

# Orthogonal Schedules

# Maximal orthogonal schedules

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TU Eindhoven



Figure 1: Brabant



## Best of Brabant competition (BOB):



**Figure 3:** Participating teams

## Best of Brabant competition (BOB):



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Round:	1	2	3
	H	A	H
	H	A	A
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(a) Home/Away patterns (HAP-set)

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(b) Match schedule

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(a) Home/Away patterns (HAP-set)



(b) Match schedule

What if we want NAC-Willem II in a different round, without changing the HAP-set?



R1	R2	R3
 - 	 - 	 - 
 - 	 - 	 - 





















(a) Match schedule

Round 3 is 'xed' with this HAP-set.

R1	R2	R3
 - 	 - 	 - 
 - 	 - 	 - 

(a) Match schedule

Round 3 is 'xed' with this HAP-set. Could have done better:

R1	R2	R3	R1	R2	R3
 - 	 - 	 - 	 - 	 - 	 - 
 - 	 - 	 - 	 - 	 - 	 - 

(a) Alt schedule 1

(b) Alt schedule 2

Round:	1	2	3
	H	H	H
	H	A	A
	A	H	A
	A	A	H

Figure 7: Alt HAP-Set

## Definition

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Higher width  $\Rightarrow$  more exhibability.



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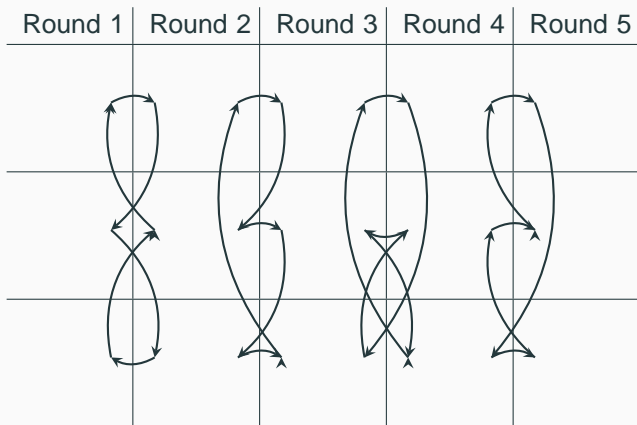
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,	,	,	,	,
,	,	,	,	,
,	,	,	,	,

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,	,	,	,	,
,	,	,	,	,
,	,	,	,	,

Round	1	2	3	4	5
	H	H	H	H	H
	A	A	H	A	H
	H	A	A	H	A
	A	H	A	A	H
	H	A	H	A	A
	A	H	A	H	A

Table 1: HAP-set with width 2

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In any HAP-set, there must be 2 teams for which the HAP-set differs only in two columns.

Definition

Two vectors  $v; v^0 \in \{0, 1\}^{2^n - 1}$  have  $\text{opp}(v; v^0) = \#\{i : v_i \neq v_i^0\} = k$ . A

HAP-set  $H$  has  $\text{opp}(H)$  defined as:

$$\text{opp}(H) = \min_{v; v^0 \in H} \text{opp}(v; v^0) \quad \text{and} \quad \text{opp}_n = \max_H \text{opp}(H)$$

Clearly,  $\text{width}(H) = \text{opp}(H)$  and  $w_n = \text{opp}_n = n$ .

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Definition

Two schedules  $S; S^0$  are rotational orthogonal when they are orthogonal and their rounds are a permutations of one another.

The rotational width of  $H$  is the size of the largest set of rotational orthogonal schedules.

$$x_n = \max_H \text{rotw}(H)$$



Not difficult to see that:

$$\text{rotw}(H) = \text{width}(H) - \text{opp}(H) = 2x_n - w_n - o_n = n$$

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When  $2n = 6$ :  $x_n = w_n = o_n = 2$ . What happens with other values of  $n$ ?

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$$\text{rotw}(H) = \text{width}(H) = \text{opp}(H) = 2^{x_n} w_n o_n n$$

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Theorem

When  $n = 2^k$ ,  $x_n = w_n = o_n = n$ .

Proof has two steps.

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- ^ First: show  $o_n = n$  by constructing  $H$  with  $\text{opp}(H) = n$ .
- ^ Second: Construct rotational orthogonal schedules compatible with  $n$ .

Constructing HAP-set  $H$  with  $\text{opp}(H) = n$  when  $n = 2^k$ .

Team 0	1	
1	0	

Above HAP-set has  $\text{opp}(H) = n$ .

NB: This procedure can be generalized to create HAP-sets with  $\text{opp}(H) = n$  even in cases when  $n \neq 2^k$ .

Constructing HAP-set  $H$  with  $\text{opp}(H) = n$  when  $n = 2^k$ .

	Round 1	2	3
Team 0	1	1	1
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2	1	0	0
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	Round 1	2	3	4	5	6	7
Team 0	1	1	1	1	1	1	1
1	0	1	0	1	0	1	0
2	1	0	0	1	1	0	0
3	0	0	1	1	0	0	1
4	1	1	1	0	0	0	0
5	0	1	0	0	1	0	1
6	1	0	0	0	0	1	1
7	0	0	1	0	1	1	0

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## Constructing the HAP-set:

- ^ Set teams  $T = \mathbb{Z}_2^n$  and rounds  $R = \mathbb{Z}_2^n \setminus \{0\}$  - i.e., each team  $t$  and round  $r$  is an  $n$ -bit  $b(t); b(r)$ .
- ^ In HAP-set  $H$ , team  $t$  gets assigned Home in round  $r$  when:

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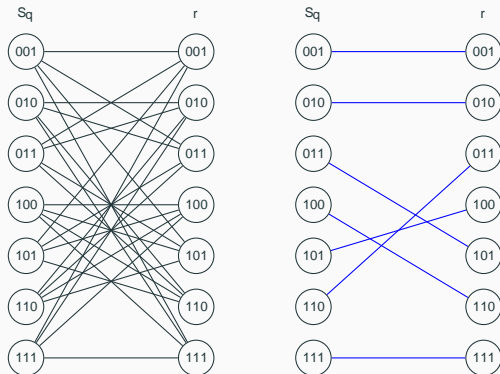
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Insight:  $S_q$  can be scheduled in round  $r$  when  $r \cdot q = 1$ .

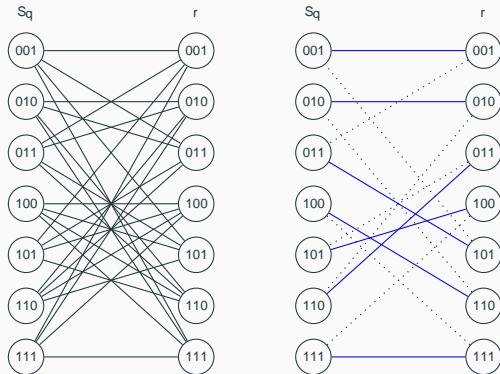
Bipartite graph  $G = (R \cup R; E)$  with  $(r; q) \in E$  when  $|r - q| = 1$ , is regular with degree  $n$ .

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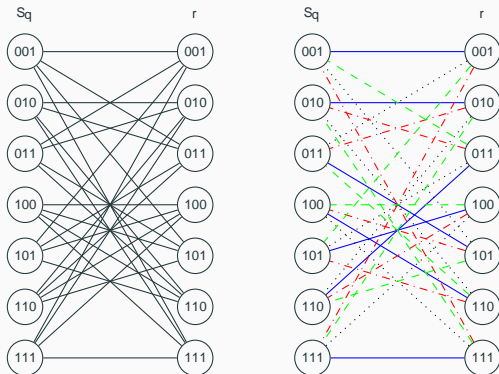
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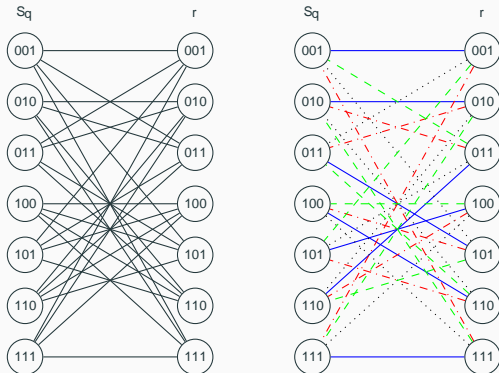


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# Conclusion

What we know:

n	2	3	4	5	6	7	8	...
$o_n$	2	2	4	4	6	6	8	...
$w_n$	2	2	4	3	?	?	8	...
$x_n$	2	2	4	3	4?5?	?	8	...

**Table 2:** Known values

More questions remain:

- Can we find  $n$  for which  $x_n \notin w_n$ ?
- Is it true that  $o_n = 2b_{\frac{n}{2}}c$ ?