Transversal polynomial of graphs

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Korteweg-De Vries Institute, University of Amsterdam

Semidefinite & Polynomial Optimization
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Polynomials in Combinatorics

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Alternatively, we can write this as a sum of elements of $\mathcal{C}$, where $w(C)$ is its order.

$$G(x) = \sum_{C \in \mathcal{C}} t^{w(C)}$$

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$$\mu(G,t) = \sum_{M} (-1)^{w(M)} t^{n-2w(M)}$$

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By decomposing the objects being counted, we find

$$\sum_{v \in V(G)} \phi(G \setminus v, t) = \frac{d}{dt}\phi(G,t)$$

and other identities, including the Christoffel-Darboux identity.

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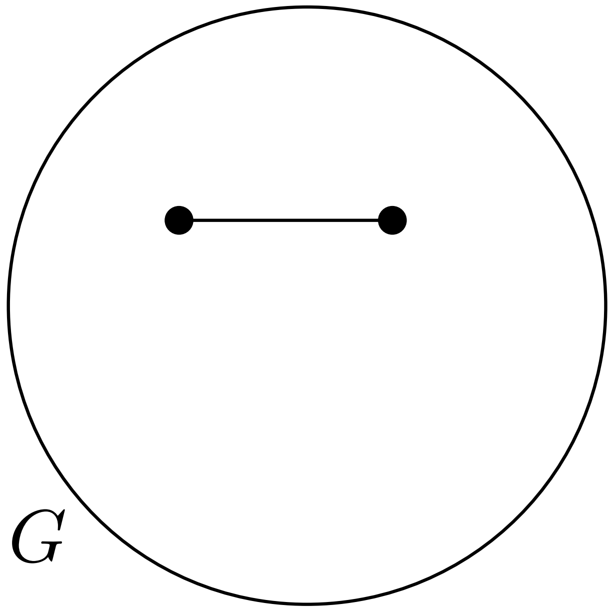

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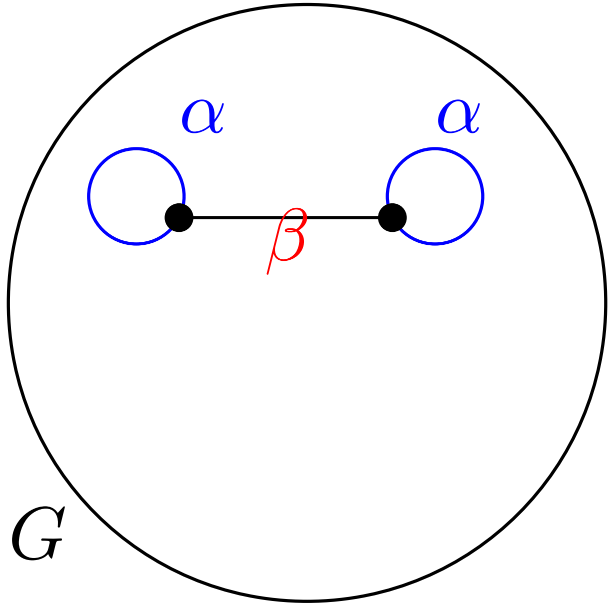

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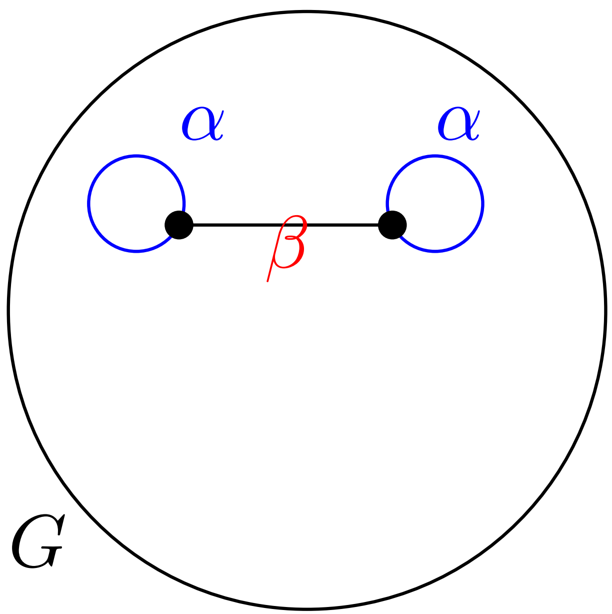

Made use of Christoffel-Darboux type identities to find eigenvalues of edge-perturbations of graphs for the purposes of quantum walks.

In this talk

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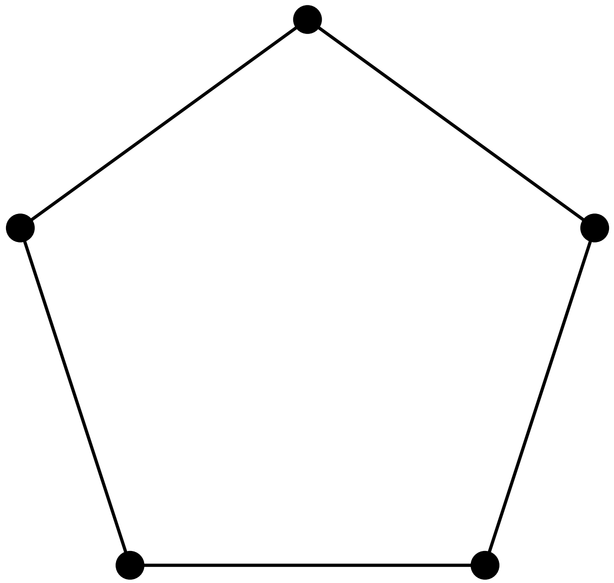
We define a generating function for these objects.

We prove basic properties about this polynomial, including the evaluation of a point.

Colourings of graphs



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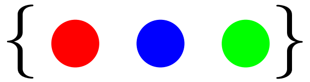
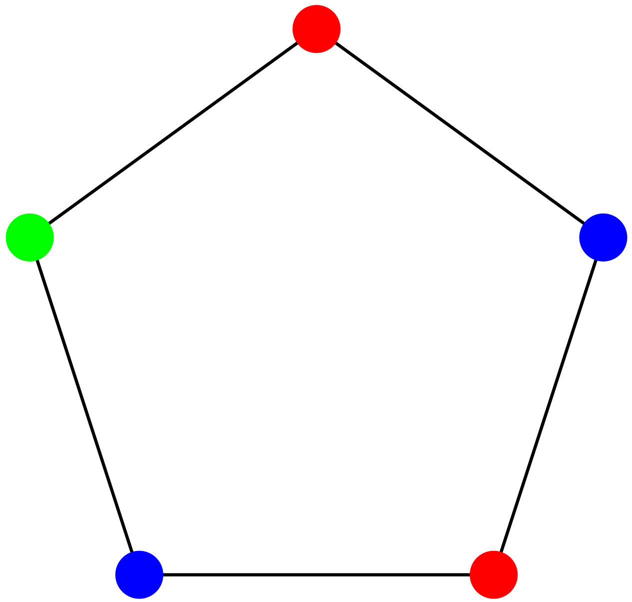
k-colouring of graphs 



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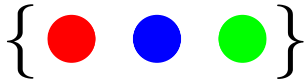
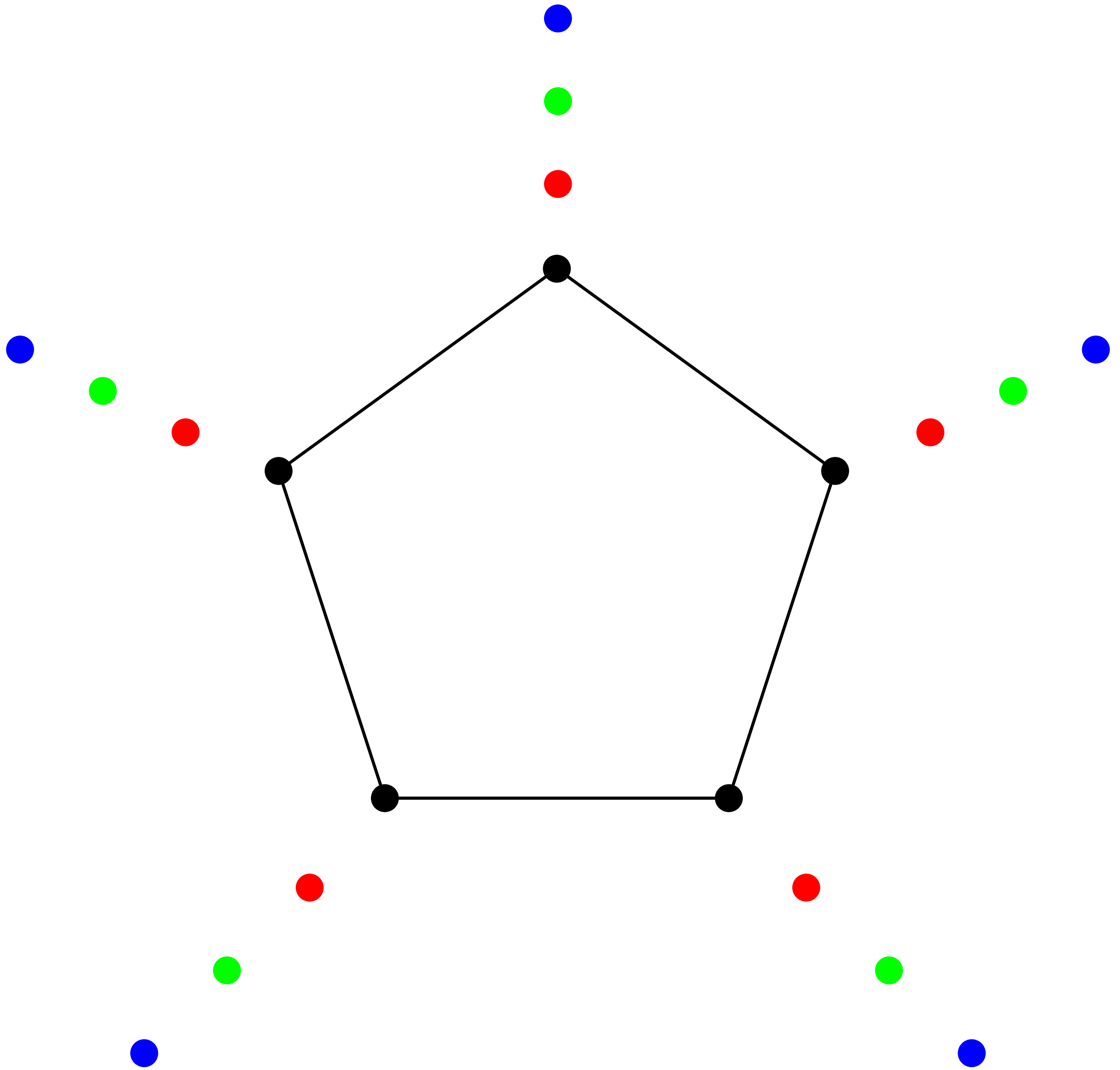
k-list colouring





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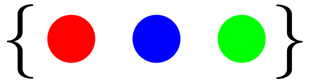

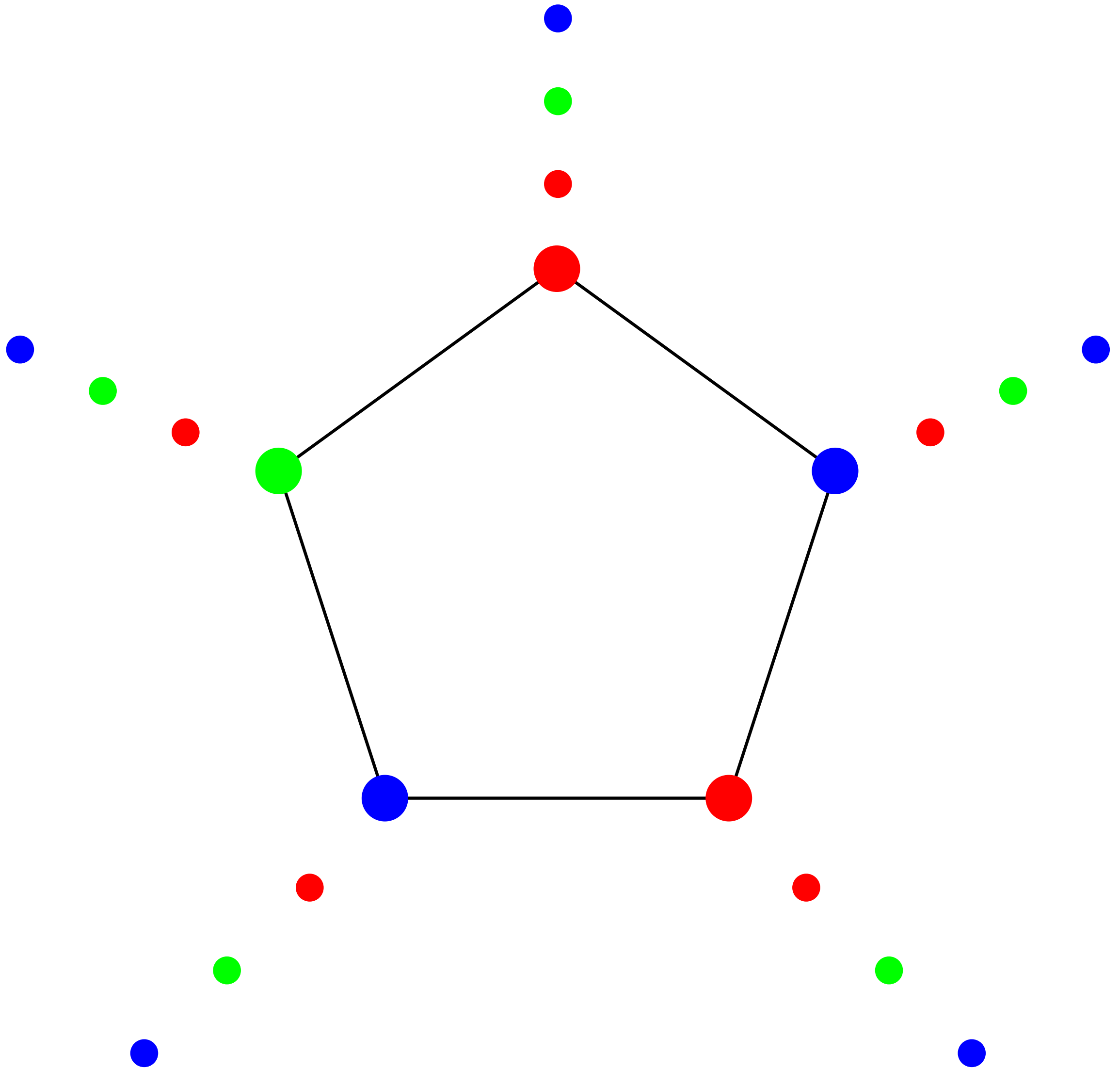

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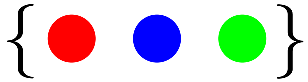

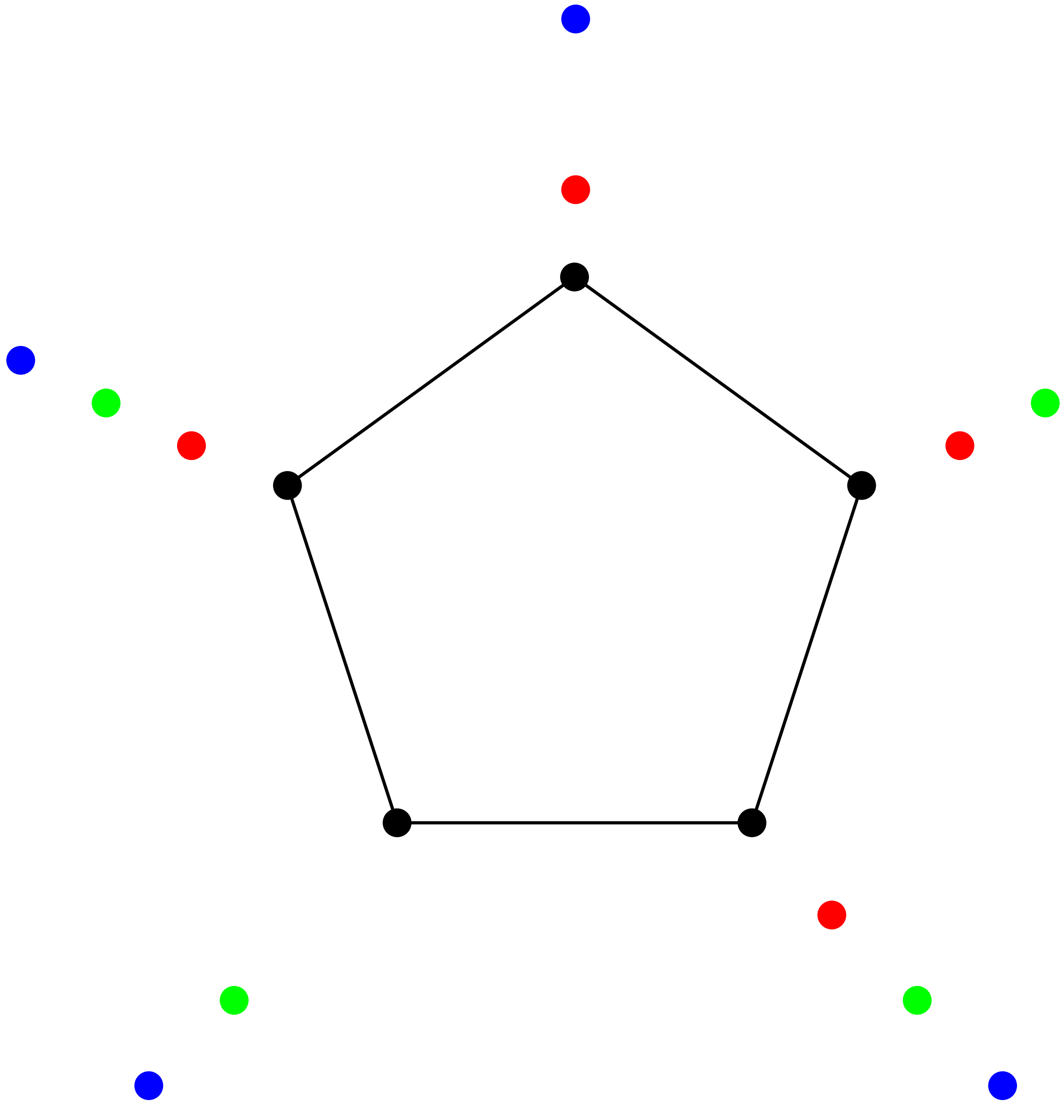

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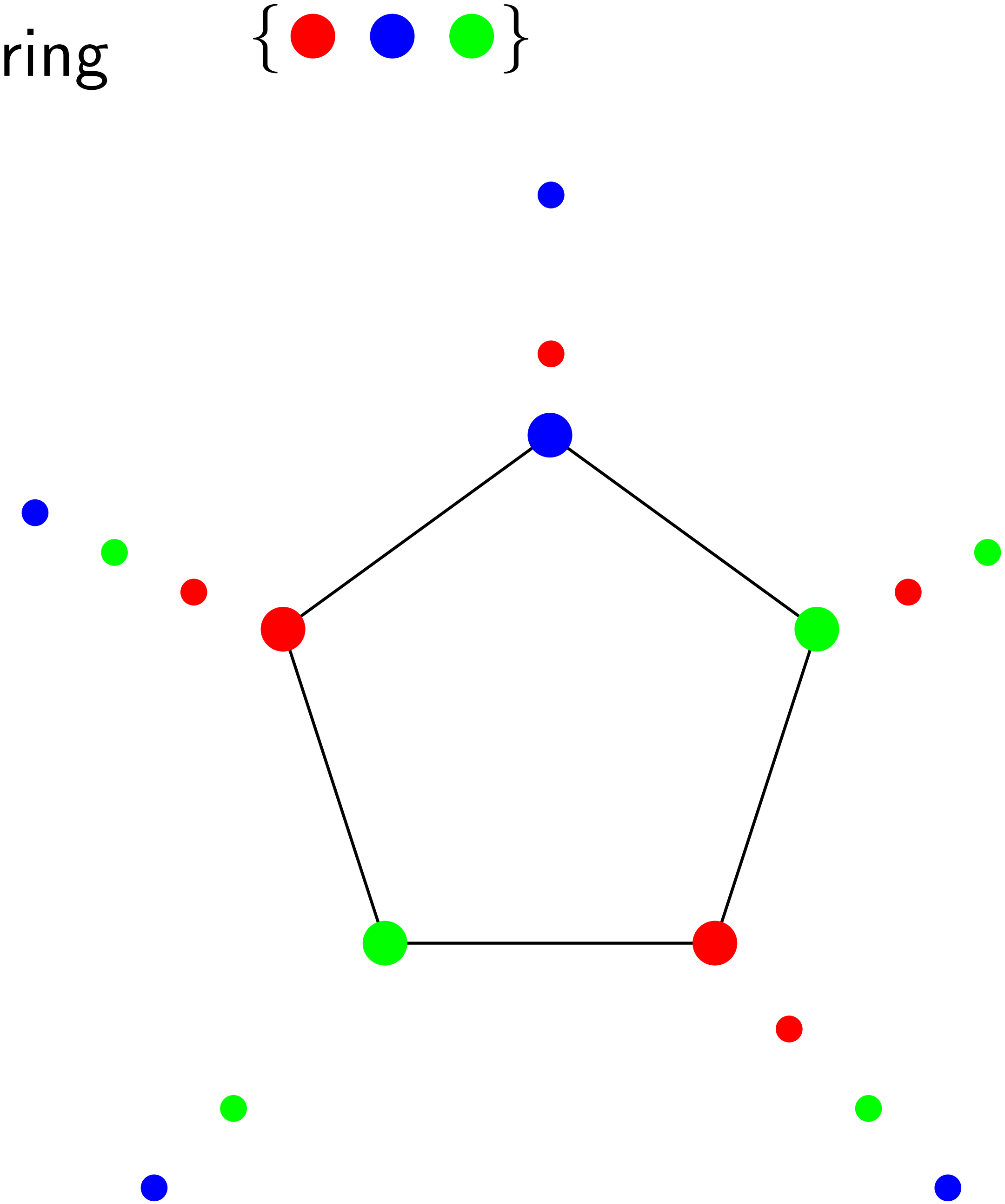

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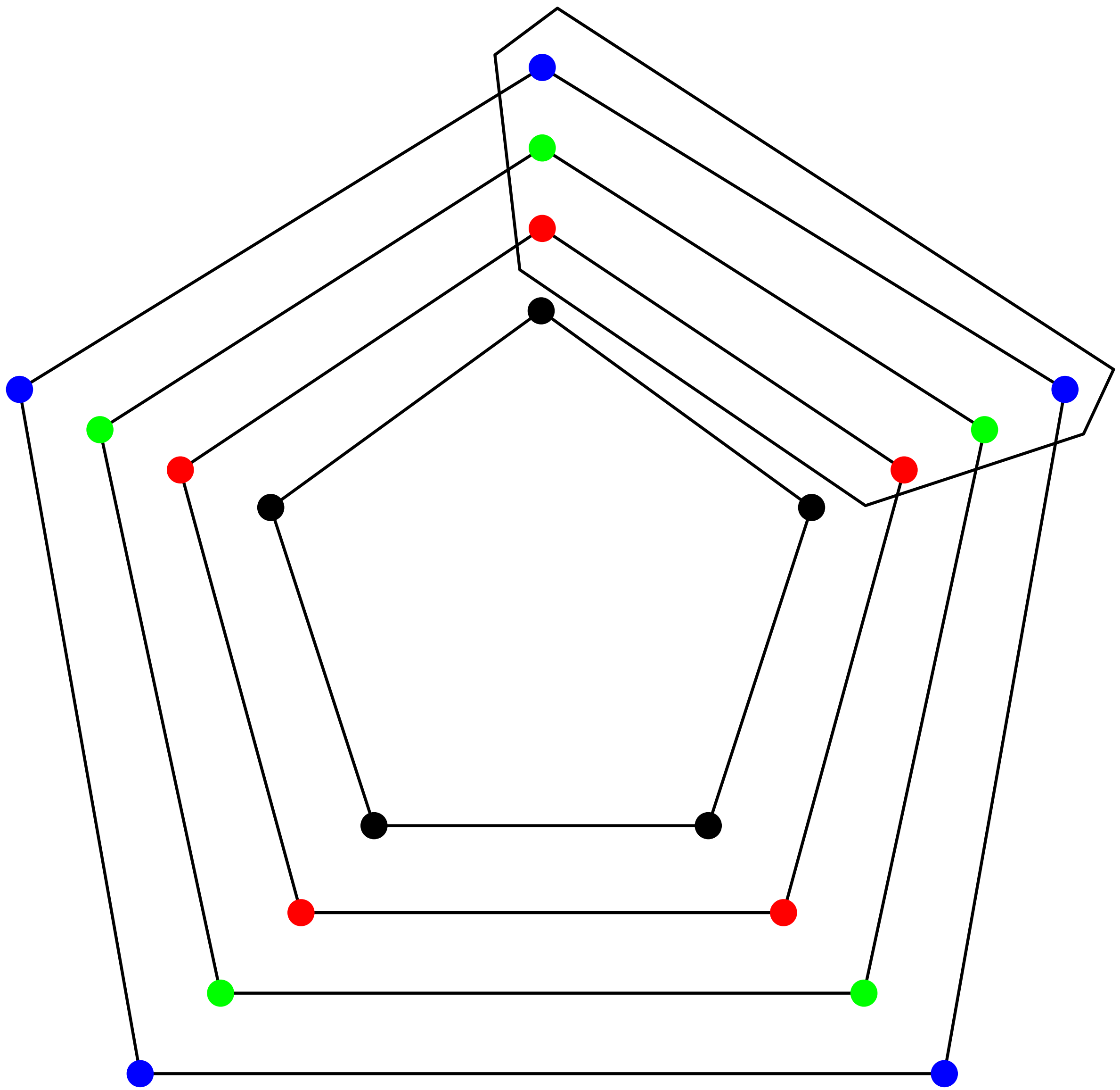

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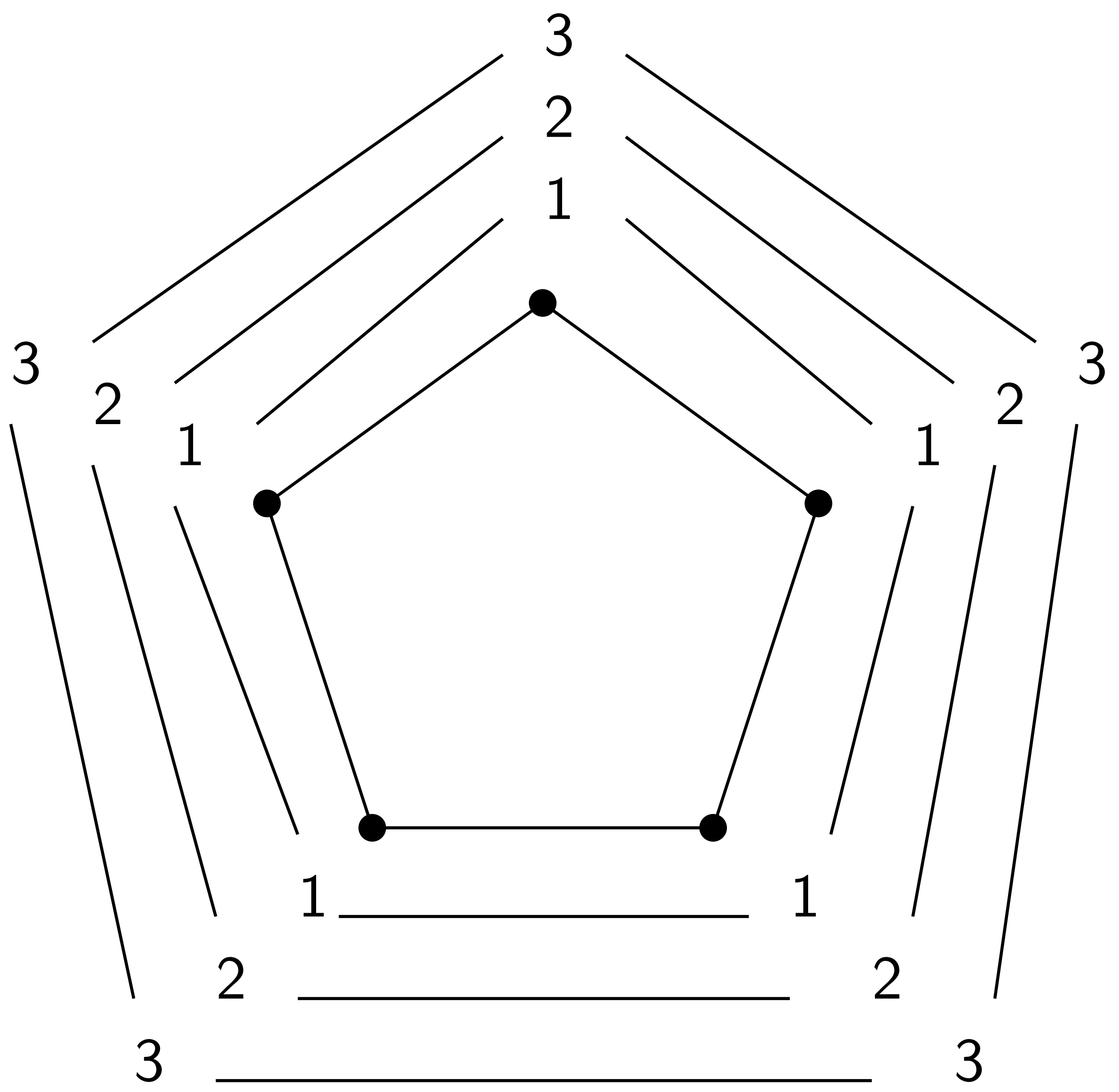

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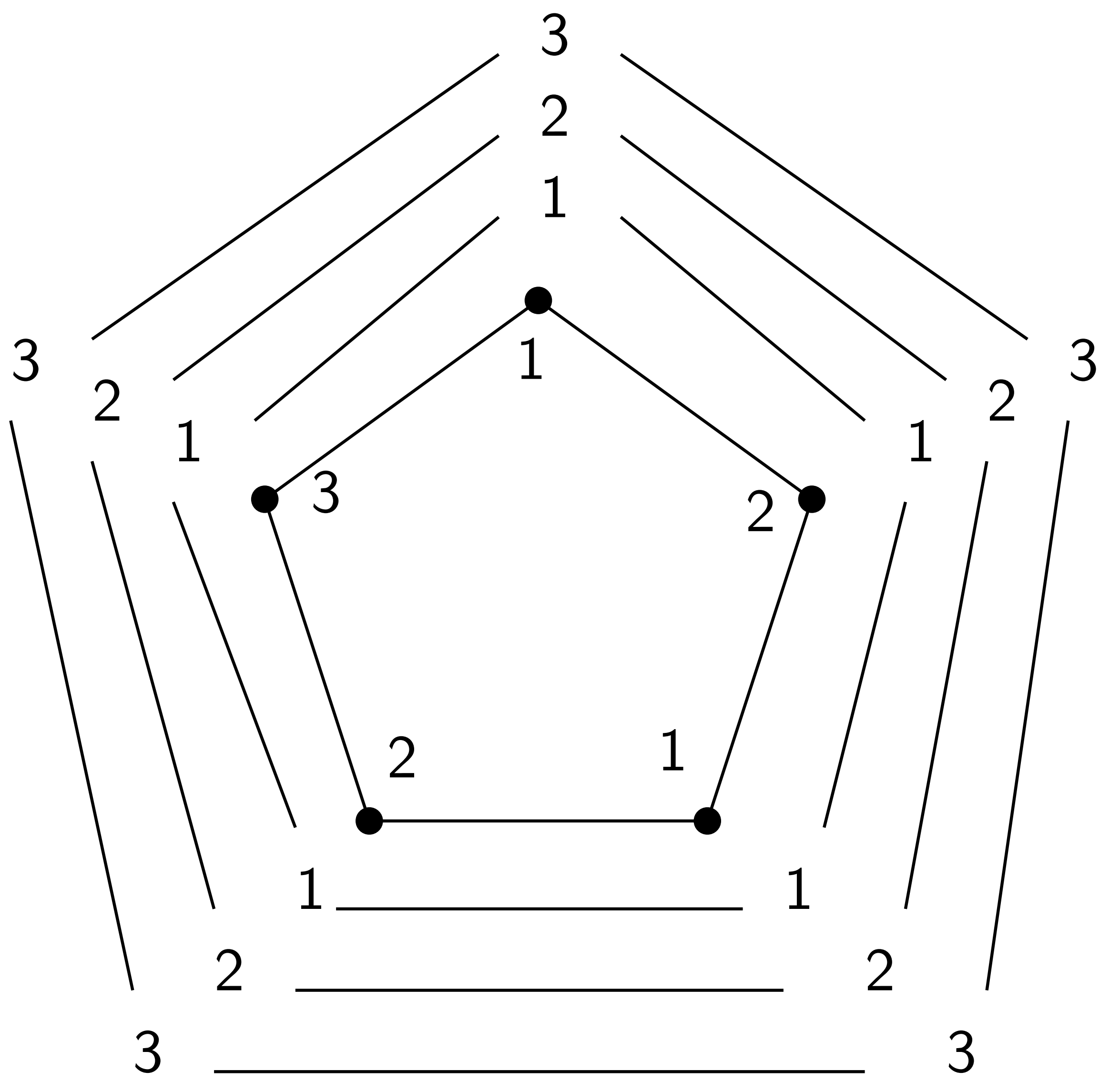

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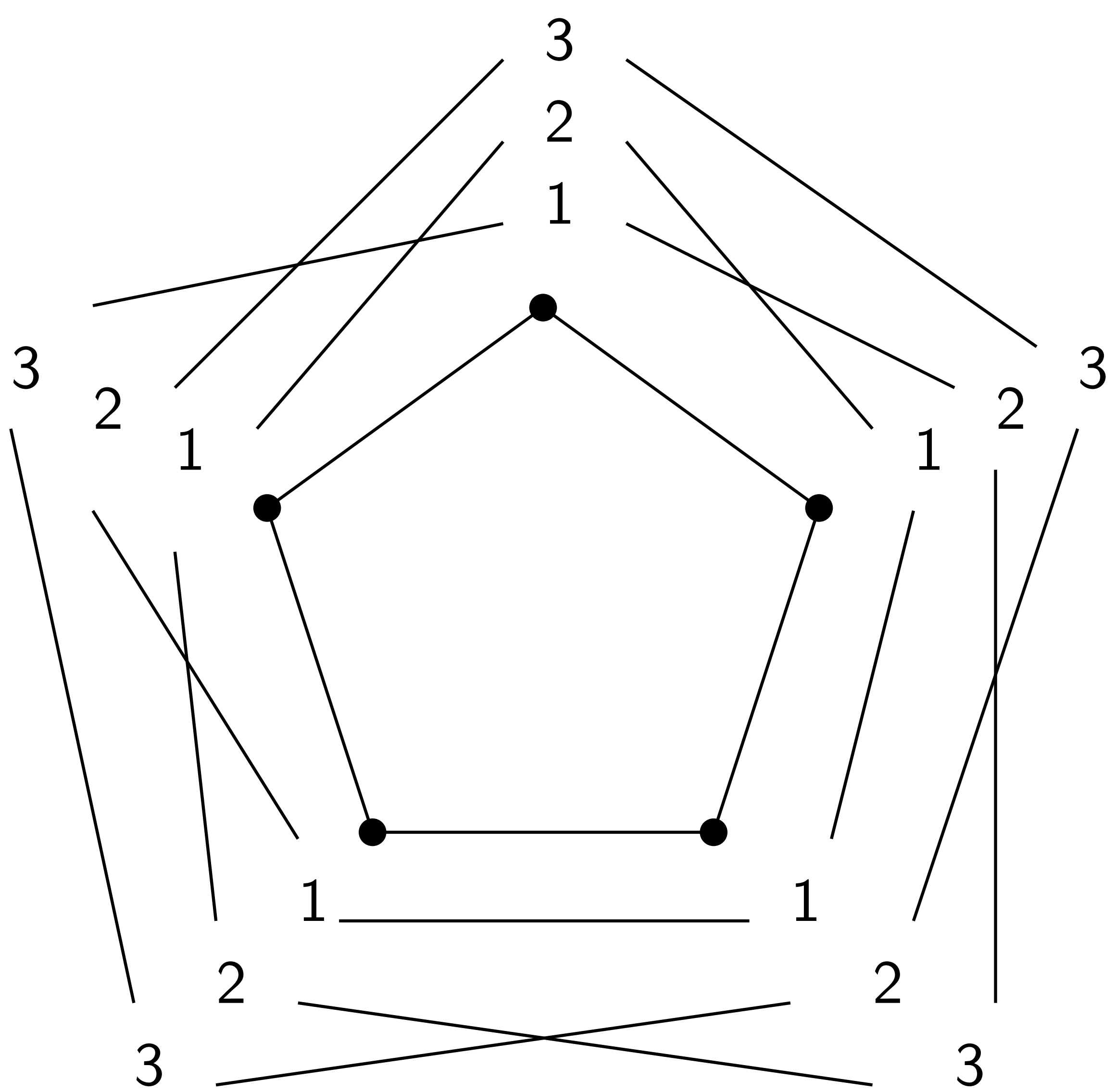

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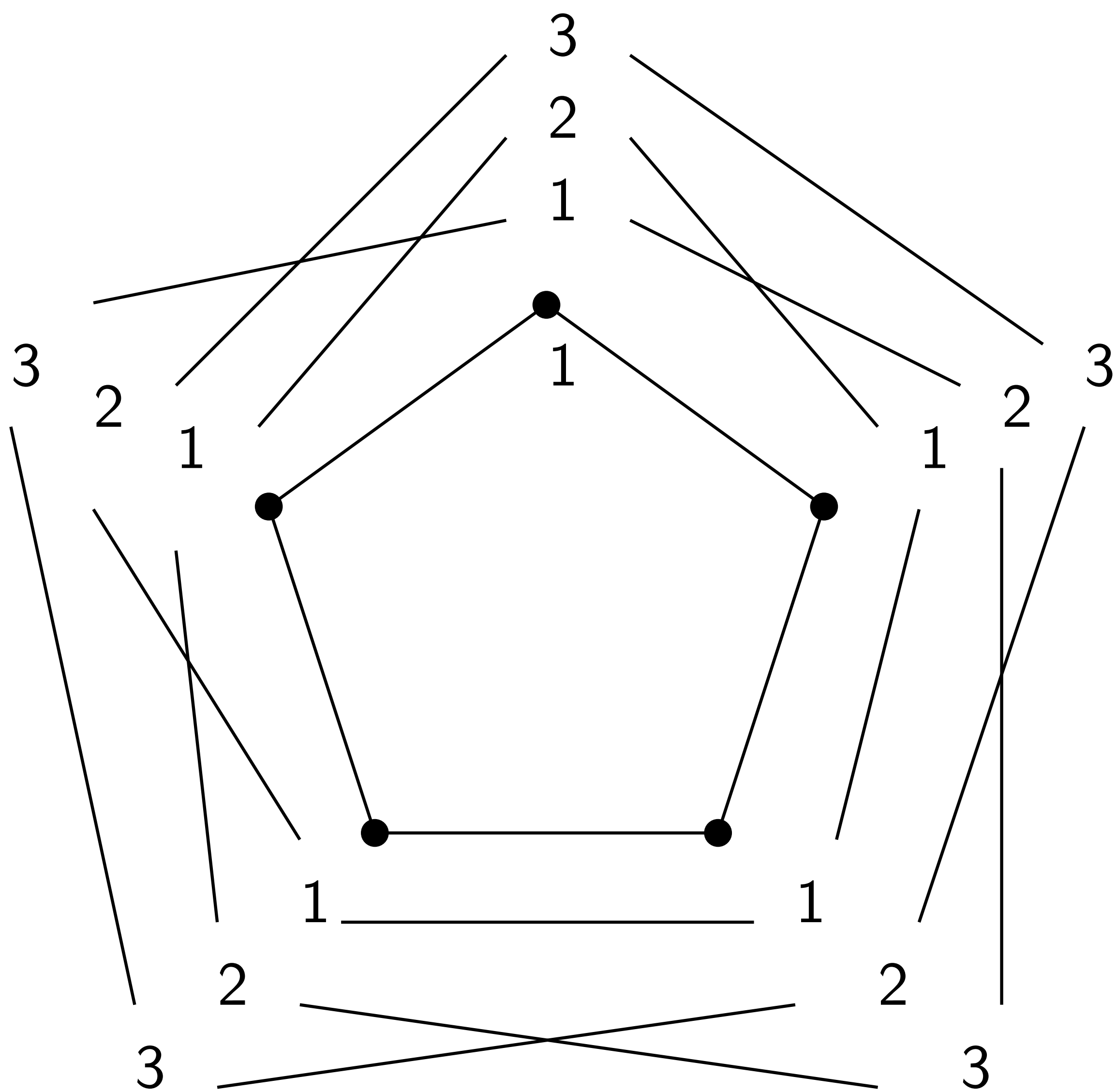

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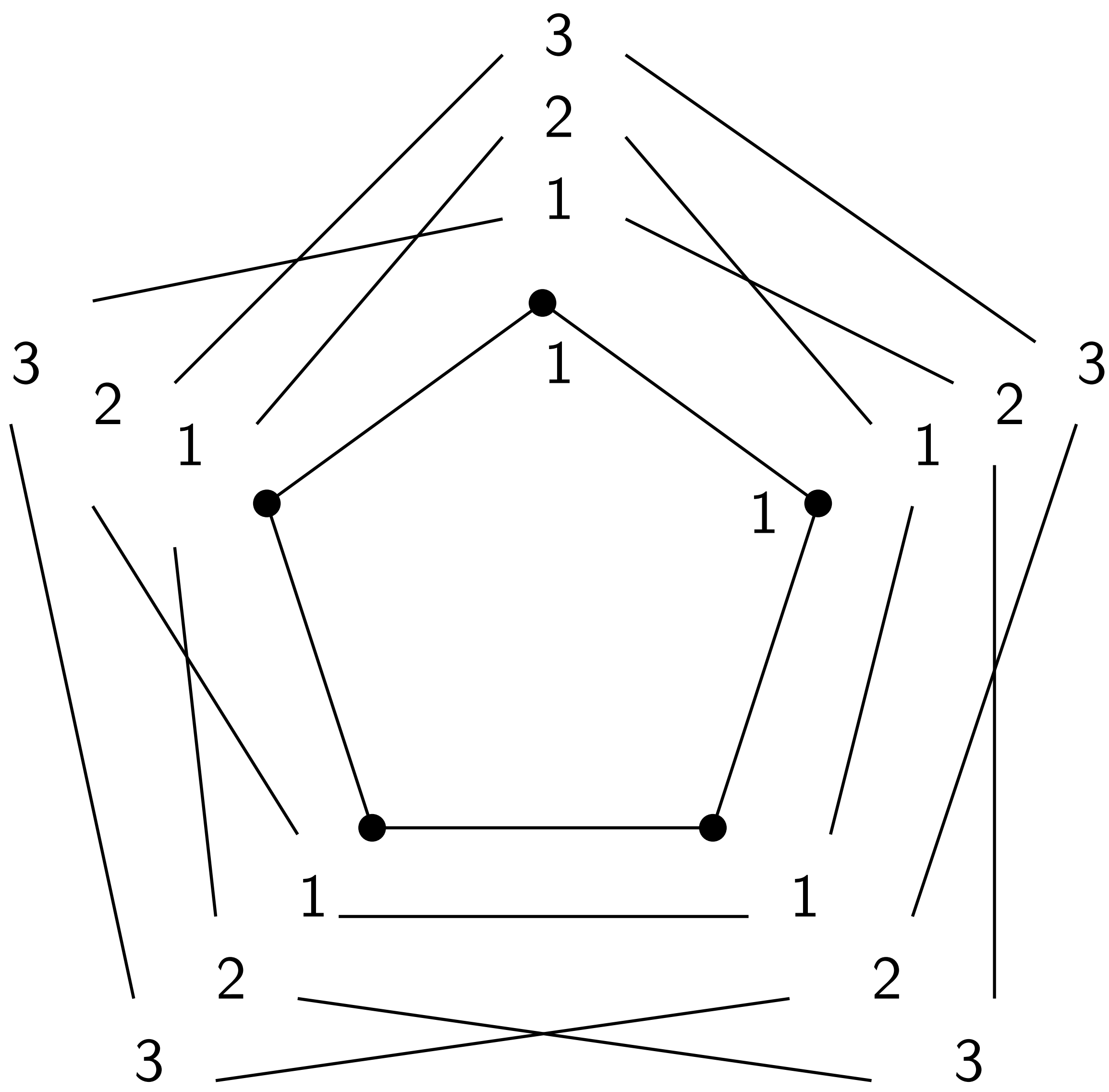

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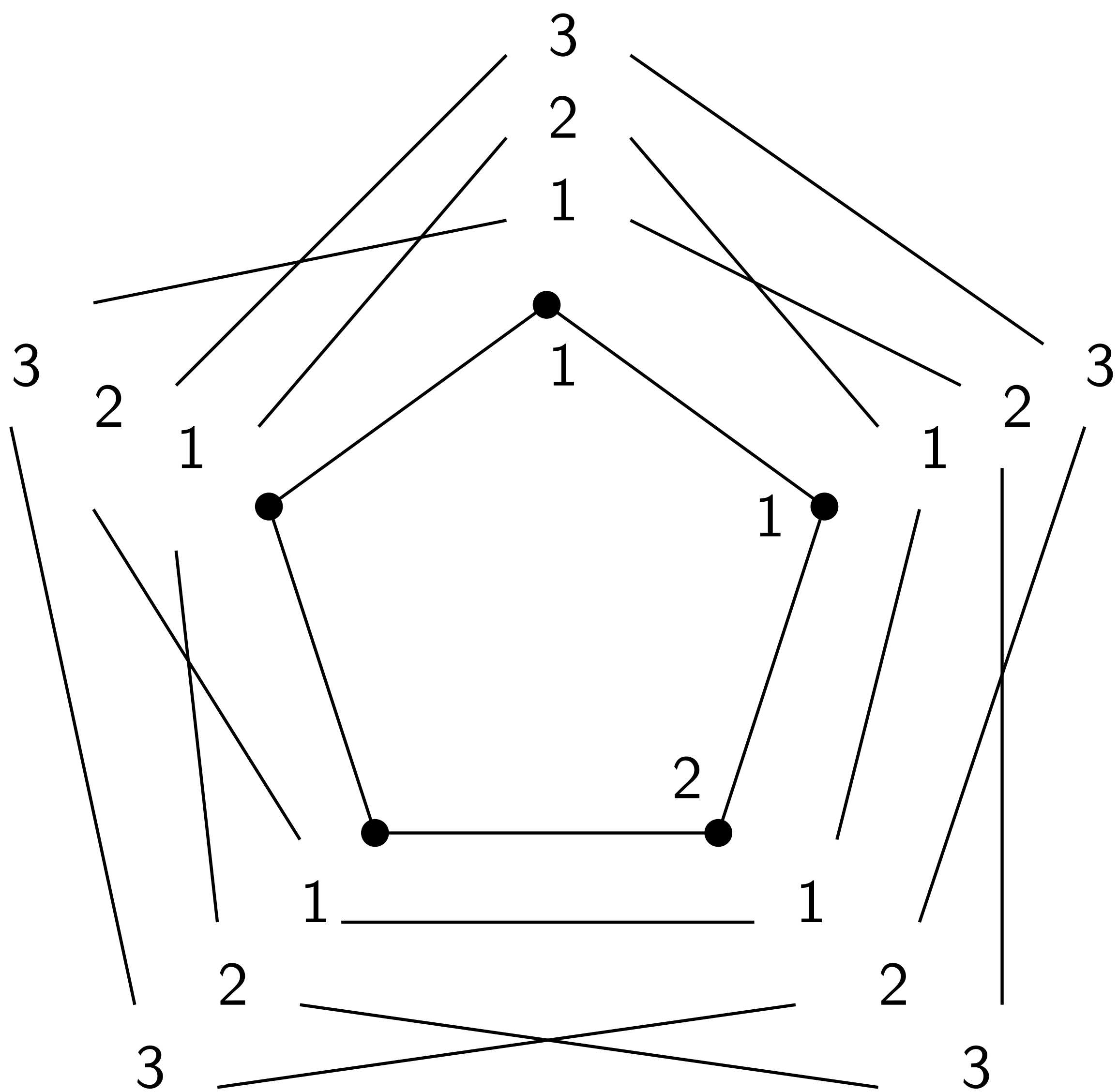

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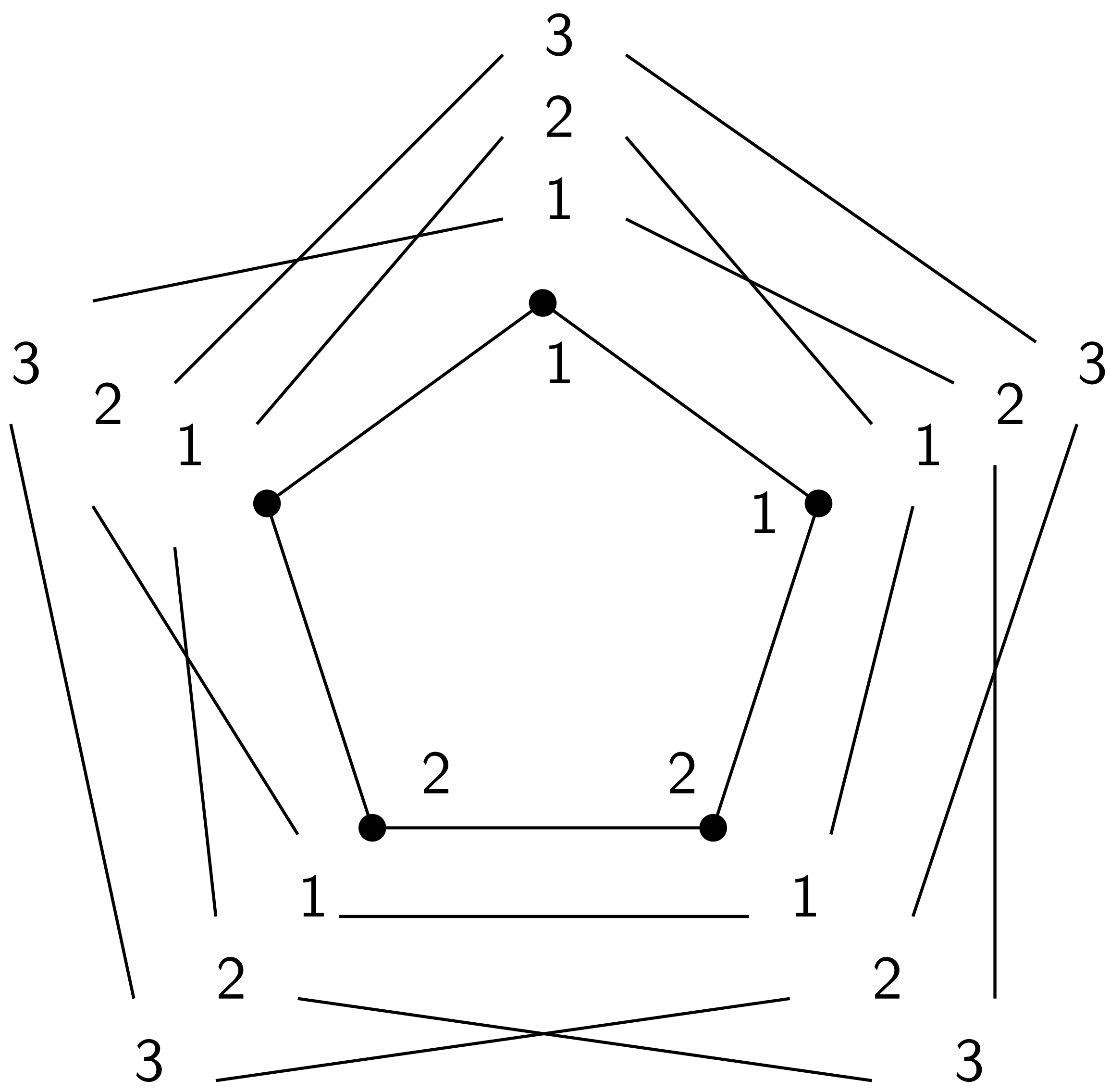

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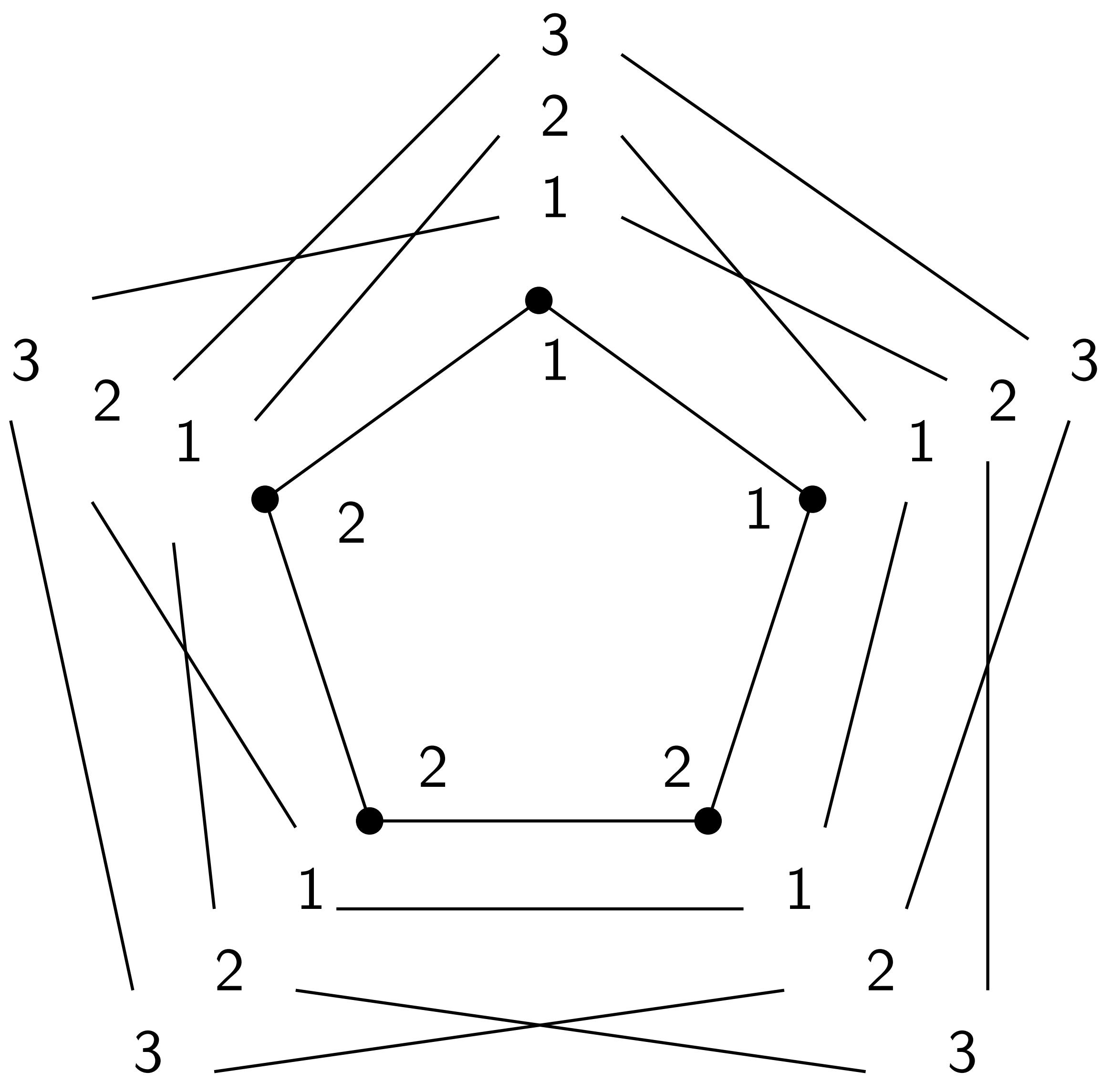

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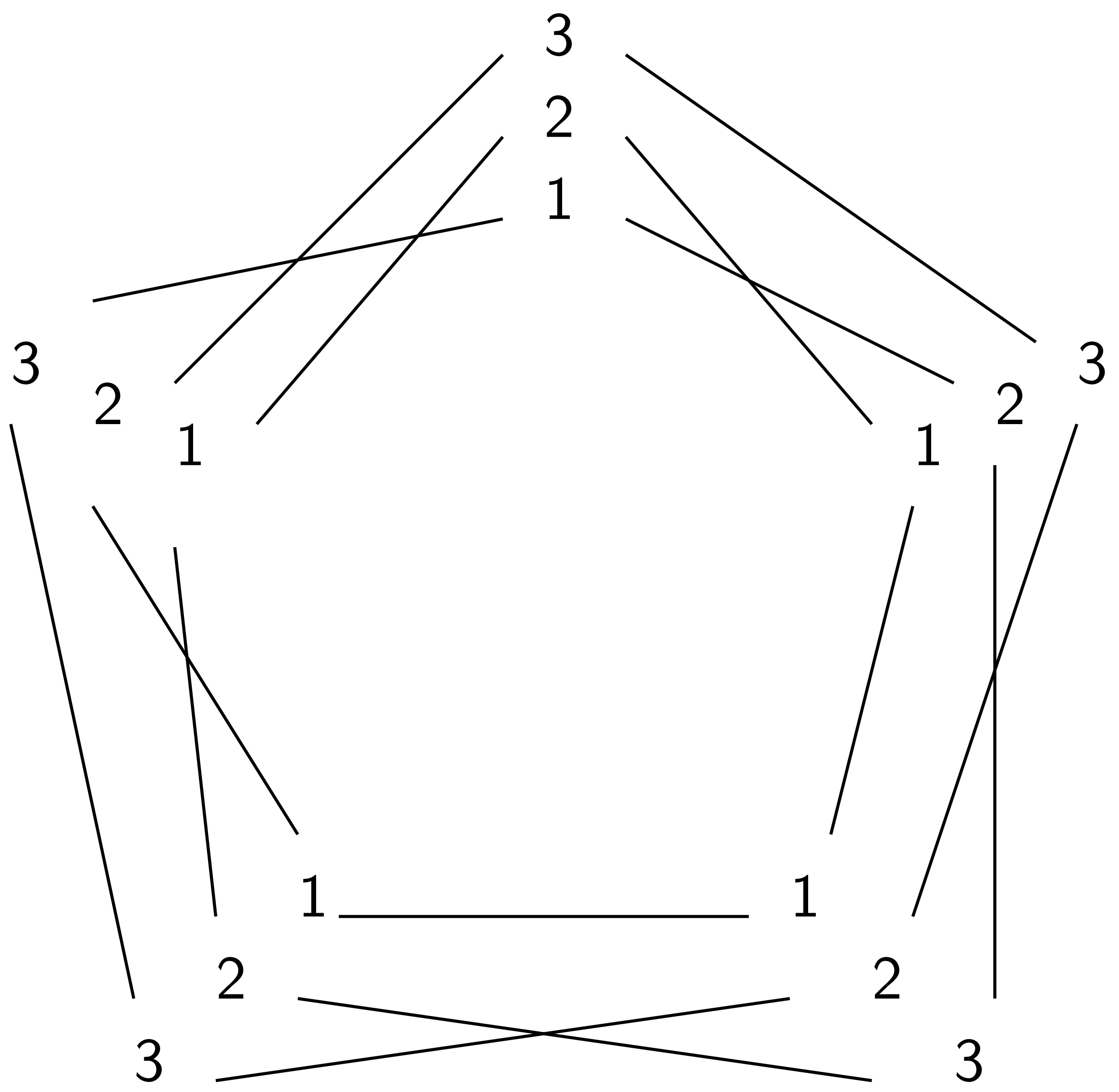

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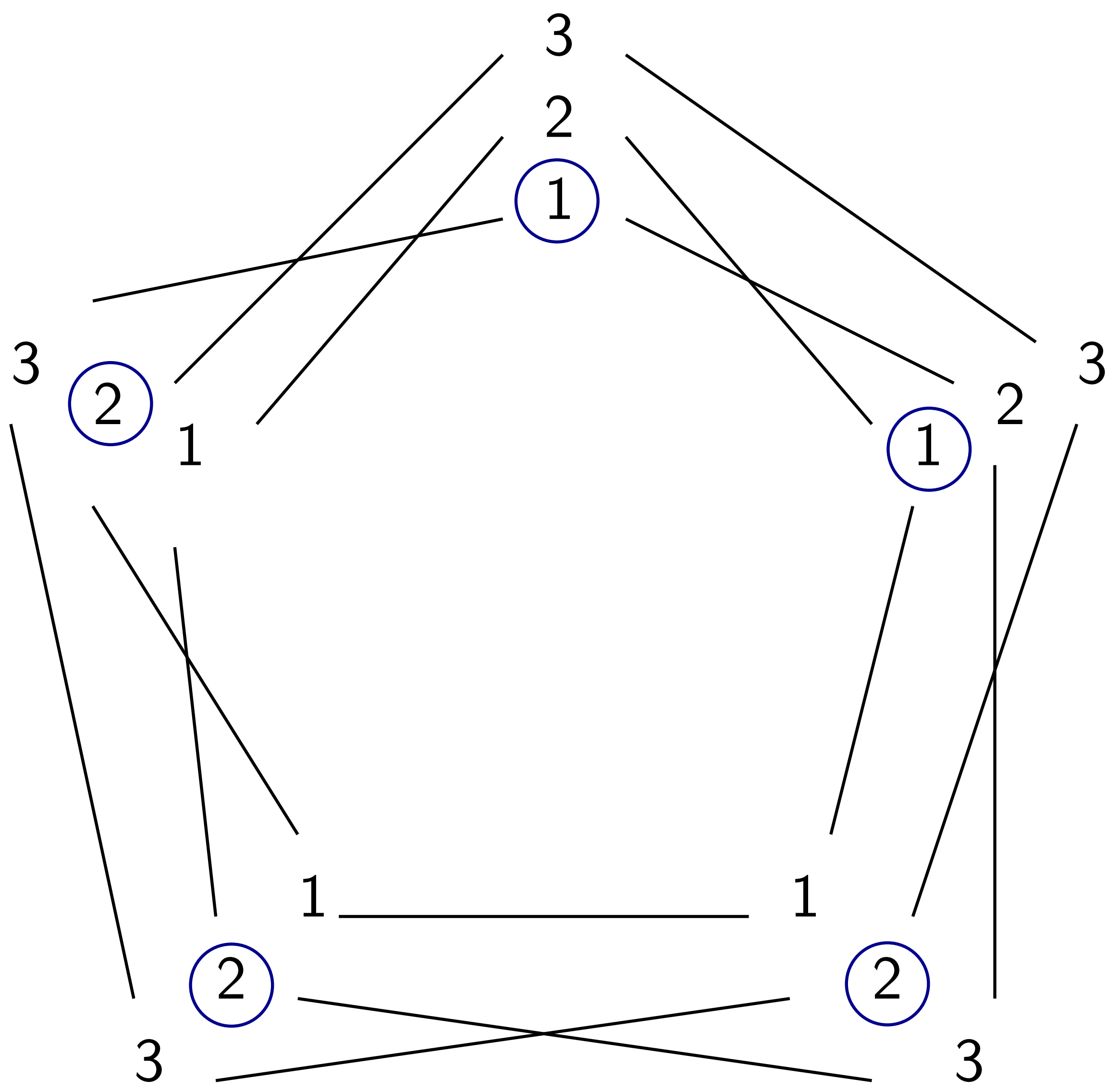

Historical perspective of colouring

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Theorem (Loeb, Rolek, Liu, Yu 2018)

Every planar graph without cycles of lengths $4, a, b, 9$ for $a, b \in \{6, 7, 8\}$ is DP-3-colourable.

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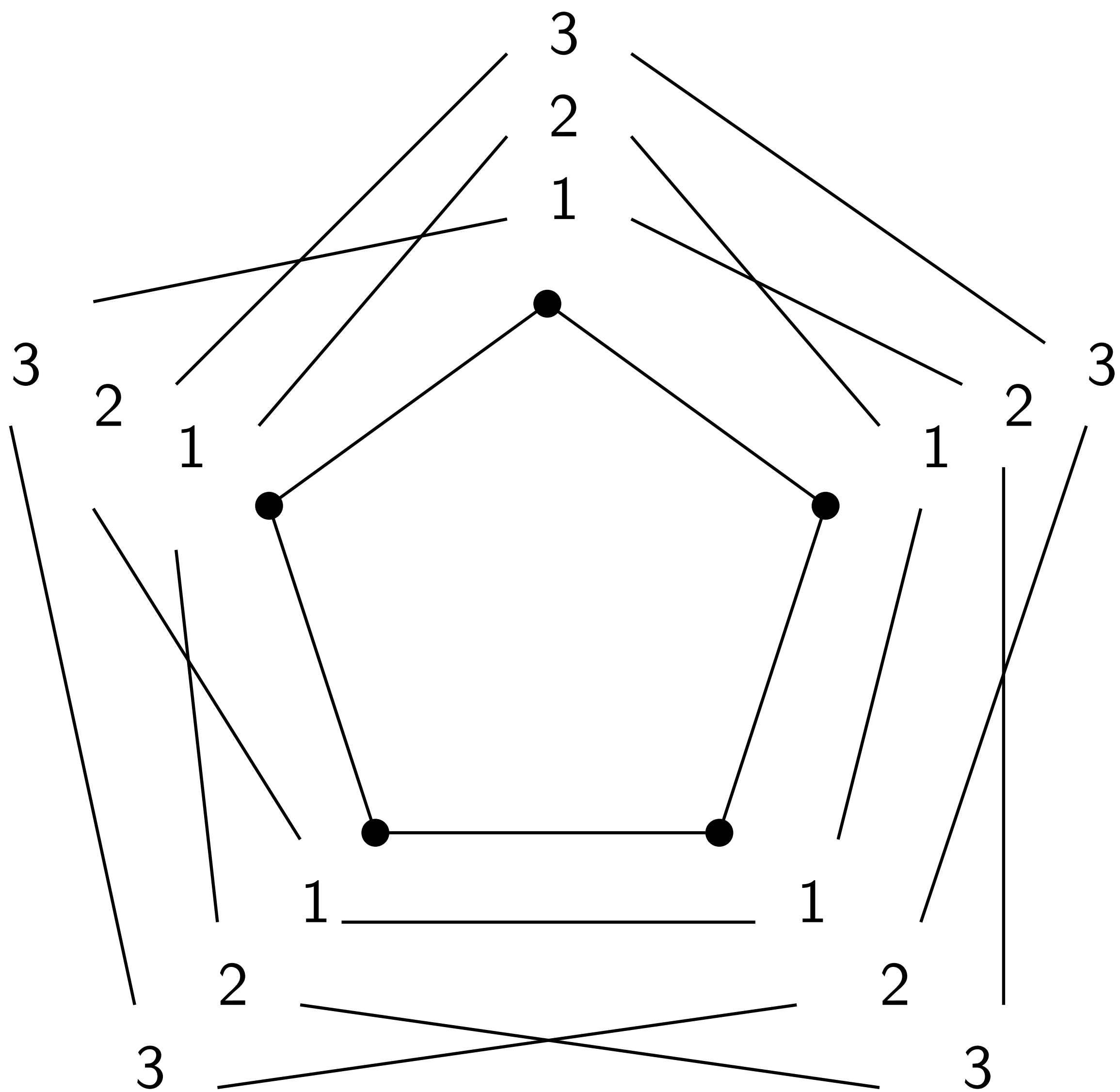

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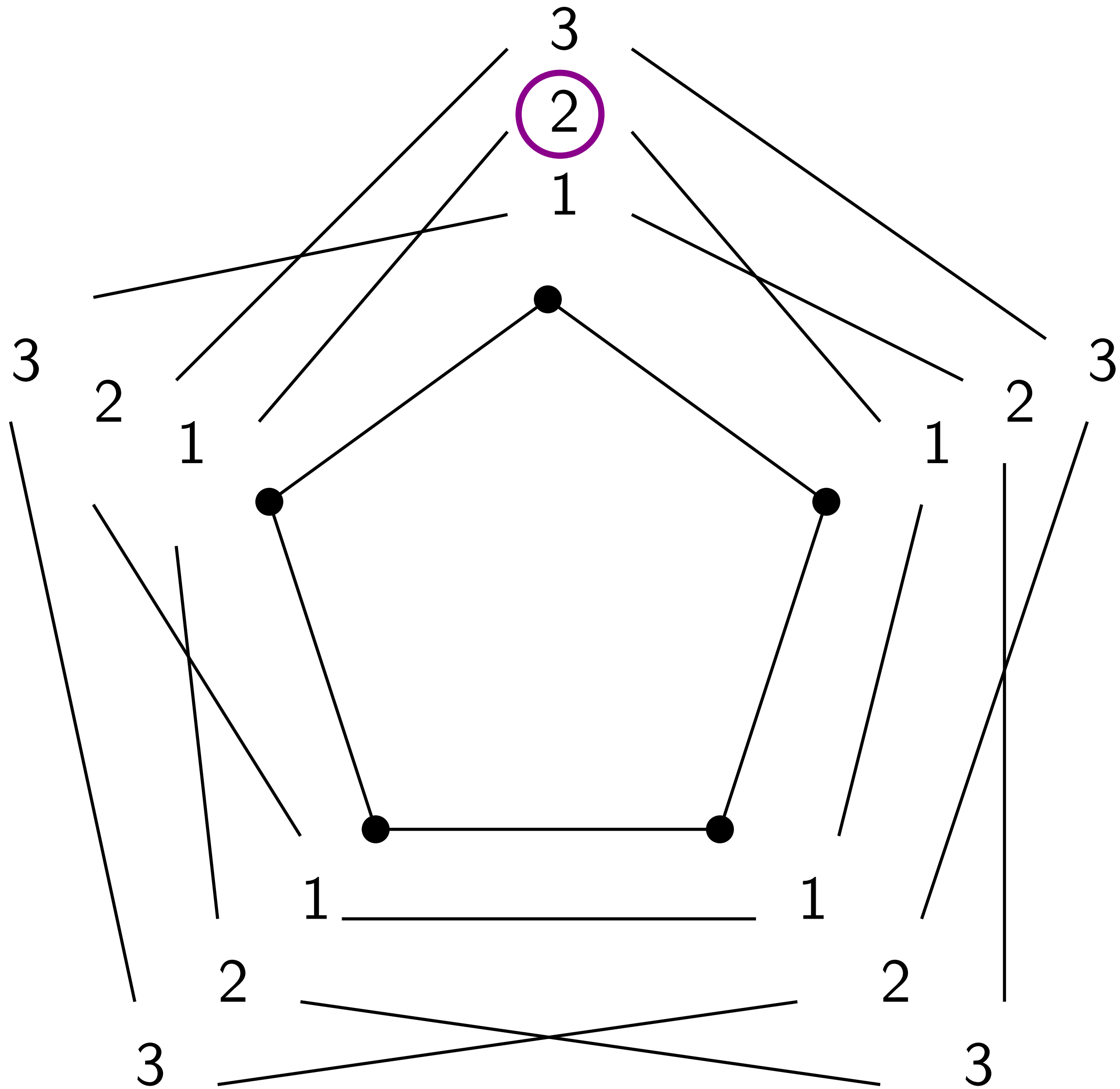

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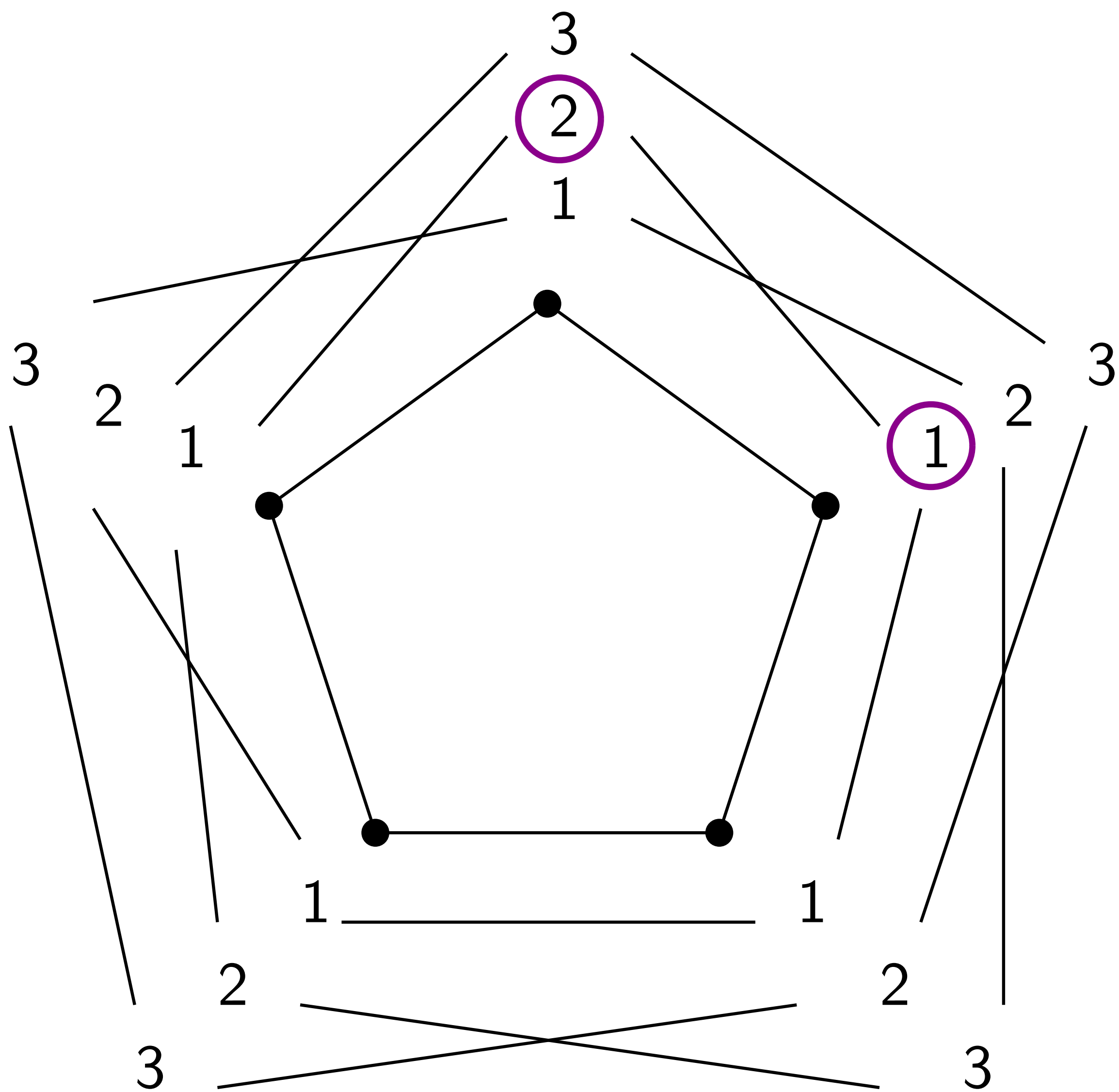

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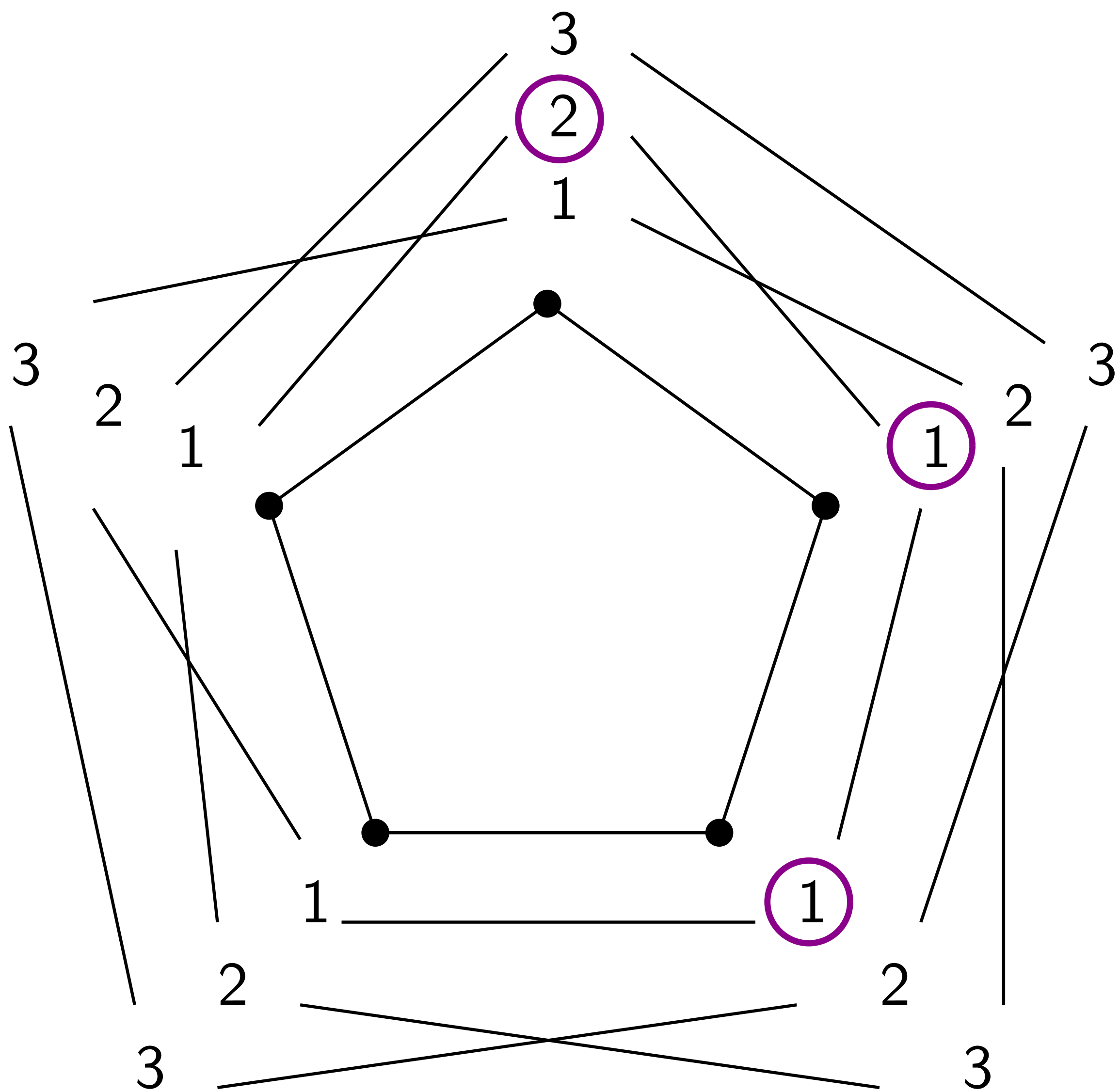

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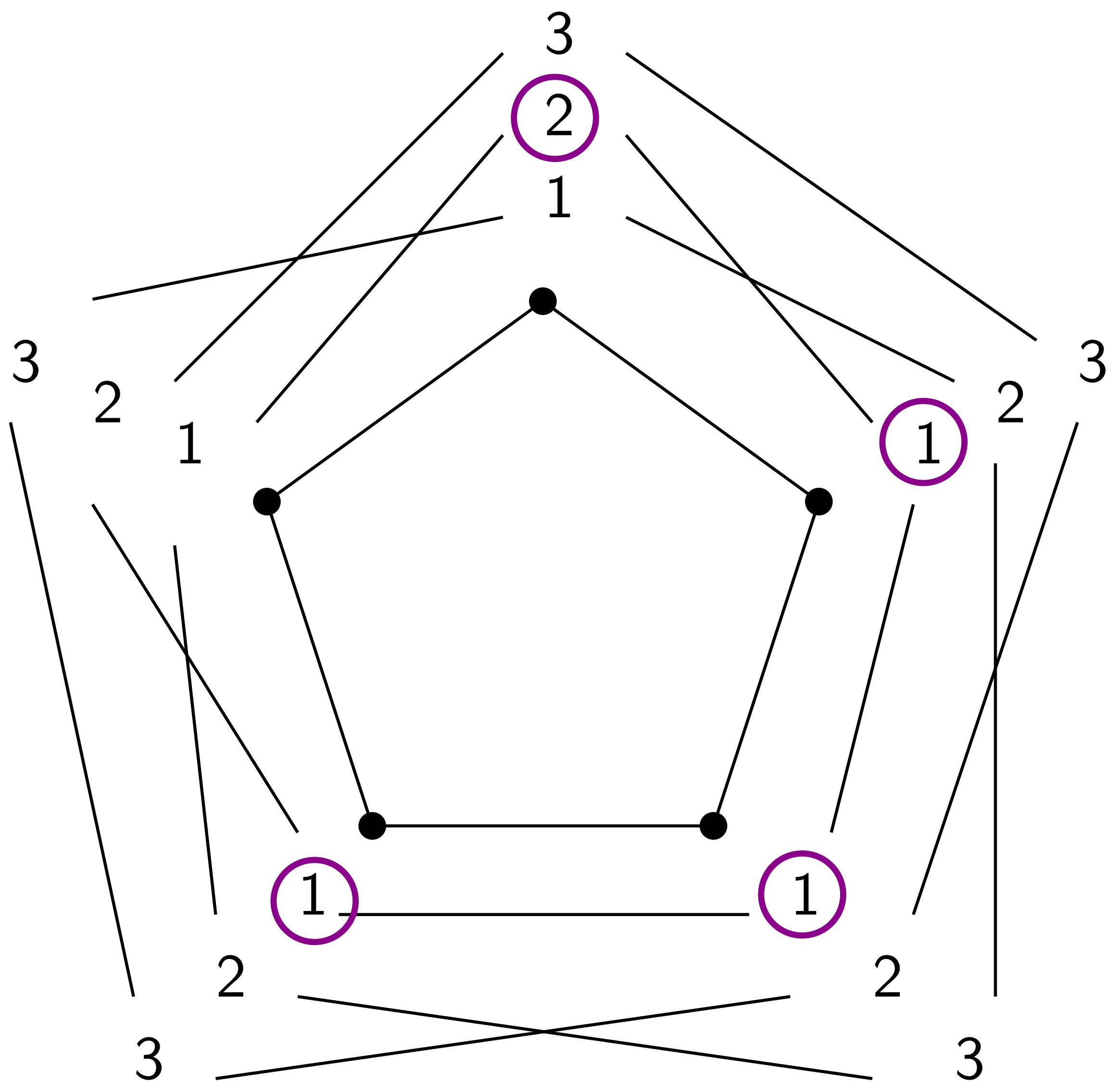

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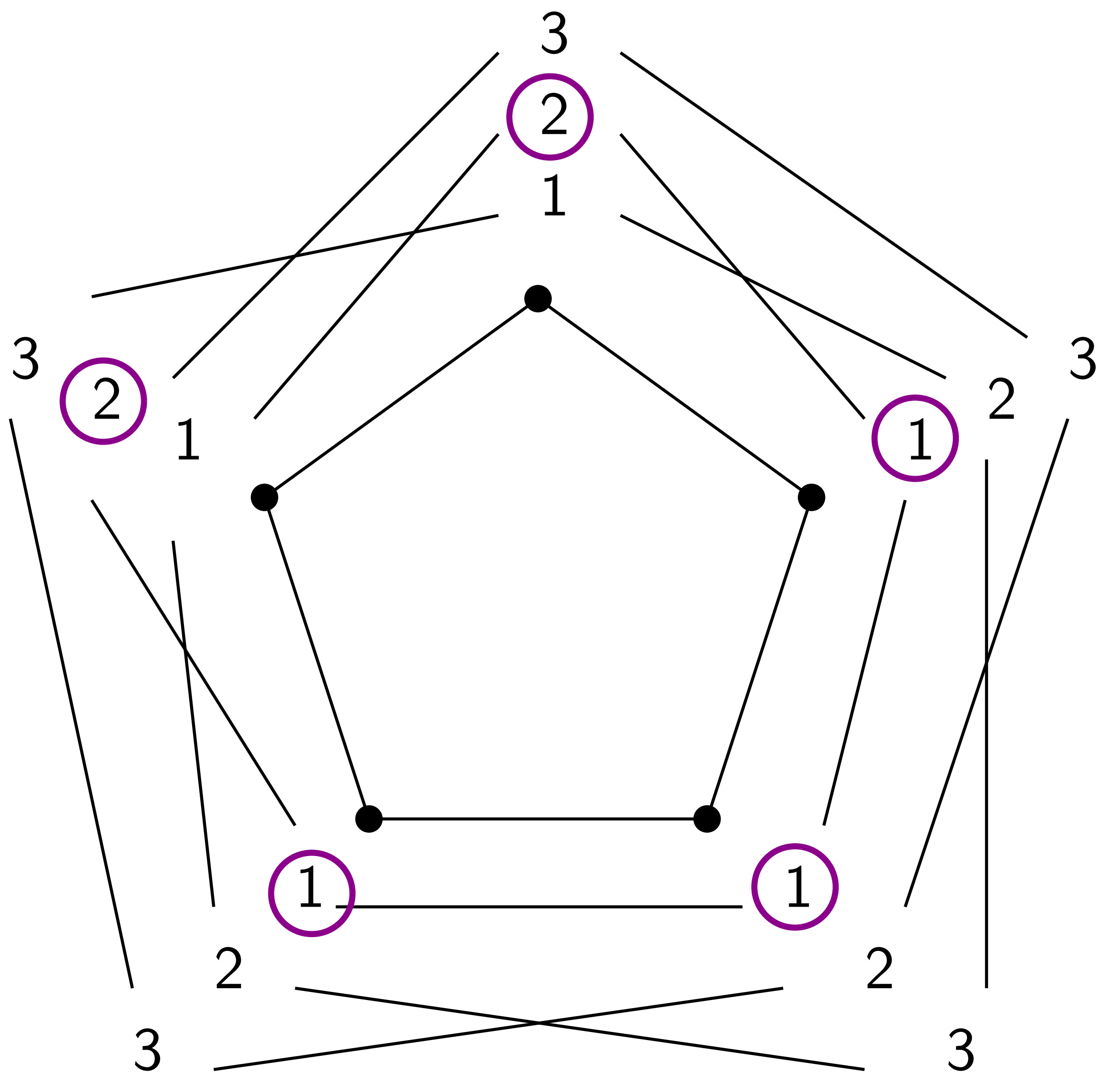

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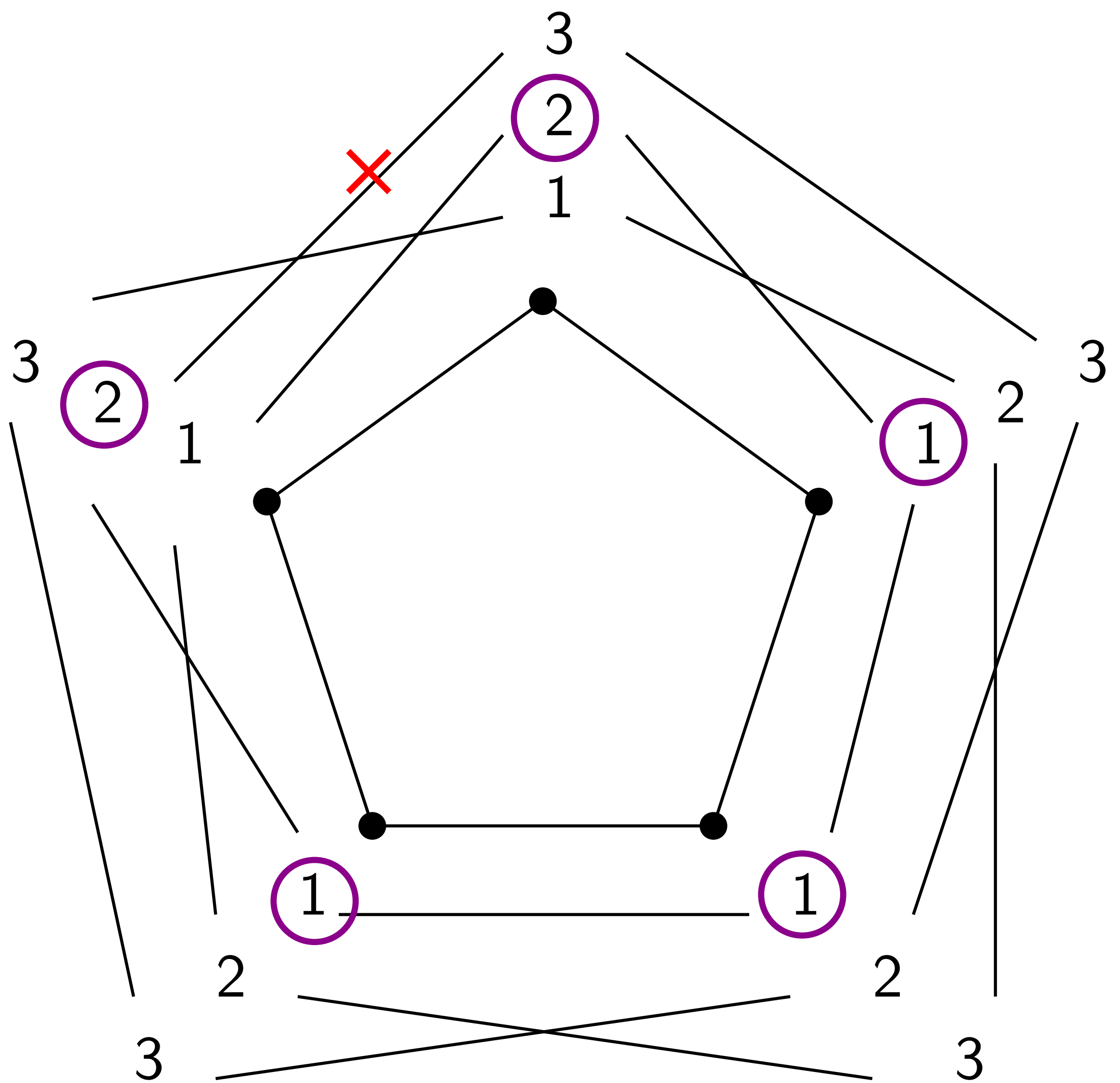

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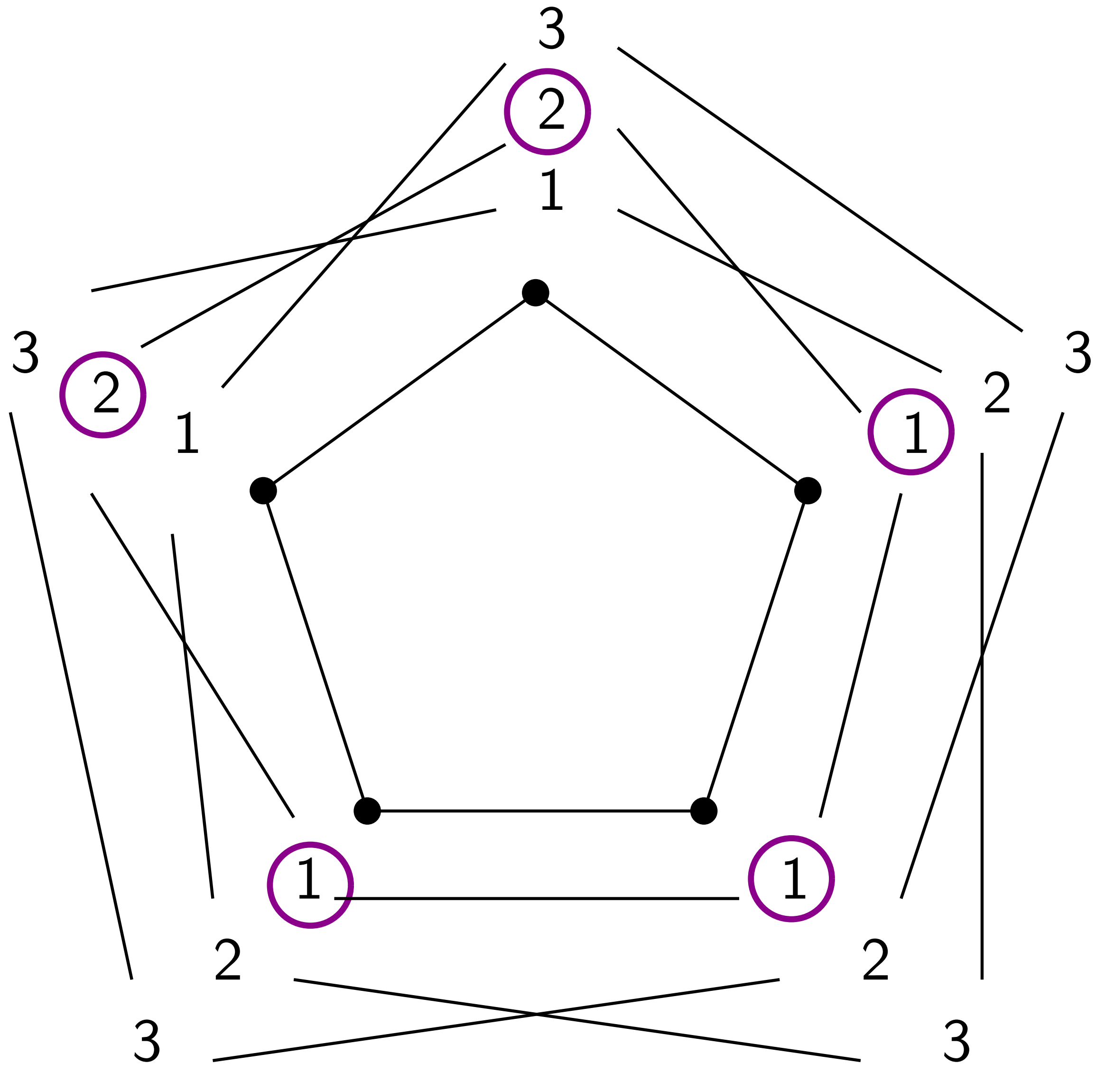

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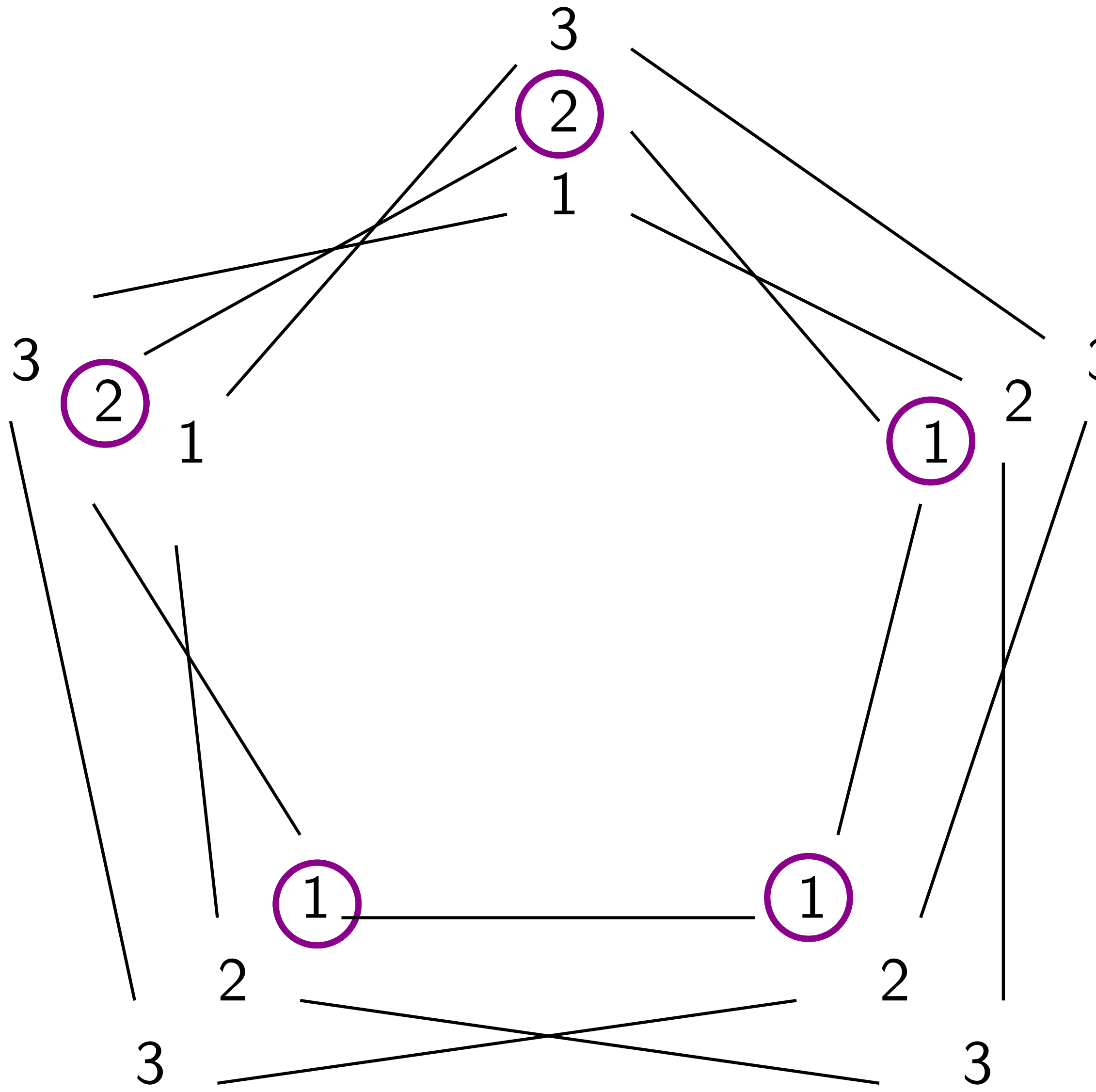

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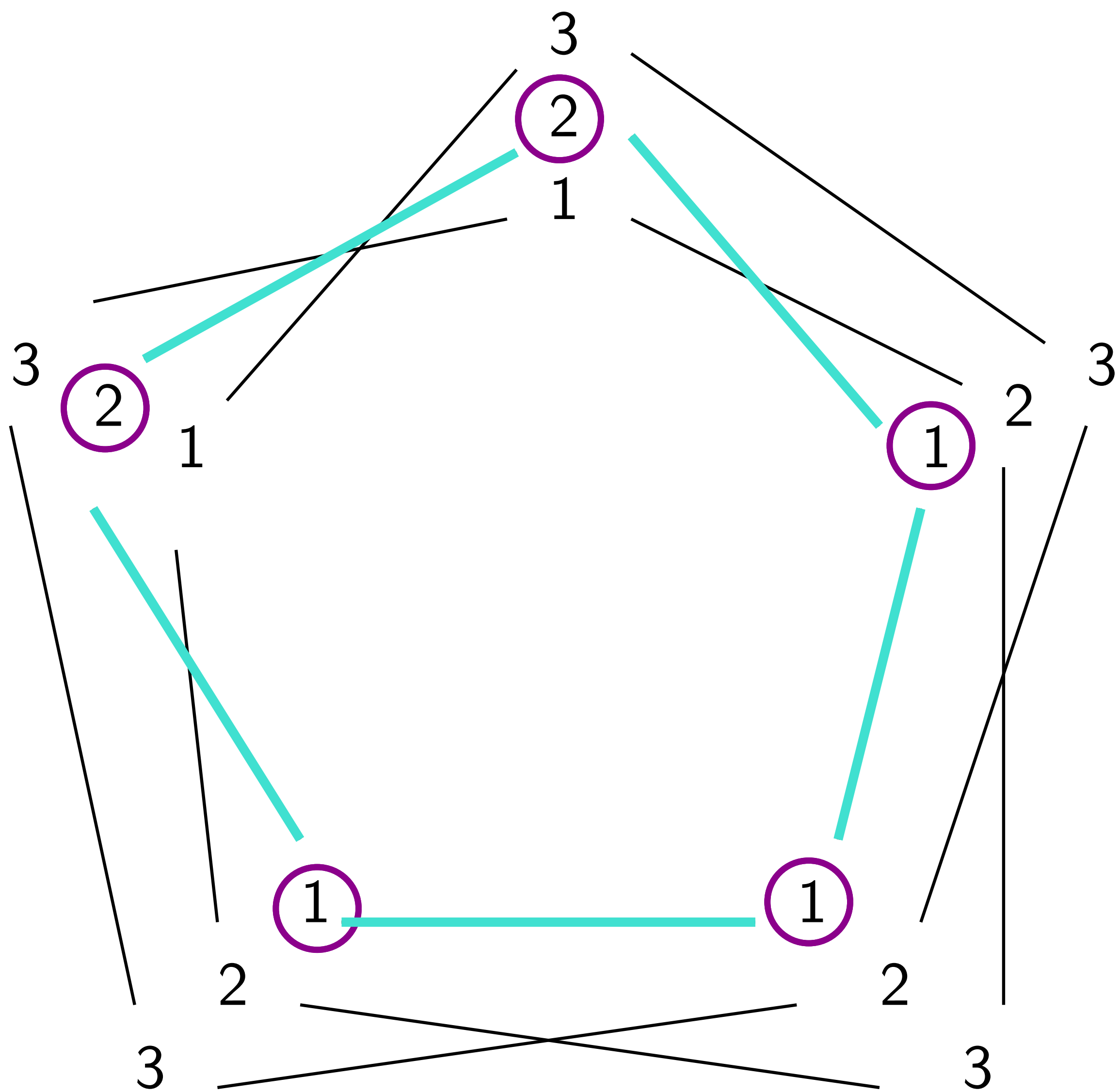

Unique Games Conjecture

Given: a unique label cover instance where there either exists a cover using $(1-\epsilon)|E|$ edges or there is no cover using more than $\delta|E|$ edges.

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Recently, Khot, Minzer and Safra solved the 2-2 Games Conjecture and it is strong evidence that UGC is true.

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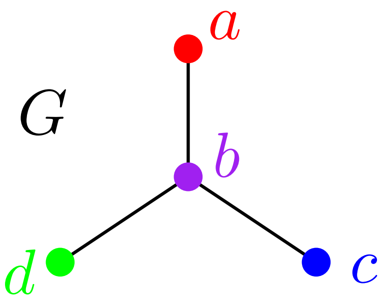
Examples:

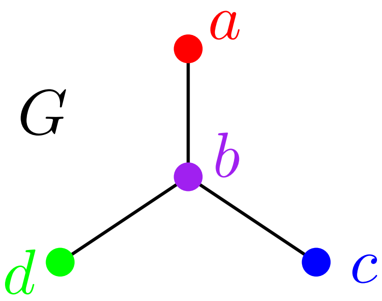
- Vertex Cover
- Max Cut
- Max acyclic subgraph

Covers of Graphs



Covers of Graphs

$$\alpha : E(G) \to S_k$$



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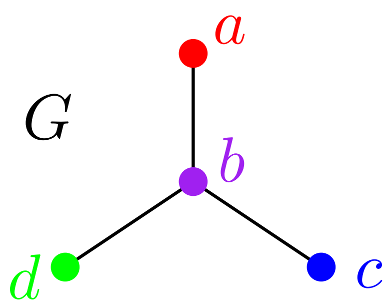

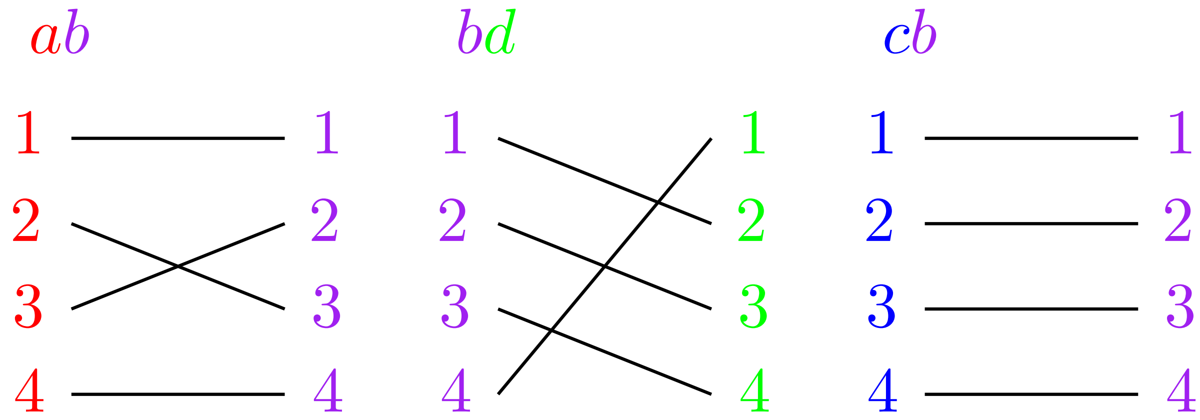

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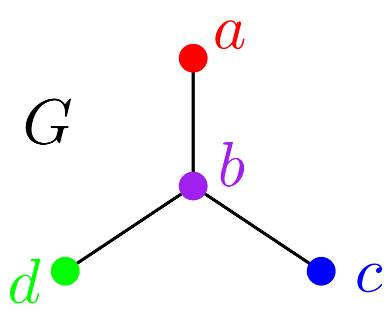

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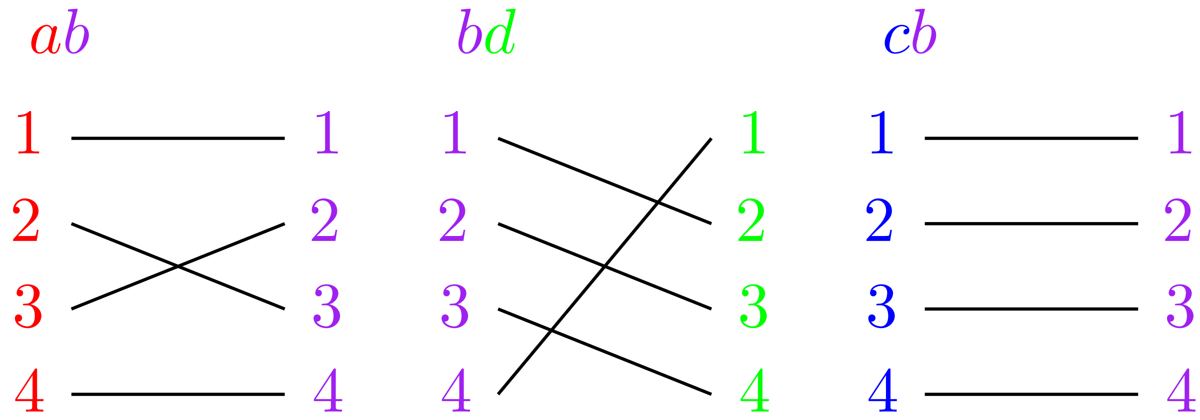

cover graph of G w.r.t. α

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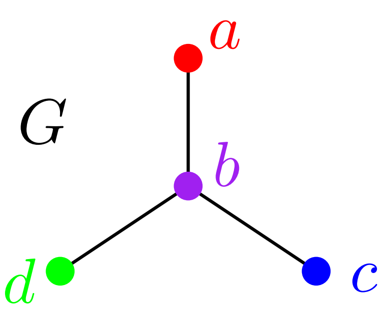

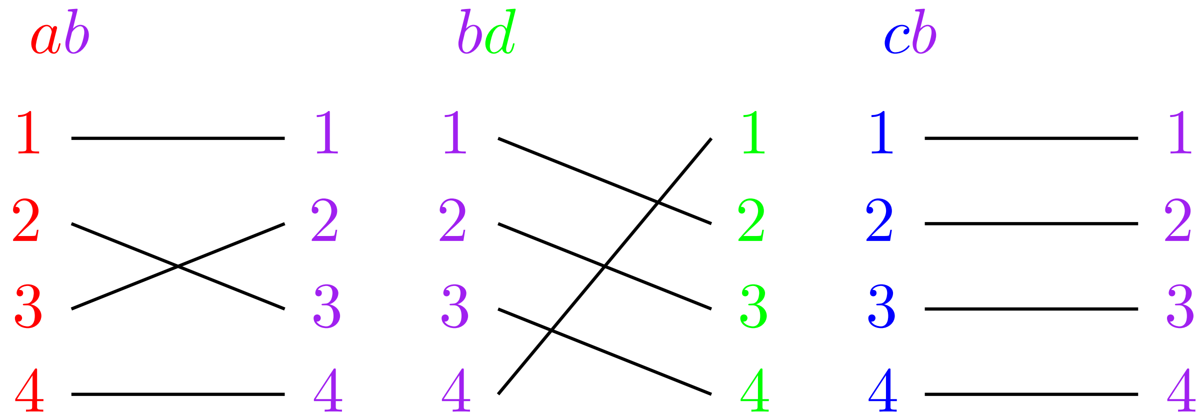

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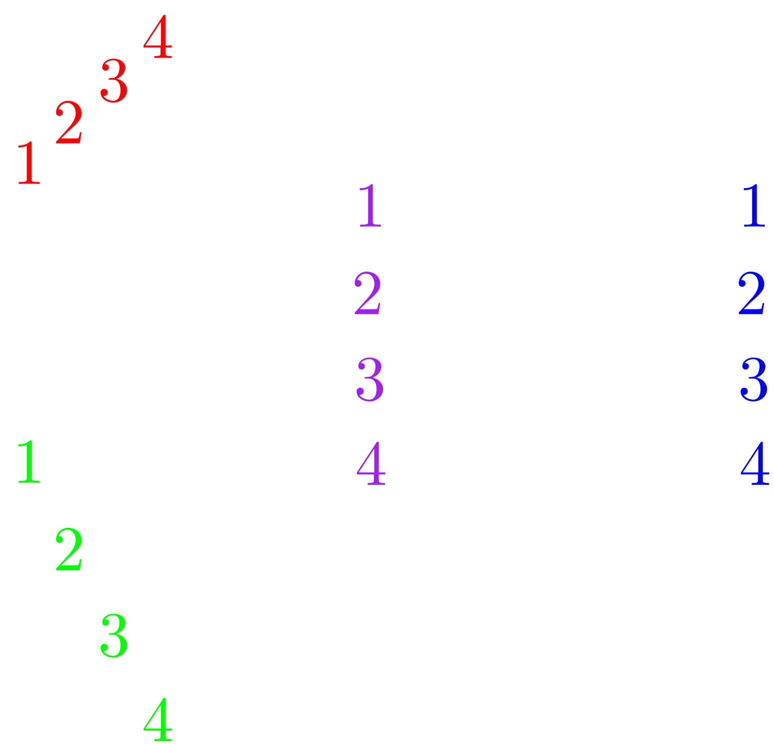

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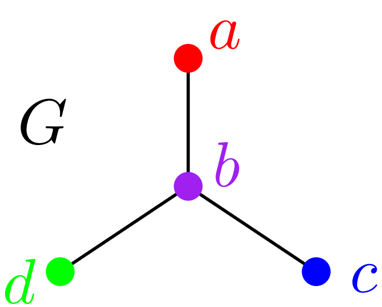

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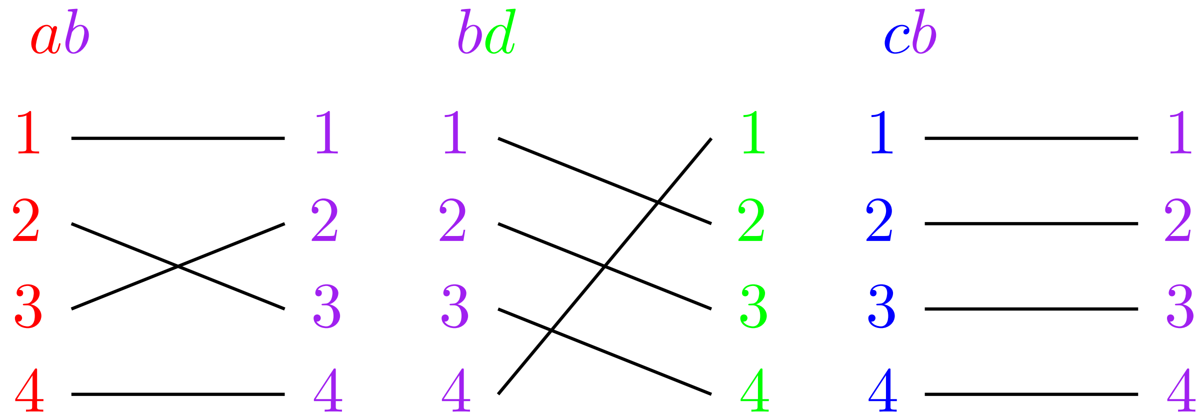

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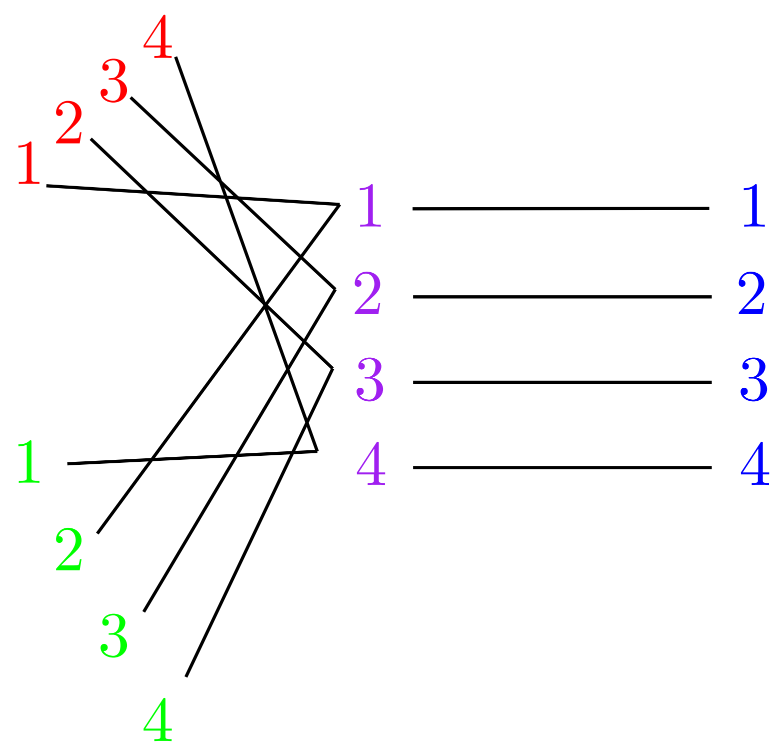
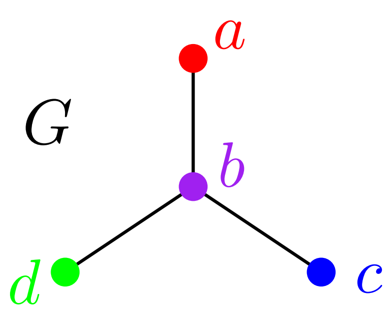
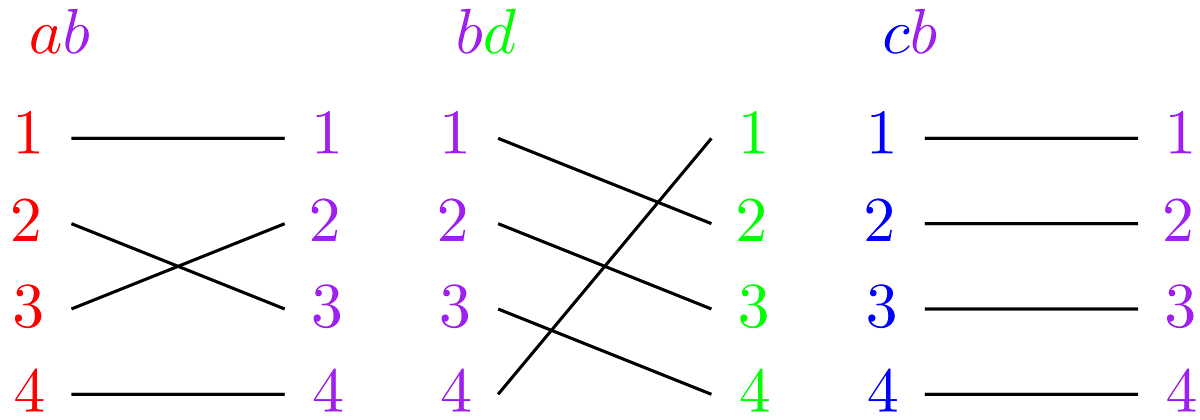


Covers of Graphs

$$\alpha : E(G) \to S_4$$





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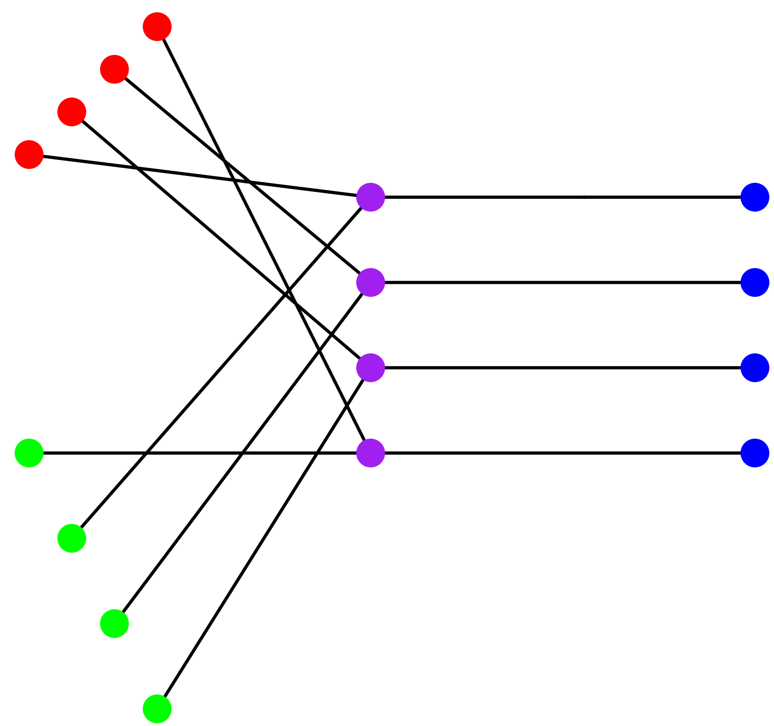

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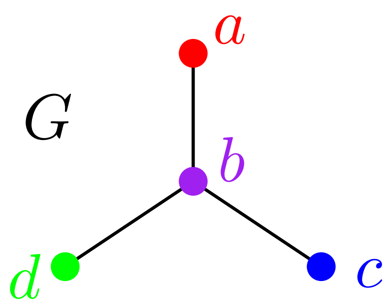

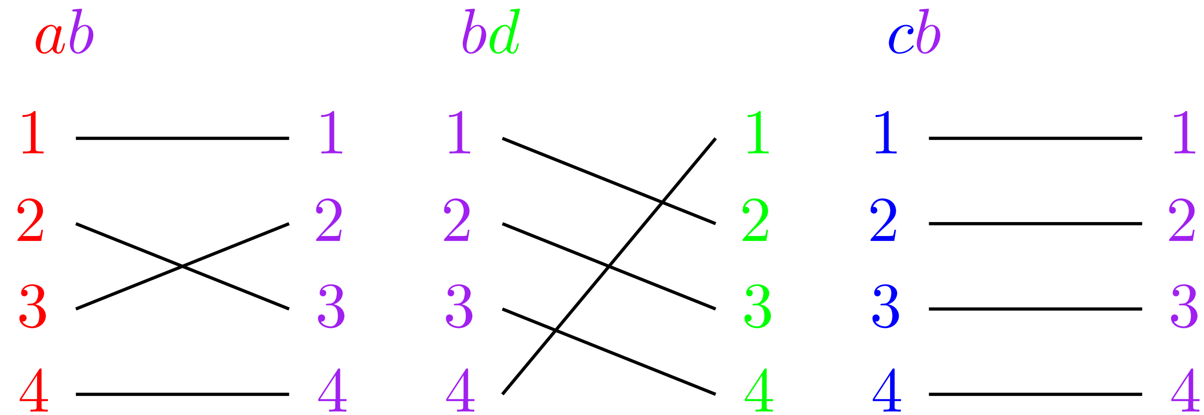

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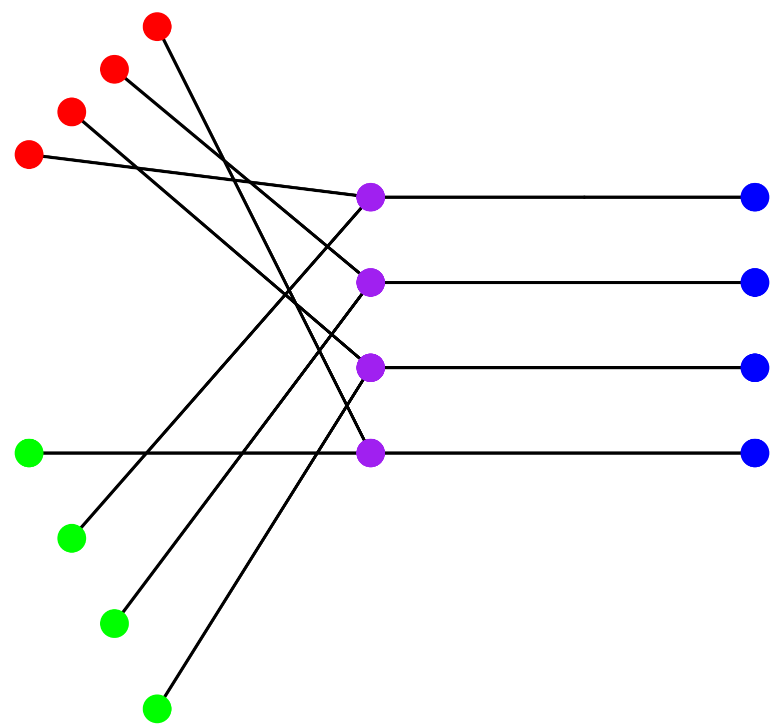

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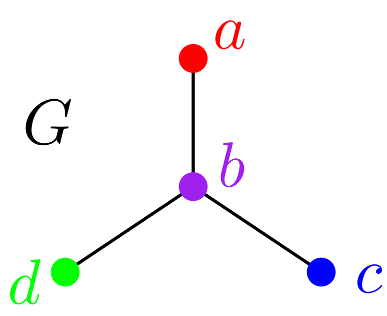
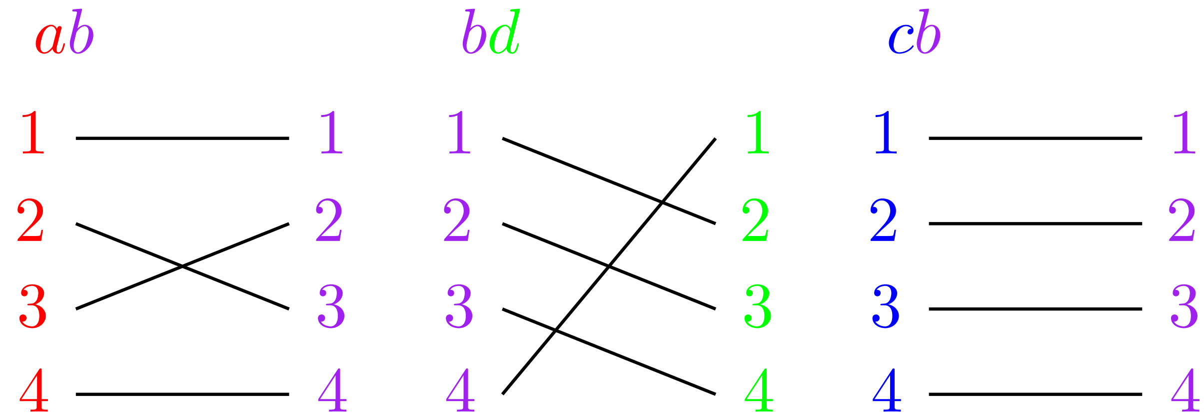
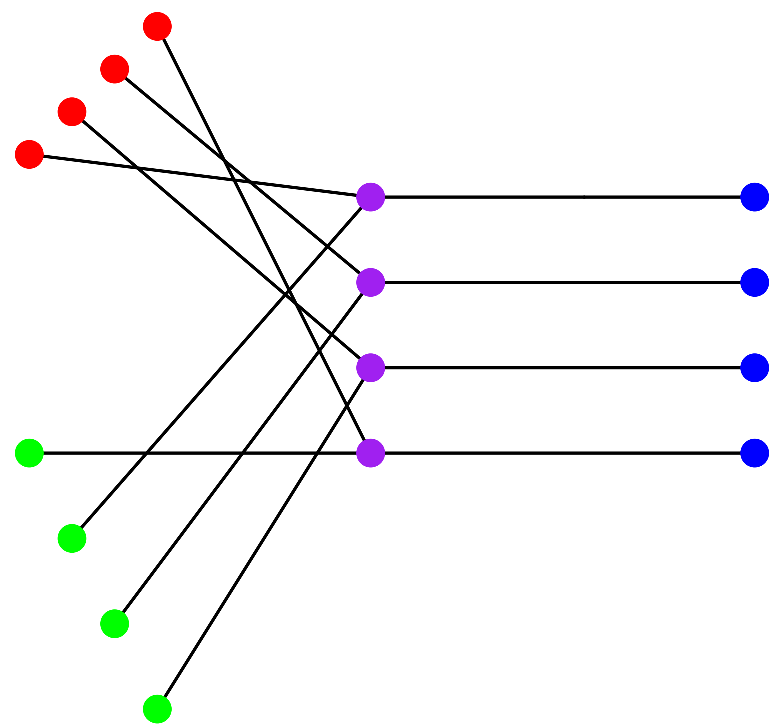




Transversal subgraph:

Induced subgraph
using one vertex
from each fibre



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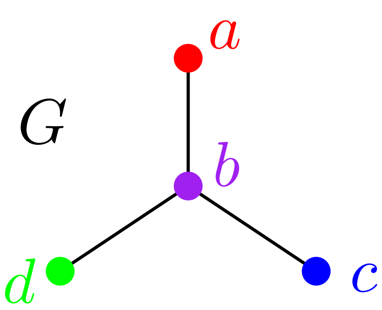

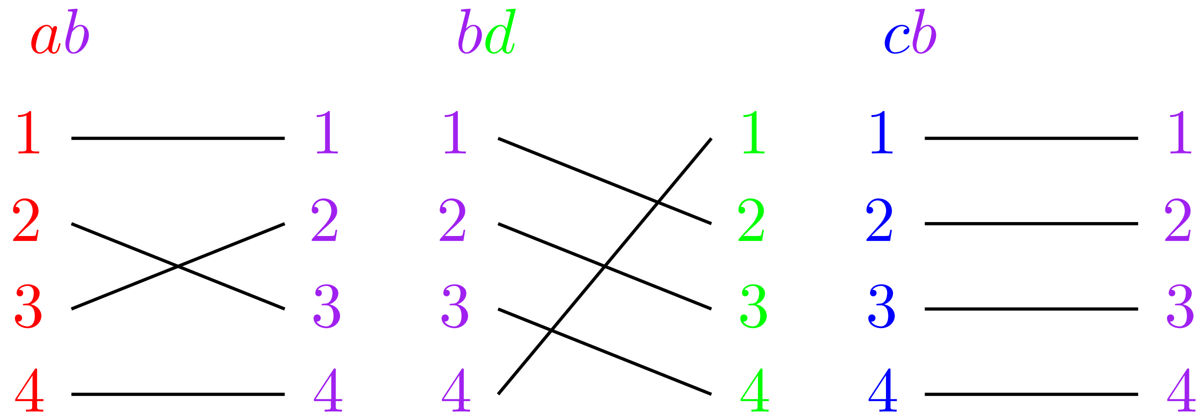

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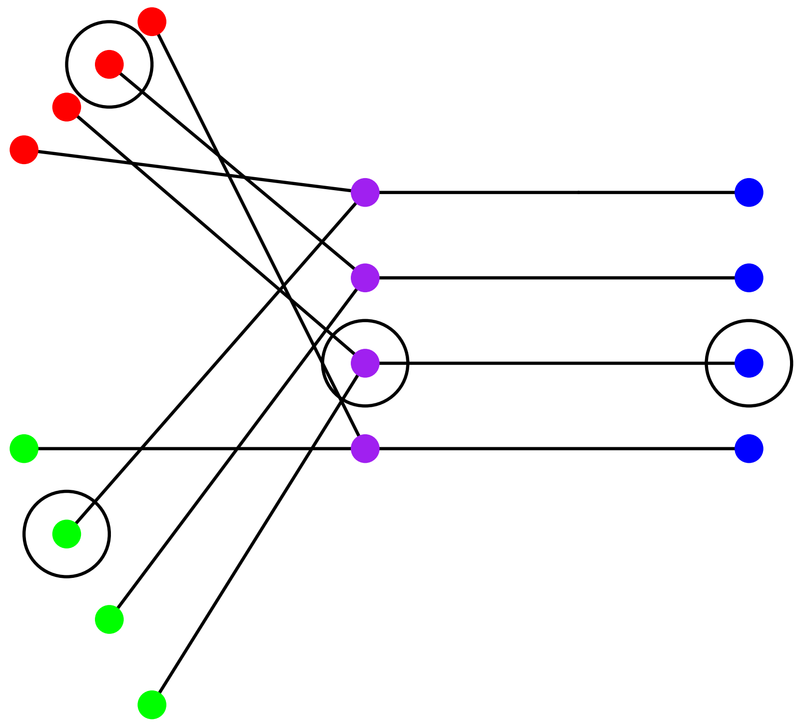

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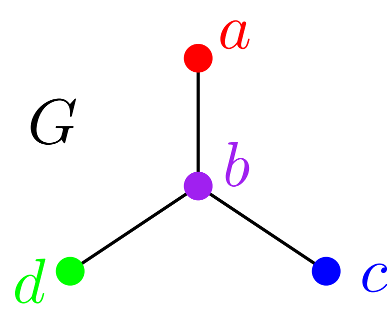

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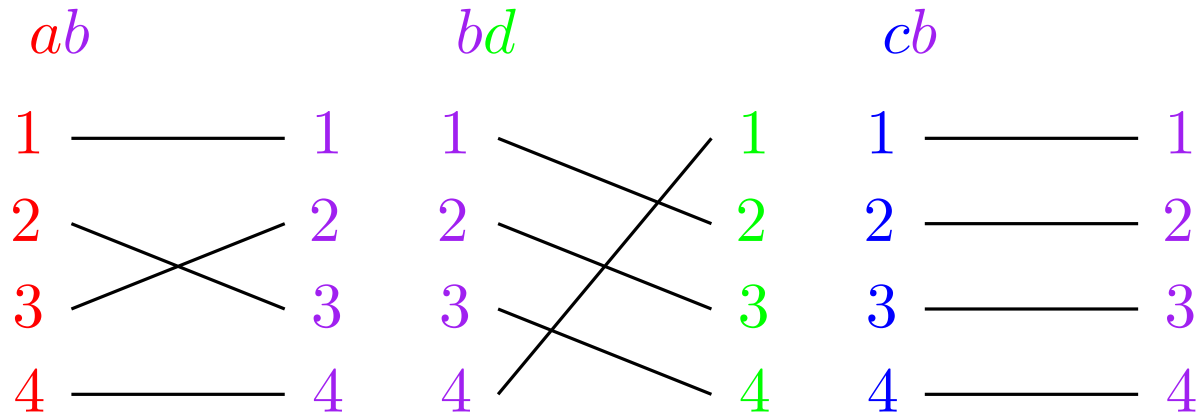

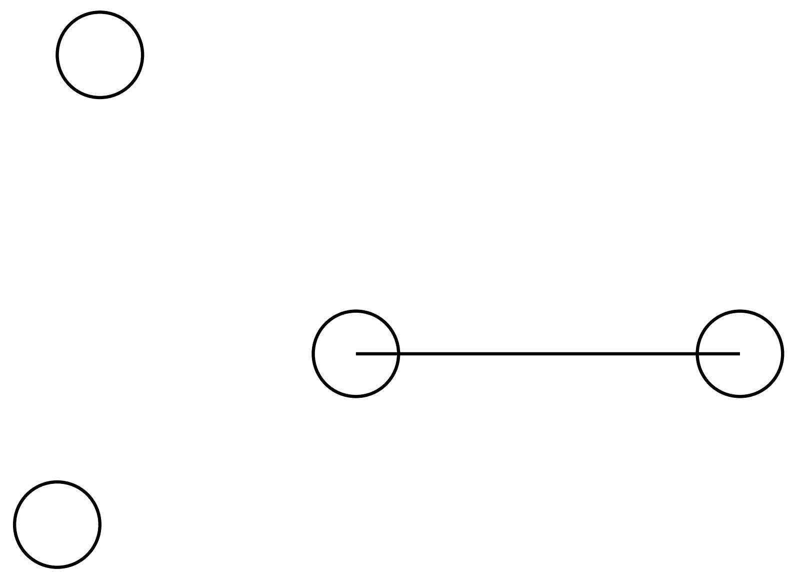

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Transversal polynomial

We define a polynomial whose coefficients count the number of transversal subgraphs with a given number of edges.

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Degree of ξ gives the maximum number of constraints satisfied by any assignment

Theorem (Godsil, Guo, Royle)

For any r-fold cover of a graph G with n vertices,

$$\xi(G^{\alpha}, -(r-1)) \cong 0 \mod r^{n}.$$

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The Tutte polynomial is also NP-hard to compute but one can evaluate it at certain points in poly-time. For example, evaluating at $(2, 1)$ gives the number of spanning forests.

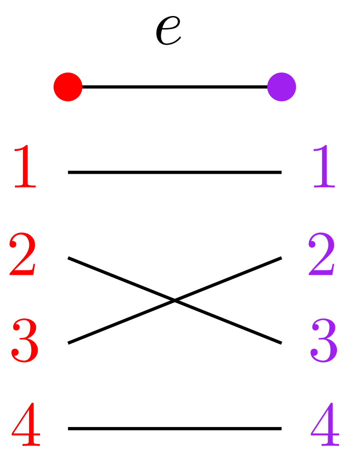
A contraction/deletion formula

Deletion

A contraction/deletion formula

Deletion

in G^α:



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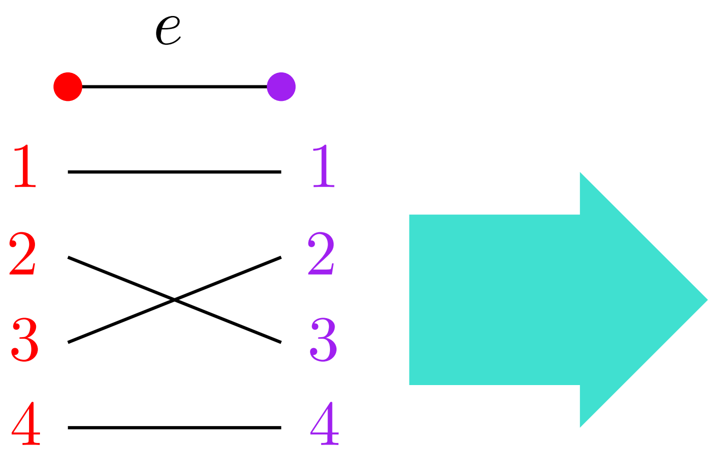

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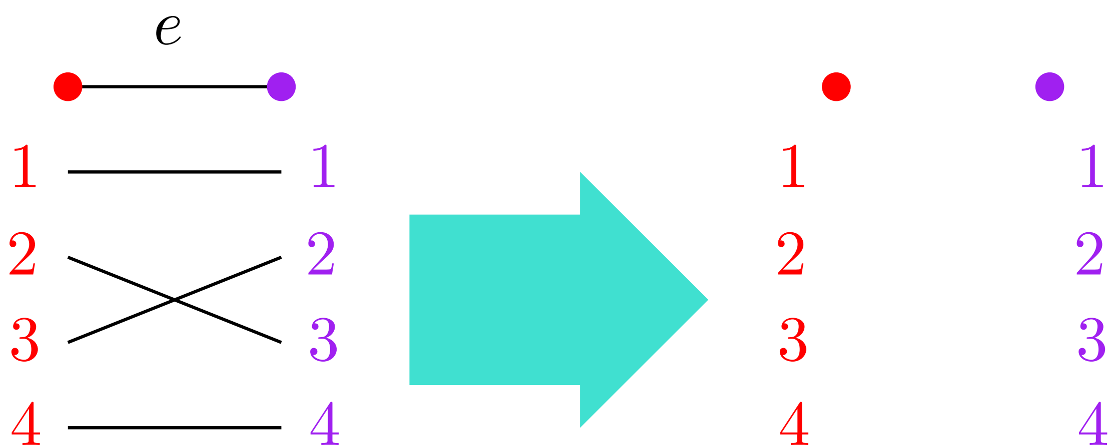

in $G^\alpha \setminus e$

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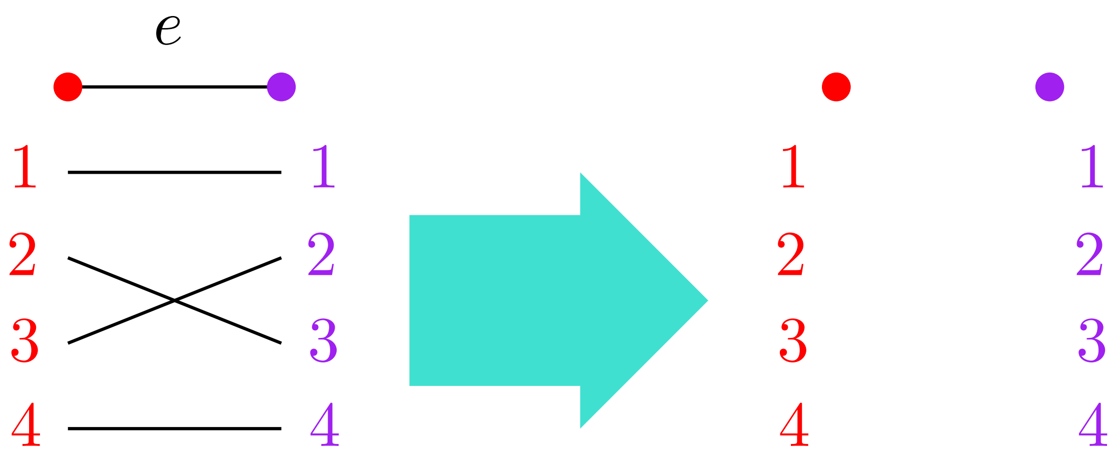

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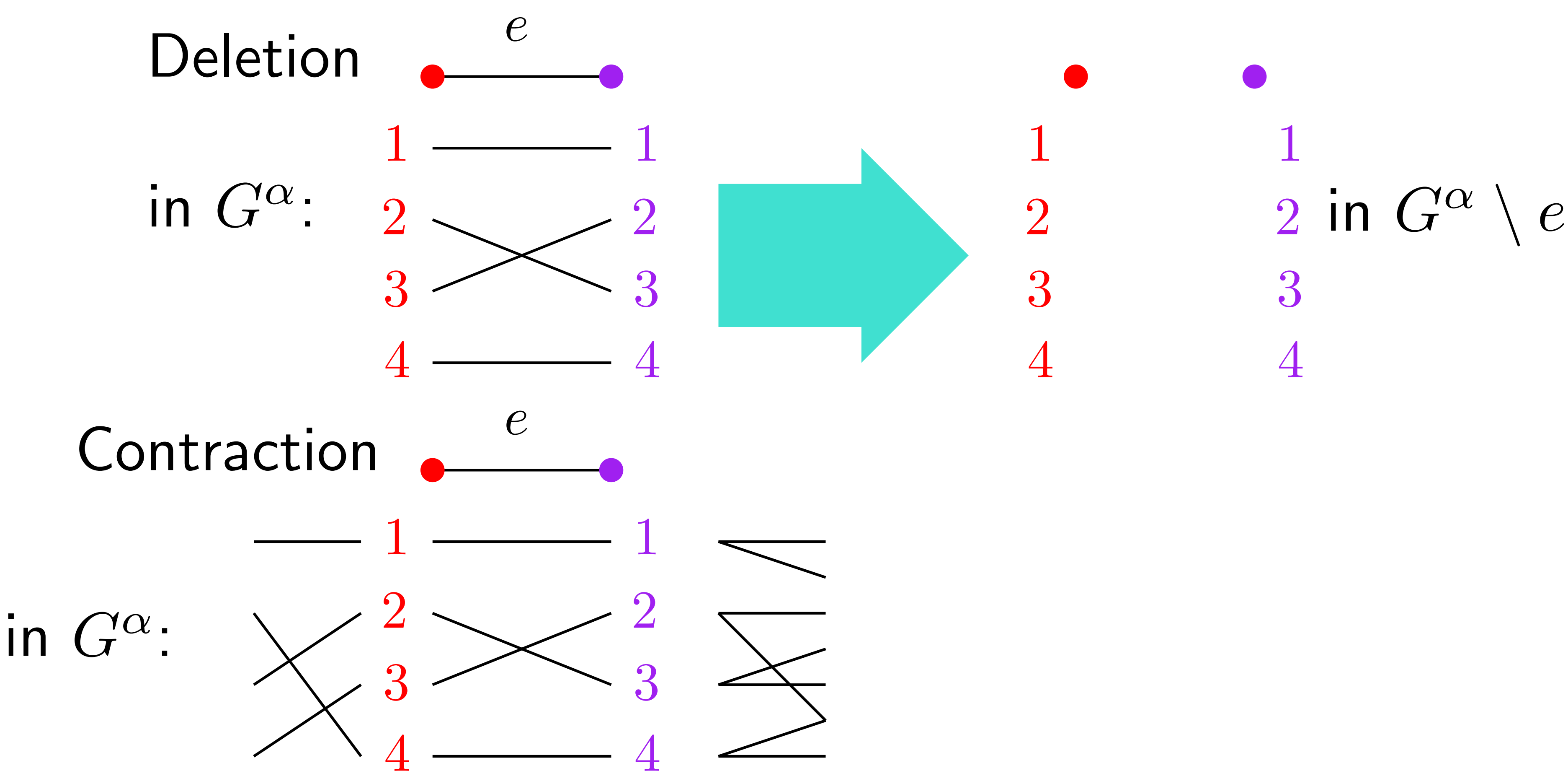
Contraction

A contraction/deletion formula



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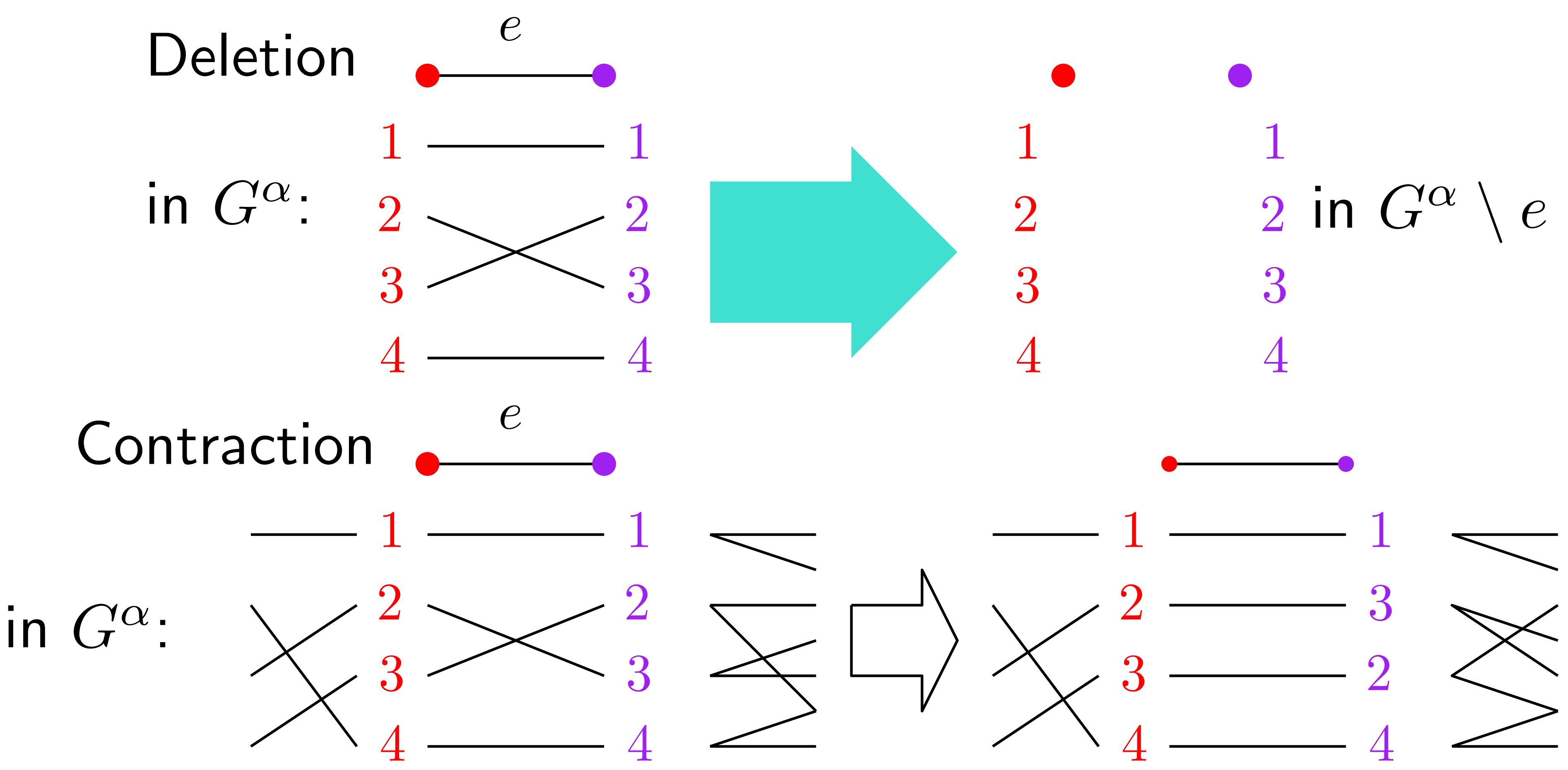

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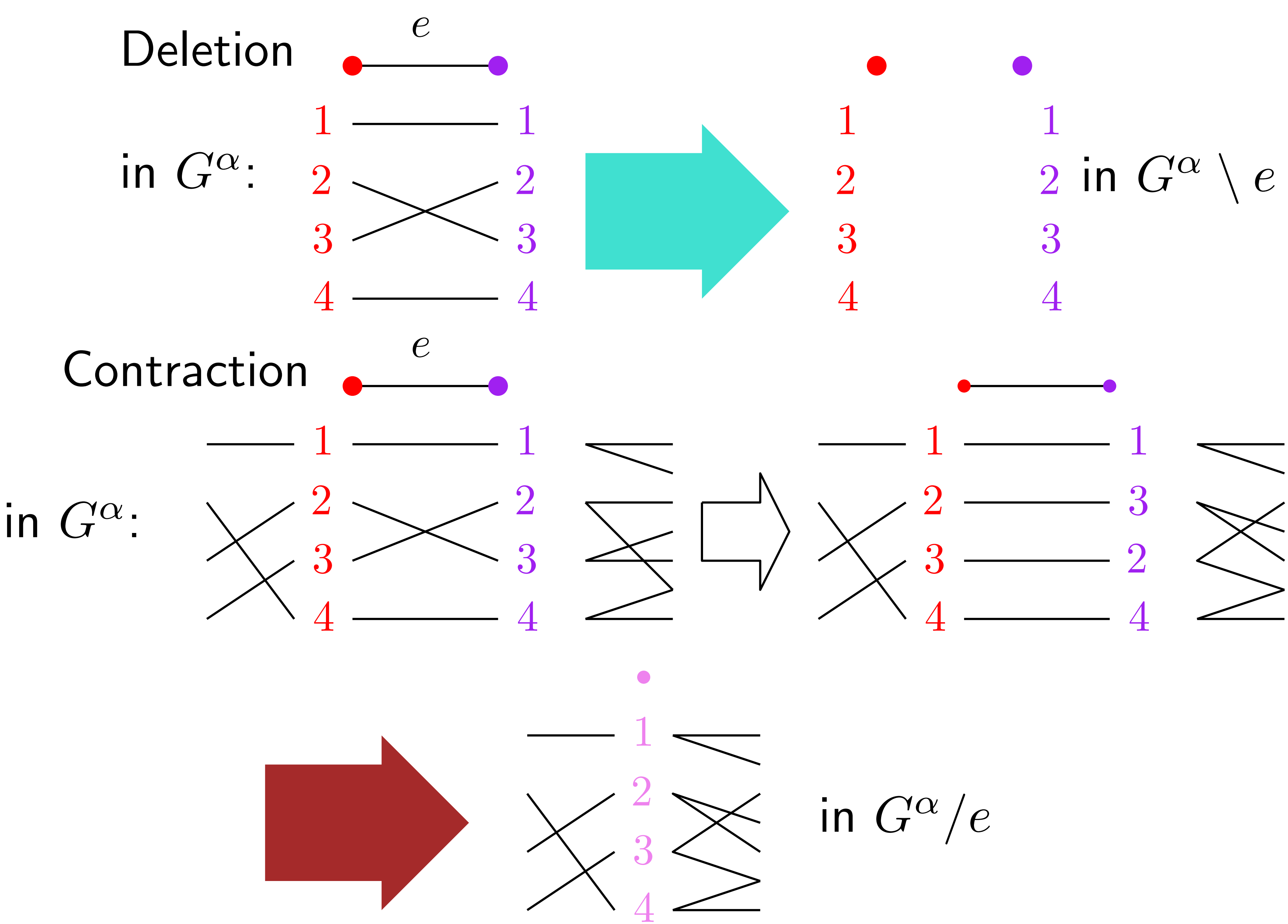

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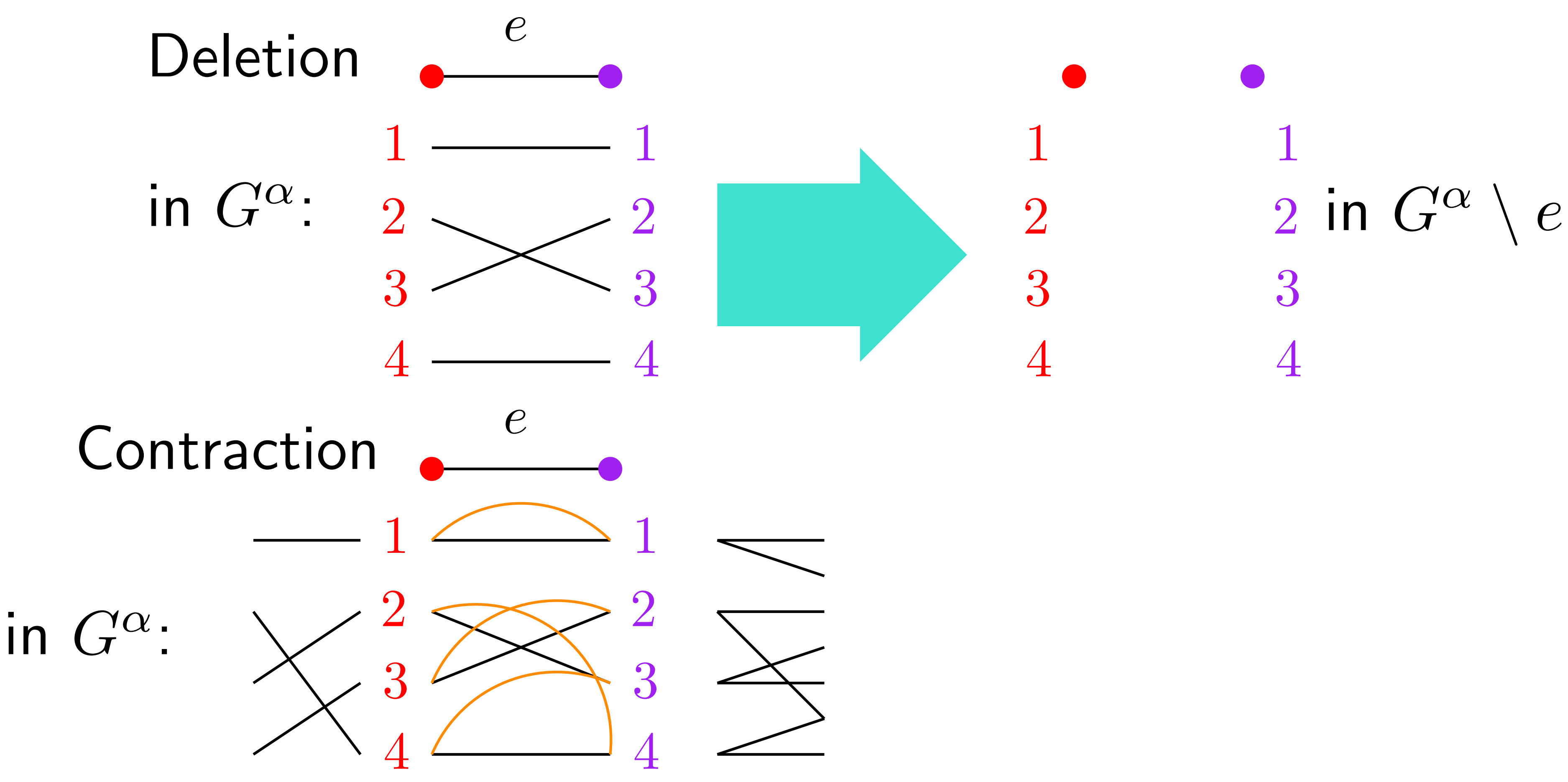

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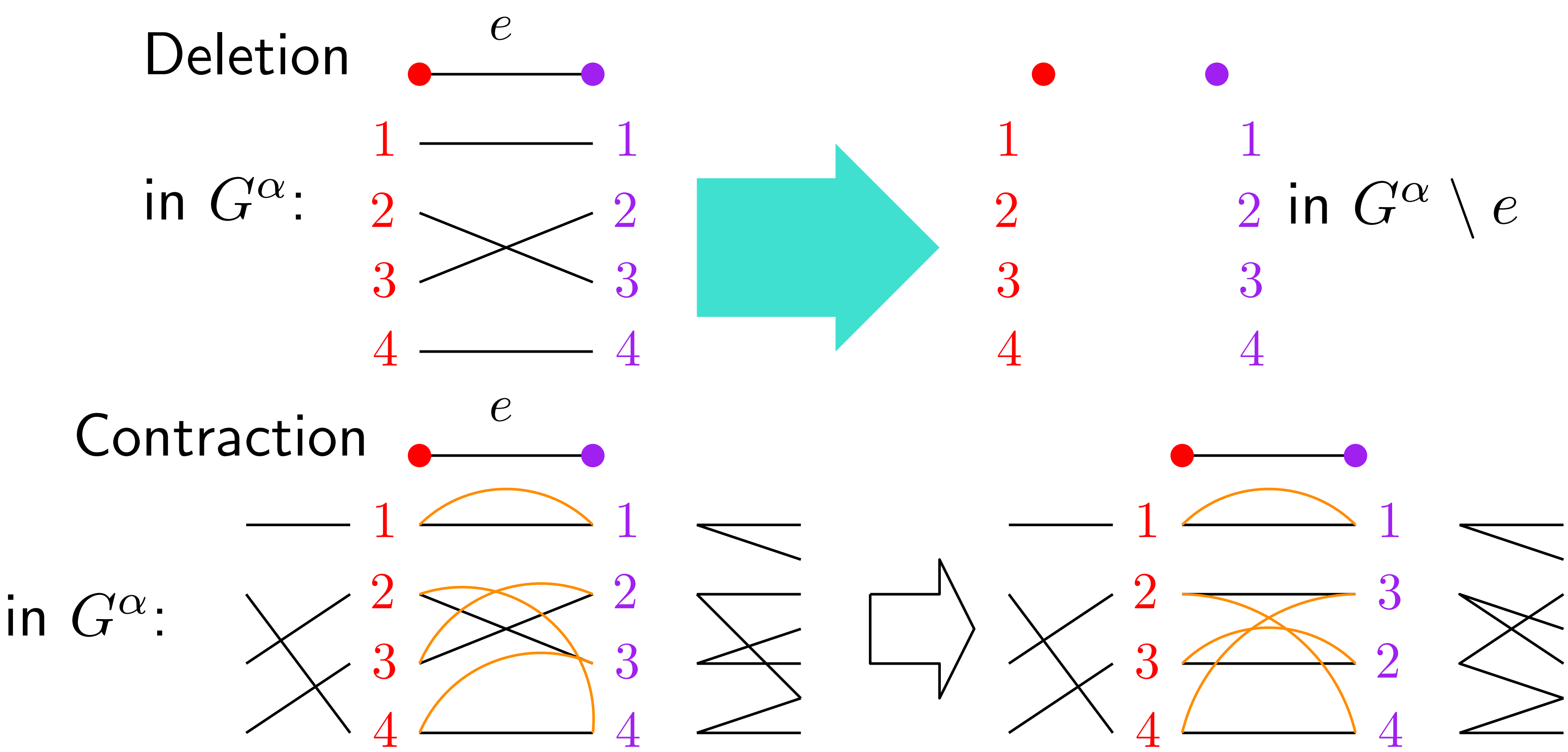

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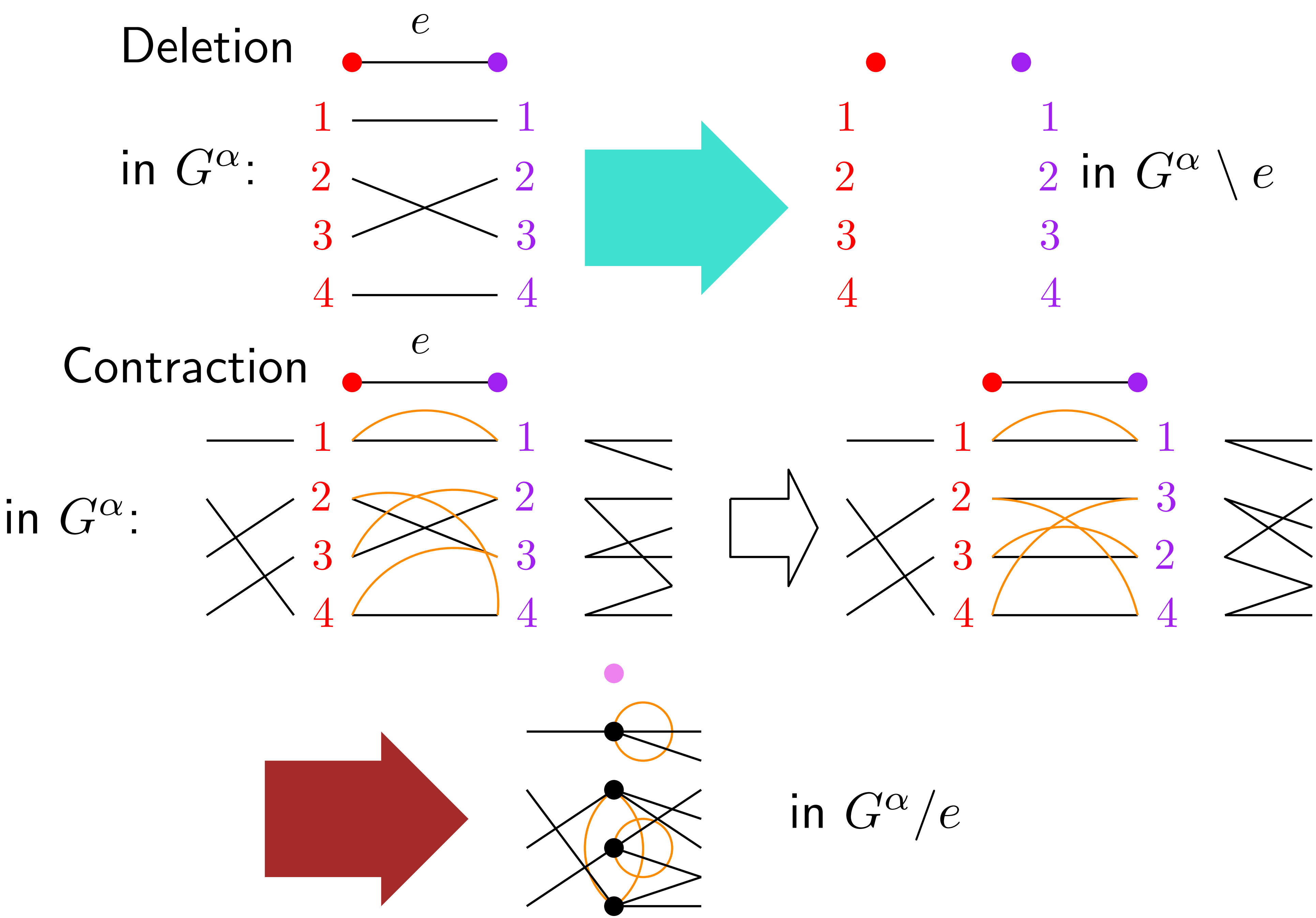

A Contraction/Deletion Formula

Theorem (Godsil, Guo, Royle)

$$\xi(G^\alpha, t) = (t-1)\xi(G^\alpha/e, t) + \xi(G^\alpha \setminus e, t)$$

Main result

Theorem (Godsil, Guo, Royle)

For any r-fold cover of a graph G with n vertices,

$$\xi(G^\alpha, -(r-1)) \cong 0 \mod r^n.$$

Theorem (Godsil, Guo, Royle)

For any 2-fold cover of a graph G with n vertices,

$$\xi(G^\alpha, 1) = \begin{cases} 2^n, \text{ if } G \text{ is Eulerian;} \\ 0, \text{ otherwise.} \end{cases}$$

Open problems

- Are there other points which we can "evaluate" the transversal polynomial?
- Are there classes of graphs for which we can compute the transversal polynomial in polynomial time?
  - For example, choosability is linear time in the number of vertices, for graphs of bounded tree-width.

Thanks!