# Orthogonal Schedules

# Maximal orthogonal schedules

Roel Lambers
Joint work with Frits Spieksma, Mehmet Akif Yılmız, Viresh Patel, Jop Briët

31-08-2022

TU Eindhoven

**Figure 1:** Brabant

**Figure 2:** Clubs in Brabant

Problem: They never win.

Best of Brabant competition (BOB):


Figure 3: Participating teams

Best of Brabant competition (BOB):


**Figure 3:** Participating teams

| Round: | 1 | 2 | 3 |
|---|---|---|---|
| | H | A | H |
| | H | A | A |
| | A | H | A |
| | A | H | H |

**(a)** Home/Away patterns (HAP-set)

Best of Brabant competition (BOB):


**Figure 3:** Participating teams

| Round: | 1 | 2 | 3 |
|---|---|---|---|
| NAC | H | A | H |
| Willem II | H | A | A |
| RKC | A | H | A |
| PSV | A | H | H |

**(a)** Home/Away patterns (HAP-set)

| R1 | R2 | R3 |
|---|---|---|
| NAC – RKC | PSV – NAC | NAC – Willem II |
| Willem II – PSV | RKC – Willem II | PSV – RKC |

**(b)** Match schedule

Best of Brabant competition (BOB):


**Figure 3:** Participating teams

| Round: | 1 | 2 | 3 |
|---|---|---|---|
| NAC | H | A | H |
| Willem II | H | A | A |
| RKC | A | H | A |
| PSV | A | H | H |

**(a)** Home/Away patterns (HAP-set)

| R1 | R2 | R3 |
|---|---|---|
| NAC - RKC | PSV - NAC | NAC - Willem II |
| Willem II - PSV | RKC - Willem II | PSV - RKC |

**(b)** Match schedule

What if we want NAC-Willem II in a different round, without changing the HAP-set?

| R1 | R2 | R3 |
|---|---|---|
| - | - | - |
| - | - | - |

(a) Match schedule

Round 3 is 'fixed' with this HAP-set.

**(a)** Match schedule

Round 3 is 'fixed' with this HAP-set. Could have done better:

**(a)** Alt schedule 1

**(b)** Alt schedule 2

| Round: | 1 | 2 | 3 |
|---|---|---|---|
| | H | H | H |
| | H | A | A |
| | A | H | A |
| | A | A | H |

**Figure 7:** Alt HAP-Set

**Definition**

- HAP-set: set of $2n$ $0-1$-vectors of length $2n-1$.
- Schedule: set of $2n-1$ perfect matchings on $2n$ teams.

**Definition**

- HAP-set: set of $2n$ $0-1$-vectors of length $2n-1$.
- Schedule: set of $2n-1$ perfect matchings on $2n$ teams.

Two schedules $S, S'$ are *orthogonal* - $S \perp S'$ if for every match $\{t, t'\}$, scheduled in rounds $r, r'$ respectively, we have $r \neq r'$.

**Definition**

- HAP-set: set of $2n$ $0-1$-vectors of length $2n-1$.
- Schedule: set of $2n-1$ perfect matchings on $2n$ teams.

Two schedules $S, S'$ are *orthogonal* - $S \perp S'$ if for every match $\{t, t'\}$, scheduled in rounds $r, r'$ respectively, we have $r \neq r'$.

Given HAP-set $\mathcal{H}$, let $\mathcal{S}(\mathcal{H})$ be all schedules *compatible* with $\mathcal{H}$.

**Definition**

- HAP-set: set of $2n$ $0-1$-vectors of length $2n-1$.
- Schedule: set of $2n-1$ perfect matchings on $2n$ teams.

Two schedules $S, S'$ are *orthogonal* - $S \perp S'$ if for every match $\{t, t'\}$, scheduled in rounds $r, r'$ respectively, we have $r \neq r'$.

Given HAP-set $\mathcal{H}$, let $\mathcal{S}(\mathcal{H})$ be all schedules *compatible* with $\mathcal{H}$. We define *width* of $\mathcal{H}$ as:

$$\text{width}(\mathcal{H}) = \max_{\mathcal{S} \subset \mathcal{S}(\mathcal{H})} \#\{S \in \mathcal{S} : S \perp S' \quad \forall S, S' \in \mathcal{S}\}$$

**Definition**

- HAP-set: set of $2n$ $0-1$-vectors of length $2n-1$.
- Schedule: set of $2n-1$ perfect matchings on $2n$ teams.

Two schedules $S, S'$ are *orthogonal* - $S \perp S'$ if for every match $\{t, t'\}$, scheduled in rounds $r, r'$ respectively, we have $r \neq r'$.

Given HAP-set $\mathcal{H}$, let $\mathcal{S}(\mathcal{H})$ be all schedules *compatible* with $\mathcal{H}$. We define *width* of $\mathcal{H}$ as:

$$\mathrm{width}(\mathcal{H}) = \max_{\mathcal{S} \subset \mathcal{S}(\mathcal{H})} \#\{S \in \mathcal{S} : S \perp S' \quad \forall S, S' \in \mathcal{S}\}$$

The width of a HAP-set $\mathcal{H}$ is the order of the largest set of pair-wise orthogonal schedules compatible with $\mathcal{H}$.

**Definition**

- HAP-set: set of $2n$ $0-1$-vectors of length $2n-1$.
- Schedule: set of $2n-1$ perfect matchings on $2n$ teams.

Two schedules $S, S'$ are *orthogonal* - $S \perp S'$ if for every match $\{t, t'\}$, scheduled in rounds $r, r'$ respectively, we have $r \neq r'$.

Given HAP-set $\mathcal{H}$, let $\mathcal{S}(\mathcal{H})$ be all schedules *compatible* with $\mathcal{H}$. We define *width* of $\mathcal{H}$ as:

$$\text{width}(\mathcal{H}) = \max_{\mathcal{S} \subset \mathcal{S}(\mathcal{H})} \#\{S \in \mathcal{S} : S \perp S' \quad \forall S, S' \in \mathcal{S}\}$$

The width of a HAP-set $\mathcal{H}$ is the order of the largest set of pair-wise orthogonal schedules compatible with $\mathcal{H}$.

Higher width $\implies$ more flexibility.

Central questions:

- Given HAP-set $\mathcal{H}$, what is width($\mathcal{H}$)?

Central questions:

- Given HAP-set $\mathcal{H}$, what is width($\mathcal{H}$)?
- Given number of teams $N = 2n$, what is $w_{2n}$, given by:

$$w_n = \max_{\mathcal{H}} \text{width}(\mathcal{H}) \ \mathcal{H} \text{ is HAP-set on } 2n \text{ teams}$$

Central questions:

- Given HAP-set $\mathcal{H}$, what is width($\mathcal{H}$)?
- Given number of teams $N = 2n$, what is $w_{2n}$, given by:

$$w_n = \max_{\mathcal{H}} \text{width}(\mathcal{H}) \; \mathcal{H} \text{ is HAP-set on } 2n \text{ teams}$$

We already saw that when $2n = 4$, we could create a HAP-set with width 2.

Central questions:

- Given HAP-set $\mathcal{H}$, what is width($\mathcal{H}$)?
- Given number of teams $N = 2n$, what is $w_{2n}$, given by:

$$w_n = \max_{\mathcal{H}} \text{width}(\mathcal{H}) \; \mathcal{H} \text{ is HAP-set on } 2n \text{ teams}$$

We already saw that when $2n = 4$, we could create a HAP-set with width 2. Do we have $w_n \geq 2$?

Yes.

Central questions:

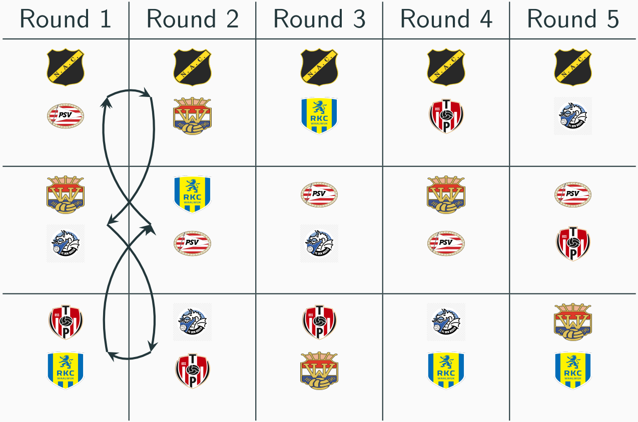
- Given HAP-set $\mathcal{H}$, what is width($\mathcal{H}$)?
- Given number of teams $N = 2n$, what is $w_{2n}$, given by:

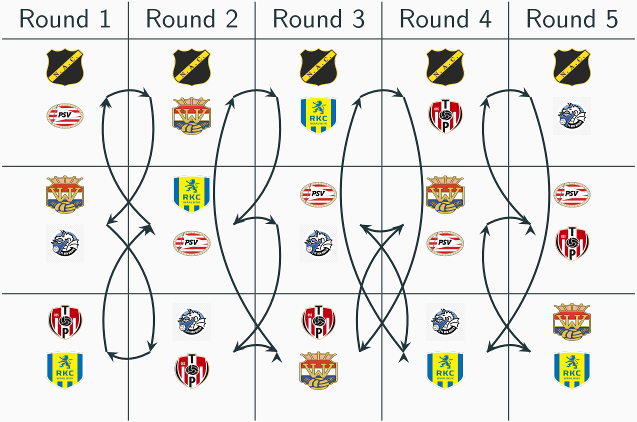
$$w_n = \max_{\mathcal{H}} \text{width}(\mathcal{H}) \ \mathcal{H} \text{ is HAP-set on } 2n \text{ teams}$$

We already saw that when $2n = 4$, we could create a HAP-set with width 2. Do we have $w_n \geq 2$?

Yes.

| Round 1 | Round 2 | Round 3 | Round 4 | Round 5 |
|---|---|---|---|---|
| | | | | |
| | | | | |
| | | | | |


Round 1
Round 2
Round 3
Round 4
Round 5


Round 1
Round 2
Round 3
Round 4
Round 5

| Round 1 | Round 2 | Round 3 | Round 4 | Round 5 |
| --- | --- | --- | --- | --- |
| , | , | , | , | , |
| , | , | , | , | , |
| , | , | , | , | , |

| Round 1 | Round 2 | Round 3 | Round 4 | Round 5 |
|---|---|---|---|---|
| N.A.C., PSV | N.A.C., W | N.A.C., RKC | N.A.C., TOP | N.A.C., FC Den Bosch |
| W, FC Den Bosch | RKC, PSV | PSV, FC Den Bosch | W, PSV | PSV, TOP |
| TOP, RKC | FC Den Bosch, TOP | TOP, W | FC Den Bosch, RKC | W, RKC |

| Round | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| N.A.C. | H | H | H | H | H |
| PSV | A | A | H | A | H |
| W | H | A | A | H | A |
| RKC | A | H | A | A | H |
| TOP | H | A | H | A | A |
| FC Den Bosch | A | H | A | H | A |

**Table 1:** HAP-set with width 2

For every even $n \geq 4$, the above method creates HAP-set $\mathcal{H}$ with width($\mathcal{H}$) $= 2$ - two compatible orthogonal schedules can be found by rotating the rounds.

For every even $n \geq 4$, the above method creates HAP-set $\mathcal{H}$ with width$(\mathcal{H}) = 2$ - two compatible orthogonal schedules can be found by rotating the rounds.

For $2n = 6$, impossible to do better. Why?

For every even $n \geq 4$, the above method creates HAP-set $\mathcal{H}$ with width$(\mathcal{H}) = 2$ - two compatible orthogonal schedules can be found by rotating the rounds.

For $2n = 6$, impossible to do better. Why?
In any HAP-set, there must be 2 teams for which the HAP-set differs only in two columns.

**Definition**
Two vectors $v, v' \in \{0,1\}^{2n-1}$ have $\text{opp}(v, v') = \#\{i : v_i \neq v_i'\} = k$. A HAP-set $\mathcal{H}$ has $\text{opp}(\mathcal{H})$ defined as:

$$\text{opp}(\mathcal{H}) = \min_{\{v,v'\} \subset \mathcal{H}} \text{opp}(v, v') \qquad\qquad o_n = \max_{\mathcal{H}} \text{opp}(\mathcal{H})$$

Clearly, $\text{width}(\mathcal{H}) \leq \text{opp}(\mathcal{H})$ and $w_n \leq o_n \leq n$.

**Definition**

Two vectors $v, v' \in \{0,1\}^{2n-1}$ have $\text{opp}(v, v') = \#\{i : v_i \neq v'_i\} = k$. A HAP-set $\mathcal{H}$ has $\text{opp}(\mathcal{H})$ defined as:

$$\text{opp}(\mathcal{H}) = \min_{\{v,v'\} \subset \mathcal{H}} \text{opp}(v, v') \qquad\qquad o_n = \max_{\mathcal{H}} \text{opp}(\mathcal{H})$$

Clearly, $\text{width}(\mathcal{H}) \leq \text{opp}(\mathcal{H})$ and $w_n \leq o_n \leq n$.

**Definition**

Two schedules $S, S'$ are rotational orthogonal when they are orthogonal and their rounds are a permutations of one another.

**Definition**
Two vectors $v, v' \in \{0,1\}^{2n-1}$ have $\text{opp}(v, v') = \#\{i : v_i \neq v'_i\} = k$. A HAP-set $\mathcal{H}$ has $\text{opp}(\mathcal{H})$ defined as:

$$\text{opp}(\mathcal{H}) = \min_{\{v,v'\}\subset\mathcal{H}} \text{opp}(v, v') \qquad\qquad o_n = \max_{\mathcal{H}} \text{opp}(\mathcal{H})$$

Clearly, $\text{width}(\mathcal{H}) \leq \text{opp}(\mathcal{H})$ and $w_n \leq o_n \leq n$.

**Definition**
Two schedules $S, S'$ are rotational orthogonal when they are orthogonal and their rounds are a permutations of one another.

The rotational width of $\mathcal{H}$ is the size of the largest set of rotational orthogonal schedules.

$$x_n = \max_{\mathcal{H}} \text{rotw}(\mathcal{H})$$

Not difficult to see that:

$$\mathrm{rotw}(\mathcal{H}) \leq \mathrm{width}(\mathcal{H}) \leq \mathrm{opp}(\mathcal{H}) \qquad 2 \leq x_n \leq w_n \leq o_n \leq n$$

Not difficult to see that:

$$\mathrm{rotw}(\mathcal{H}) \leq \mathrm{width}(\mathcal{H}) \leq \mathrm{opp}(\mathcal{H}) \qquad 2 \leq x_n \leq w_n \leq o_n \leq n$$

In fact, when $n$ is odd, $o_n \leq n - 1$!

Not difficult to see that:

$$\text{rotw}(\mathcal{H}) \leq \text{width}(\mathcal{H}) \leq \text{opp}(\mathcal{H}) \qquad 2 \leq x_n \leq w_n \leq o_n \leq n$$

In fact, when $n$ is odd, $o_n \leq n - 1$!

When $2n = 6$: $x_n = w_n = o_n = 2$. What happens with other values of $n$?

Not difficult to see that:

$$\text{rotw}(\mathcal{H}) \leq \text{width}(\mathcal{H}) \leq \text{opp}(\mathcal{H}) \qquad 2 \leq x_n \leq w_n \leq o_n \leq n$$

In fact, when $n$ is odd, $o_n \leq n - 1$!

When $2n = 6$: $x_n = w_n = o_n = 2$. What happens with other values of $n$?

**Theorem**
*When $n = 2^k$, $x_n = w_n = o_n = n$.*

Proof has two steps.

- First: show $o_n = n$ by constructing $\mathcal{H}$ with $\text{opp}(\mathcal{H}) = n$.

Not difficult to see that:

$$\text{rotw}(\mathcal{H}) \leq \text{width}(\mathcal{H}) \leq \text{opp}(\mathcal{H}) \qquad 2 \leq x_n \leq w_n \leq o_n \leq n$$

In fact, when $n$ is odd, $o_n \leq n - 1$!

When $2n = 6$: $x_n = w_n = o_n = 2$. What happens with other values of $n$?

**Theorem**
*When $n = 2^k$, $x_n = w_n = o_n = n$.*

Proof has two steps.

- First: show $o_n = n$ by constructing $\mathcal{H}$ with $\text{opp}(\mathcal{H}) = n$.
- Second: Construct $n$ rotational orthogonal schedules compatible with $n$.

Constructing HAP-set $\mathcal{H}$ with $\text{opp}(\mathcal{H}) = n$ when $n = 2^k$.

| | | |
|---:|:---:|---|
| Team 0 | 1 | |
| 1 | 0 | |

Above HAP-set has $\text{opp}(\mathcal{H}) = n$.

NB: This procedure can be generalized to create HAP-sets with $\text{opp}(\mathcal{H}) = n$ even in cases when $n \neq 2^k$.

Constructing HAP-set $\mathcal{H}$ with $\text{opp}(\mathcal{H}) = n$ when $n = 2^k$.

| | Round 1 | 2 | 3 | |
|---|---|---|---|---|
| Team 0 | 1 | 1 | 1 | |
| 1 | 0 | 1 | 0 | |
| 2 | 1 | 0 | 0 | |
| 3 | 0 | 0 | 1 | |

Above HAP-set has $\text{opp}(\mathcal{H}) = n$.

NB: This procedure can be generalized to create HAP-sets with $\text{opp}(\mathcal{H}) = n$ even in cases when $n \neq 2^k$.

## Constructing HAP-set $\mathcal{H}$ with $\text{opp}(\mathcal{H}) = n$ when $n = 2^k$.

| | Round 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| Team 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |
| 2 | 1 | 0 | 0 | 1 | 1 | 0 | 0 |
| 3 | 0 | 0 | 1 | 1 | 0 | 0 | 1 |
| 4 | 1 | 1 | 1 | 0 | 0 | 0 | 0 |
| 5 | 0 | 1 | 0 | 0 | 1 | 0 | 1 |
| 6 | 1 | 0 | 0 | 0 | 0 | 1 | 1 |
| 7 | 0 | 0 | 1 | 0 | 1 | 1 | 0 |

Above HAP-set has $\text{opp}(\mathcal{H}) = n$.

NB: This procedure can be generalized to create HAP-sets with $\text{opp}(\mathcal{H}) = n$ even in cases when $n \neq 2^k$.

Constructing the HAP-set:

- Set teams $T = \mathbb{Z}_2^n$ and rounds $R = \mathbb{Z}_2^n \setminus \{0\}$ - i.e., each team $t$ and round $r$ is an $n$-bit $b(t), b(r)$.
- In HAP-set $\mathcal{H}$, team $t$ gets assigned Home in round $r$ when:

$$r \cdot t = \sum_i b_i(r) b_i(t) = 0 \mod 2$$

Constructing the HAP-set:

- Set teams $T = \mathbb{Z}_2^n$ and rounds $R = \mathbb{Z}_2^n \setminus \{0\}$ - i.e., each team $t$ and round $r$ is an $n$-bit $b(t), b(r)$.
- In HAP-set $\mathcal{H}$, team $t$ gets assigned Home in round $r$ when:

$$r \cdot t = \sum_i b_i(r) b_i(t) = 0 \mod 2$$

Only need $n$ orthogonal schedules. Construct schedule $S = (S_q)_{q \in R}$ as follows:

$$S_q = \{\{i, j\} \colon i + j = q\}$$

Constructing the HAP-set:

- Set teams $T = \mathbb{Z}_2^n$ and rounds $R = \mathbb{Z}_2^n \setminus \{0\}$ - i.e., each team $t$ and round $r$ is an $n$-bit $b(t), b(r)$.
- In HAP-set $\mathcal{H}$, team $t$ gets assigned Home in round $r$ when:

$$r \cdot t = \sum_i b_i(r) b_i(t) = 0 \mod 2$$

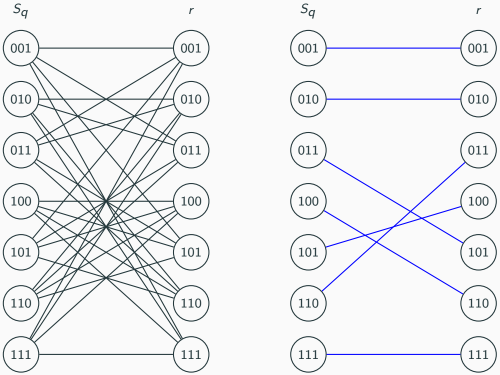
Only need $n$ orthogonal schedules. Construct schedule $S = (S_q)_{q \in R}$ as follows:

$$S_q = \{\{i, j\} \colon i + j = q\}$$

Insight: $S_q$ *can be scheduled in round* $r$ *when* $r \cdot q = 1$.

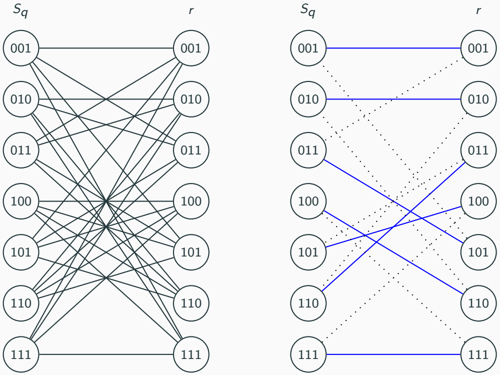
Bipartite graph $G = (R \times R, E)$ with $(r, q) \in E$ when $r \cdot q = 1$, is regular with degree $n$.

Bipartite graph $G = (R \times R, E)$ with $(r, q) \in E$ when $r \cdot q = 1$, is regular with degree $n$.



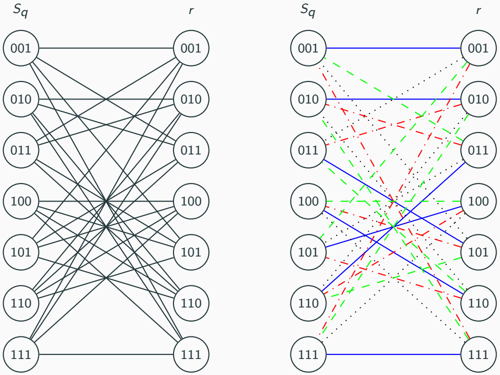

Only possible when $n = 2^k$! Otherwise, $x_n < n$.

Bipartite graph $G = (R \times R, E)$ with $(r, q) \in E$ when $r \cdot q = 1$, is regular with degree $n$.

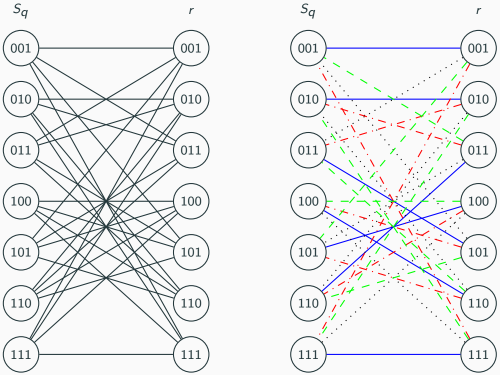



Only possible when $n = 2^k$! Otherwise, $x_n < n$.

Bipartite graph $G = (R \times R, E)$ with $(r, q) \in E$ when $r \cdot q = 1$, is regular with degree $n$.



Bipartite graph $G = (R \times R, E)$ with $(r, q) \in E$ when $r \cdot q = 1$, is regular with degree $n$.




Only possible when $n = 2^k$! Otherwise, $x_n < n$.

# Conclusion

What we know:

| n | 2 | 3 | 4 | 5 | 6 | 7 | 8 | ... |
|---|---|---|---|---|---|---|---|---|
| $o_n$ | 2 | 2 | 4 | 4 | 6 | $\leq 6$ | 8 | ... |
| $w_n$ | 2 | 2 | 4 | 3 | ? | ? | 8 | ... |
| $x_n$ | 2 | 2 | 4 | 3 | 4?5? | ? | 8 | ... |

**Table 2:** Known values

More questions remain:

- Can we find $n$ for which $x_n \neq w_n$?
- Is it true that $o_n = 2\lfloor \frac{n}{2} \rfloor$?