Orthogonal Schedules
Maximal orthogonal schedules

Roel Lambers
Joint work with Frits Spieksma, Mehmet Akif Yılmız, Viresh Patel, Jop Briët
31-08-2022

TU Eindhoven
Figure 1: Brabant
Figure 2: Clubs in Brabant

Problem: They never win.
Best of Brabant competition (BOB):

**Figure 3:** Participating teams
Best of Brabant competition (BOB):

**Figure 3: Participating teams**

<table>
<thead>
<tr>
<th>Round</th>
<th>1</th>
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<th>3</th>
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</thead>
<tbody>
<tr>
<td>H</td>
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<tr>
<td>H</td>
<td>A</td>
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<td>A</td>
<td>H</td>
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</tbody>
</table>

(a) Home/Away patterns (HAP-set)
Best of Brabant competition (BOB):

![BOB (4 teams)](image)

**Figure 3:** Participating teams

<table>
<thead>
<tr>
<th>Round</th>
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<tr>
<td>(a) Home/Away patterns (HAP-set)</td>
<td>H</td>
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<table>
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<th></th>
<th>R1</th>
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<tbody>
<tr>
<td>(b) Match schedule</td>
<td><img src="image" alt="Match schedule" /></td>
<td><img src="image" alt="Match schedule" /></td>
<td><img src="image" alt="Match schedule" /></td>
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</tbody>
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What if we want NAC-Willem II in a different round, without changing the HAP-set?
Best of Brabant competition (BOB):

Figure 3: Participating teams

(a) Home/Away patterns (HAP-set)

What if we want NAC-Willem II in a different round, without changing the HAP-set?
Round 3 is 'fixed' with this HAP-set.
Round 3 is 'fixed' with this HAP-set. Could have done better:

(a) Match schedule

(b) Alt schedule 1

(b) Alt schedule 2

Figure 7: Alt HAP-Set
Definition

- HAP-set: set of $2n$ 0–1-vectors of length $2n - 1$.
- Schedule: set of $2n - 1$ perfect matchings on $2n$ teams.
Definition

- HAP-set: set of $2n$ 0 – 1-vectors of length $2n – 1$.
- Schedule: set of $2n – 1$ perfect matchings on $2n$ teams.

Two schedules $S, S'$ are orthogonal - $S \perp S'$ if for every match $\{t, t'\}$, scheduled in rounds $r, r'$ respectively, we have $r \neq r'$. 

Given HAP-set $H$, let $S(H)$ be all schedules compatible with $H$. We define width of $H$ as:

$$\text{width}(H) = \max_{S \subset S(H)} \# \{S \in S(H): S \perp S' \forall S, S' \in S \}$$

The width of a HAP-set $H$ is the order of the largest set of pair-wise orthogonal schedules compatible with $H$. Higher width $\Rightarrow$ more flexibility.
Definition

- HAP-set: set of $2n$ 0–1-vectors of length $2n - 1$.
- Schedule: set of $2n - 1$ perfect matchings on $2n$ teams.

Two schedules $S$, $S'$ are orthogonal - $S \perp S'$ if for every match $\{t, t'\}$, scheduled in rounds $r, r'$ respectively, we have $r \neq r'$.

Given HAP-set $\mathcal{H}$, let $S(\mathcal{H})$ be all schedules compatible with $\mathcal{H}$.
**Definition**

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- **Schedule**: set of $2n−1$ perfect matchings on $2n$ teams.

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The width of a HAP-set $\mathcal{H}$ is the order of the largest set of pair-wise orthogonal schedules compatible with $\mathcal{H}$. 

Higher width $\Rightarrow$ more flexibility.
Definition

- HAP-set: set of $2^n 0 - 1$-vectors of length $2^n - 1$.
- Schedule: set of $2^n - 1$ perfect matchings on $2n$ teams.

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The width of a HAP-set $\mathcal{H}$ is the order of the largest set of pair-wise orthogonal schedules compatible with $\mathcal{H}$.

Higher width $\Rightarrow$ more flexibility.
Central questions:

- Given HAP-set $\mathcal{H}$, what is $\text{width}(\mathcal{H})$?

We already saw that when $2^n = 4$, we could create a HAP-set with width 2. Do we have $w_n \geq 2$? Yes.
Central questions:

• Given HAP-set $\mathcal{H}$, what is $\text{width}(\mathcal{H})$?

• Given number of teams $N = 2n$, what is $w_{2n}$, given by:

\[
    w_n = \max_{\mathcal{H}} \text{width}(\mathcal{H}) \quad \mathcal{H} \text{ is HAP-set on } 2n \text{ teams}
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Yes.
<table>
<thead>
<tr>
<th>Round 1</th>
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<th>Round 3</th>
<th>Round 4</th>
<th>Round 5</th>
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<tbody>
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Table 1: HAP-set with width 2
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<tbody>
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<td><img src="image14" alt="Teams" /></td>
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**Table 1:** HAP-set with width 2
For every even $n \geq 4$, the above method creates HAP-set $\mathcal{H}$ with \( \text{width}(\mathcal{H}) = 2 \) - two compatible orthogonal schedules can be found by rotating the rounds.
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For $2n = 6$, impossible to do better. Why?
For every even $n \geq 4$, the above method creates HAP-set $\mathcal{H}$ with width$(\mathcal{H}) = 2$ - two compatible orthogonal schedules can be found by rotating the rounds.

For $2n = 6$, impossible to do better. Why?
In any HAP-set, there must be 2 teams for which the HAP-set differs only in two columns.
Definition
Two vectors $v, v' \in \{0, 1\}^{2n-1}$ have $\text{opp}(v, v') = \#\{i : v_i \neq v'_i\} = k$. A HAP-set $\mathcal{H}$ has $\text{opp}(\mathcal{H})$ defined as:

$$\text{opp}(\mathcal{H}) = \min_{\{v, v'\} \subset \mathcal{H}} \text{opp}(v, v') \quad \quad o_n = \max_{\mathcal{H}} \text{opp}(\mathcal{H})$$

Clearly, $\text{width}(\mathcal{H}) \leq \text{opp}(\mathcal{H})$ and $w_n \leq o_n \leq n$. 

Definition
Two schedules $S, S'$ are rotational orthogonal when they are orthogonal and their rounds are a permutation of one another.

The rotational width of $\mathcal{H}$ is the size of the largest set of rotational orthogonal schedules.
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**Definition**

Two schedules $S, S'$ are rotational orthogonal when they are orthogonal and their rounds are permutations of one another.

The rotational width of $\mathcal{H}$ is the size of the largest set of rotational orthogonal schedules.

$$x_n = \max_{\mathcal{H}} \text{rotw}(\mathcal{H})$$
Not difficult to see that:

\[
\text{rotw}(\mathcal{H}) \leq \text{width}(\mathcal{H}) \leq \text{opp}(\mathcal{H}) \quad 2 \leq x_n \leq w_n \leq o_n \leq n
\]
Not difficult to see that:

$$\text{rotw}(\mathcal{H}) \leq \text{width}(\mathcal{H}) \leq \text{opp}(\mathcal{H}) \quad 2 \leq x_n \leq w_n \leq o_n \leq n$$

In fact, when $n$ is odd, $o_n \leq n - 1!$
Not difficult to see that:

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In fact, when \( n \) is odd, \( o_n \leq n - 1 \)!

When \( 2n = 6 \): \( x_n = w_n = o_n = 2 \). What happens with other values of \( n \)?
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**Theorem**

*When \( n = 2^k \), \( x_n = w_n = o_n = n \).*

Proof has two steps.

- First: show \( o_n = n \) by constructing \( \mathcal{H} \) with \( \text{opp}(\mathcal{H}) = n \).
Not difficult to see that:

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In fact, when $n$ is odd, $o_n \leq n - 1$!

When $2n = 6$: $x_n = w_n = o_n = 2$. What happens with other values of $n$?

**Theorem**

*When $n = 2^k$, $x_n = w_n = o_n = n$.*

Proof has two steps.

- First: show $o_n = n$ by constructing $H$ with $\text{opp}(H) = n$.
- Second: Construct $n$ rotational orthogonal schedules compatible with $n$. 
Constructing HAP-set $\mathcal{H}$ with $\text{opp}(\mathcal{H}) = n$ when $n = 2^k$.


<table>
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<tr>
<th>Team 0</th>
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<tr>
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Above HAP-set has $\text{opp}(\mathcal{H}) = n$.

NB: This procedure can be generalized to create HAP-sets with $\text{opp}(\mathcal{H}) = n$ even in cases when $n \neq 2^k$. 
Constructing HAP-set $\mathcal{H}$ with $\text{opp}(\mathcal{H}) = n$ when $n = 2^k$.

<table>
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<tr>
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<table>
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<tr>
<th></th>
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</table>

Above HAP-set has $\text{opp}(\mathcal{H}) = n$.

NB: This procedure can be generalized to create HAP-sets with $\text{opp}(\mathcal{H}) = n$ even in cases when $n \neq 2^k$. 
Constructing the HAP-set:

- Set teams $T = \mathbb{Z}_2^n$ and rounds $R = \mathbb{Z}_2^n \setminus \{0\}$ - i.e., each team $t$ and round $r$ is an $n$-bit $b(t), b(r)$.
- In HAP-set $\mathcal{H}$, team $t$ gets assigned Home in round $r$ when:

$$r \cdot t = \sum_i b_i(r)b_i(t) = 0 \mod 2$$
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Only need $n$ orthogonal schedules. Construct schedule $S = (S_q)_{q \in R}$ as follows:

$$ S_q = \{\{i,j\} : i + j = q\} $$
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Only need $n$ orthogonal schedules. Construct schedule $S = (S_q)_{q \in R}$ as follows:

$$S_q = \{\{i, j\} : i + j = q\}$$

Insight: $S_q$ can be scheduled in round $r$ when $r \cdot q = 1$. 
Bipartite graph $G = (R \times R, E)$ with $(r, q) \in E$ when $r \cdot q = 1$, is regular with degree $n$. 
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Only possible when $n = 2^k$! Otherwise, $x_n < n$. 
Bipartite graph $G = (R \times R, E)$ with $(r, q) \in E$ when $r \cdot q = 1$, is regular with degree $n$.

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Only possible when $n = 2^k!$ Otherwise, $x_n < n$. 

16
Conclusion

What we know:

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Table 2: Known values

More questions remain:

- Can we find $n$ for which $x_n \neq w_n$?
- Is it true that $o_n = 2\lfloor \frac{n}{2} \rfloor$?