2009]
- Determining whether a matrix is copositive is hard

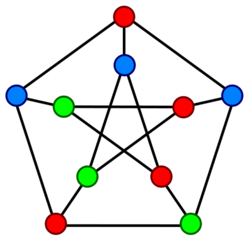
In this talk:

- Certificates for copositive matrices with sums of squares of polynomials



Example: Graph Matrices

Given a graph $G = (V, E)$, $S \subseteq V$ is **stable** if S contains no edge.



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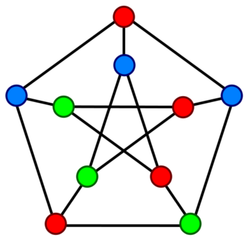

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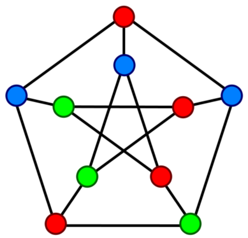

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The matrix $M_G := \alpha(G)(A_G + I) - J$ is copositive and satisfies many nice properties.

Sum-of-squares matrices

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For the $n \geq 5$, the Horn matrix:

$$H = \begin{pmatrix} 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 \end{pmatrix} \in \mathrm{COP}_5$$

and $(x^{\circ 2})^T H x^{\circ 2}$ is not SOS. [Choi-Lam 76']

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But, $(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2)(x^{\circ 2})^T H x^{\circ 2}$ is SOS [Parrilo 2000]

Observe that this implies $H \in \mathrm{COP}_5$.

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Theorem (Reznick 95')

Let p homogeneous polynomial and $p > 0$ on $\mathbb{R}^n \setminus \{0\}$ then $p(x)(\sum x_i^2)^r$ is SOS for some $r \in \mathbb{N}$.

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Based on that certificate Parrilo (2000) propose the cones $\mathcal{K}_n^{(r)}$ for approximating COP_n:

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$\boxed{\text{Are all copositive matrices SOS-certifiable?}}$

Main Result #1

Theorem (Laurent-Schweighofer-V 22'+)

Every 5×5 *copositive matrix is SOS-certifiable. That is,* $\mathrm{COP}_5 = \bigcup_{r \geq 0} \mathcal{K}_5^{(r)}$

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Some related results:

- $\mathrm{COP}_5 \neq \mathcal{K}_n^{(r)}$ for any $r \in \mathbb{N}$. [Dickinson-Dür-Gijben-Hildebrand 12']
- COP_5 is not the projection of a spectrahedra [Bodirsky-Kummer-Thom22']
- Every 5×5 copositive matrix with all-ones diagonal is SOS-certifiable (with $r = 1$) [DDGH 12']
- **Conjecture*** [DDGH 12']: Every copositive matrix with all-ones diagonal is SOS-certifiable

Constructing non SOS-certifiable copositive matrices

Theorem (L-V '21)

Let $M \in \mathrm{COP}_n$ *and assume that* $M \notin \mathcal{K}_n^{(0)}$, *i.e,* $(x^{\circ 2})^T M x^{\circ 2}$ *is not SOS. Then*

$$M' := \left(\begin{array}{c|c} M & 0 \\ \hline 0 & 0 \end{array}\right).$$

is copositive and non SOS-certifiable.

Hence, there are 6×6 *non SOS-certifiable copositive matrices.*

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Setting $y = 1$, The least degree homogeneous part is equal to $(x^{\circ 2})^T M x^{\circ 2}$, and it should be a SOS. Contradiction. $\square$

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Using a similar argument we obtain that the matrix

$$M' := \left(\begin{array}{c|cc} H & \multicolumn{2}{c}{0} \\ \hline 0 & 1 & -1 \\ & -1 & 1 \end{array}\right)$$

is not SOS-certifiable. This disproves Conjecture*.

Link to Lasserre-type SOS certificates

$$\boxed{\mathrm{COP}_n = \{M \in \mathcal{S}^n : x^T Mx \geq 0 \text{ for } x \in \Delta\}}$$

We say that M is Δ-certifiable if

$$x^T Mx = \underbrace{\sigma}_{\text{SOS}} + \sum_{i=1}^{n} x_i \underbrace{\sigma_i}_{SOS} + u(x)(\sum_{i=1}^{n} x_i - 1),$$

for some $\sigma, \sigma_i \in$ SOS and $u(x) \in \mathbb{R}[x]$

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Theorem. follows from [Parrilo-de Klerk-Laurent 02'] and [Laurent-V 22']

$$\boxed{M \text{ is } \Delta\text{-certifiable} \Longrightarrow M \text{ is } \mathbb{S}\text{-certifiable} \Longleftrightarrow M \text{ is SOS-certifiable}}$$

Important Reduction: Diagonal Scalings of H

Theorem (Laurent-V 22')

Every 5×5 *copositive matrix is SOS-certifiable if and only if DHD is SOS-certifiable for all D positive diagonal.*

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Sketch of the proof:

1. The extreme rays of COP_5 are fully described [Hildebrand 12']
2. Every extreme matrix that is not of the form *DHD* is Δ-certifiable, and thus SOS-certifiable.
3. For this we use sufficient conditions for finite convergence of Lasserre hierarchies [Marshall-Nie 09'-12']

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(!) H is SOS-certifiable but not Δ-certifiable

Pure states-reformulation

Let $D = \mathrm{diag}(d_1, d_2, \ldots, d_5)$ be a positive diagonal matrix
To show: DHD is SOS-certifiable, or equivalently, DHD is $\mathbb{S}$-certifiable, that is,

$$(x^{\circ 2})^T DHDx^{\circ 2} \in \Sigma + (\sum_{i=1}^{5} x_i^2 - 1)$$

or equivalently, after replacing $x_i \to \frac{x_i}{\sqrt{d_i}}$,

$$(x^{\circ 2})^T Hx^{\circ 2} \in \Sigma + (\sum_{i=1}^{5} d_i x_i^2 - 1) \text{ for all } d_i > 0?$$

$$\boxed{\text{Membership in preorderings?}}$$

Membership in preorderings: Theory of Pure States

Let $M = \Sigma + (\sum_{i=1}^{5} d_i x_i^2 - 1)$ $\qquad$ $S := \{x \in \mathbb{R}^n : \sum d_i x_i^2 = 1\}$

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From [Burgdof, Scheiderer, Schweighofer 12']:

- Let $f \geq 0$ on S
- Assume there exists a "unit" $u \in M$ with the "same behavior" of f around the zero set. $\implies$ $f \in M$
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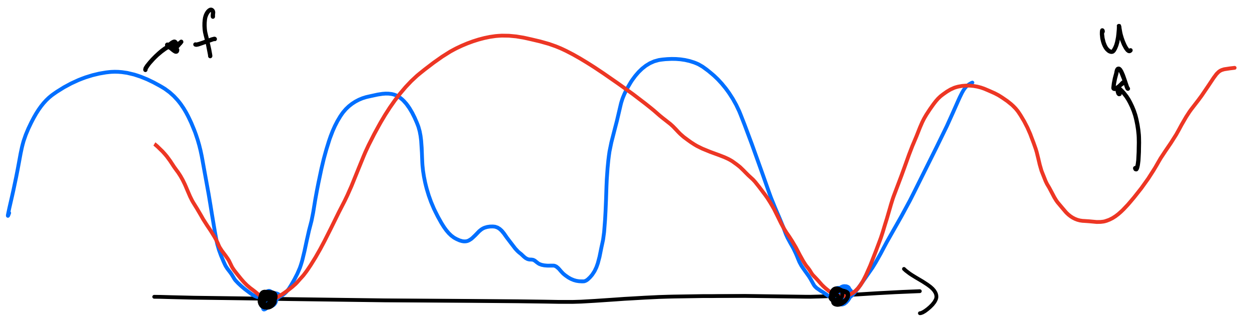

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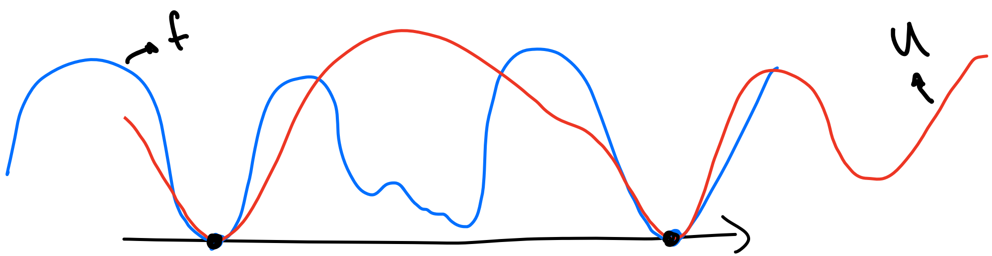

Taking $f = (x^{\circ 2})^T Hx$ and $u = (\sum_{i=1}^{5} x_i^2) f$ we obtain that $f \in M$

Theorem (Schweighofer-V 22'+)

DHD is SOS-certifiable for any positive diagonal matrix D. Hence, every 5×5 copositive matrix is SOS-certifiable.

Graph Matrices and Main Result #2

Let $G = (V = [n], E)$.

$M_G = \alpha(G)(A_G + I) - J \in \mathrm{COP}_n$ $\qquad$ $f_G(x) := (x^{\circ 2})^T M_G x^{\circ 2} \geq 0$ on $\mathbb{R}^n$.

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$(\sum_{i=1}^n x_i^2)^{\alpha(G)-1} f_G(x)$ *is SOS*

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Lemma[Laurent-V 22'] (Gvozdenović-L 07'):

If (M_H is SOS-certifiable $\Longrightarrow$ $M_{H\oplus i}$ is SOS-certifiable) then M_G is SOS-certifiable for every graph G.

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Theorem (Schweighofer-V 22'+)
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Graph Matrices and Main Result #2

Let $G = (V = [n], E)$.

$M_G = \alpha(G)(A_G + I) - J \in \mathrm{COP}_n$ $\qquad f_G(x) := (x^{\circ 2})^T M_G x^{\circ 2} \geq 0$ on $\mathbb{R}^n$.

Conjecture 1 (de Klerk-Pasechnik 2002)
$(\sum_{i=1}^n x_i^2)^{\alpha(G)-1} f_G(x)$ *is SOS*

Def. Let $G \oplus i$ the graph obtained by adding the isolated node i to G.

Lemma[Laurent-V 22'] (Gvozdenović-L 07'):

If (M_H is SOS-certifiable $\Longrightarrow$ $M_{H \oplus i}$ is SOS-certifiable) then M_G is SOS-certifiable for every graph G.

Theorem (Schweighofer-V 22'+)
M_G is SOS-certifiable, that is, $(\sum_{i=1}^n x_i^2)^r f_G$ is SOS for some $r \in \mathbb{N}$.

Thanks!