Transversal polynomial of graphs

Chris Godsil    Krystal Guo    Gordon Royle

Korteweg-De Vries Institute, University of Amsterdam

Semidefinite & Polynomial Optimization
29 August – 2 September 2022
Polynomials in Combinatorics

Polynomials arise in combinatorics, usually as a way of counting.
Polynomials in Combinatorics

Polynomials arise in combinatorics, usually as a way of counting.

Suppose we have a class of objects $C$ and we can count the number of them of order $n$. 
Polynomials in Combinatorics

Polynomials arise in combinatorics, usually as a way of counting.

Suppose we have a class of objects $C$ and we can count the number of them of order $n$.

We get a sequence $a_0, a_1, a_2, \ldots$ and we can associate with this sequence the generating function

$$G(t) = \sum_{i=0}^{\infty} a_i t^i$$
Polynomials in Combinatorics

Polynomials arise in combinatorics, usually as a way of counting.

Suppose we have a class of objects $C$ and we can count the number of them of order $n$.

We get a sequence $a_0, a_1, a_2, \ldots$ and we can associate with this sequence the generating function

$$G(t) = \sum_{i=0}^{\infty} a_i t^i$$

Alternatively, we can write this as a sum of elements of $C$, where $w(C)$ is its order.

$$G(x) = \sum_{C \in C} t^{w(C)}$$
Examples of Generating Functions

Catalan Numbers
Examples of Generating Functions

Catalan Numbers

The famous Catalan numbers $1, 1, 2, 5, 14, 42, \ldots$ enumerate many things. The generating function is the unique (formal) power series with non-negative coefficients satisfying:
Catalan Numbers

The famous Catalan numbers 1, 1, 2, 5, 14, 42, ... enumerate many things. The generating function is the unique (formal) power series with non-negative coefficients satisfying:

\[ 1 - C(t) + tC(t)^2 = 0 \]
Examples of Generating Functions

Catalan Numbers
The famous Catalan numbers 1, 1, 2, 5, 14, 42, ... enumerate many things. The generating function is the unique (formal) power series with non-negative coefficients satisfying:

\[ 1 - C(t) + tC(t)^2 = 0 \]

Matchings in graphs
Examples of Generating Functions

Catalan Numbers
The famous Catalan numbers 1, 1, 2, 5, 14, 42, ... enumerate many things. The generating function is the unique (formal) power series with non-negative coefficients satisfying:

\[ 1 - C(t) + tC(t)^2 = 0 \]

Matchings in graphs

Take a graph $G$ and let $w(M)$ for a matching $M$ be the number of edges in $M$. 
Examples of Generating Functions

Catalan Numbers
The famous Catalan numbers 1, 1, 2, 5, 14, 42, ... enumerate many things. The generating function is the unique (formal) power series with non-negative coefficients satisfying:

\[ 1 - C(t) + tC(t)^2 = 0 \]

Matchings in graphs

Take a graph $G$ and let $w(M)$ for a matching $M$ be the number of edges in $M$.

The matching polynomial is the generating function:
Examples of Generating Functions

Catalan Numbers
The famous Catalan numbers 1, 1, 2, 5, 14, 42, ... enumerate many things. The generating function is the unique (formal) power series with non-negative coefficients satisfying:

$$1 - C(t) + tC(t)^2 = 0$$

Matchings in graphs
Take a graph $G$ and let $w(M)$ for a matching $M$ be the number of edges in $M$.
The matching polynomial is the generating function:

$$M(G, t) = \sum_M t^{w(M)}$$
Examples of Generating Functions

Catalan Numbers

The famous Catalan numbers $1, 1, 2, 5, 14, 42, \ldots$ enumerate many things. The generating function is the unique (formal) power series with non-negative coefficients satisfying:

$$1 - C(t) + tC(t)^2 = 0$$

Matchings in graphs

Take a graph $G$ and let $w(M)$ for a matching $M$ be the number of edges in $M$.

The matching polynomial is the generating function:

$$\mu(G, t) = \sum_{M} (-1)^{w(M)} t^{n-2w(M)}$$
Why polynomials?

Main idea: we manipulate the polynomials with algebra or calculus and prove something about the objects we enumerate.
Why polynomials?

Main idea: we manipulate the polynomials with algebra or calculus and prove something about the objects we enumerate.

Tutte polynomial, chromatic polynomial, etc.
Why polynomials?

Main idea: we manipulate the polynomials with algebra or calculus and prove something about the objects we enumerate.

Tutte polynomial, chromatic polynomial, etc.

The characteristic polynomial of the adjacency matrix of a graph is secretly also a generating function.
Why polynomials?

Main idea: we manipulate the polynomials with algebra or calculus and prove something about the objects we enumerate.

Tutte polynomial, chromatic polynomial, etc.

The characteristic polynomial of the adjacency matrix of a graph is secretly also a generating function.

\[ \phi(G, t) = \det(tI - A(G)) \]
Why polynomials?

Main idea: we manipulate the polynomials with algebra or calculus and prove something about the objects we enumerate.

Tutte polynomial, chromatic polynomial, etc.

The characteristic polynomial of the adjacency matrix of a graph is secretly also a generating function.

\[ \phi(G, t) = \det(tI - A(G)) \]

By decomposing the objects being counted, we find

\[ \sum_{v \in V(G)} \phi(G \setminus v, t) = \frac{d}{dt} \phi(G, t) \]

and other identities, including the Christoffel-Darboux identity.
Applications

Marcus, Spielman, Srivastava 2015
Applications

Marcus, Spielman, Srivastava 2015

Made use of a connection between the matching polynomial and characteristic polynomial to find bipartite Ramanujan graphs of all degrees.
Applications

Marcus, Spielman, Srivastava 2015

Made use of a connection between the matching polynomial and characteristic polynomial to find bipartite Ramanujan graphs of all degrees.

Godsil, Guo, Kempton, Lippner 2017
Applications

Marcus, Spielman, Srivastava 2015

Made use of a connection between the matching polynomial and characteristic polynomial to find bipartite Ramanujan graphs of all degrees.

Godsil, Guo, Kempton, Lippner 2017

\[ G \]
Applications

Marcus, Spielman, Srivastava 2015

Made use of a connection between the matching polynomial and characteristic polynomial to find bipartite Ramanujan graphs of all degrees.

Godsil, Guo, Kempton, Lippner 2017
Applications

Marcus, Spielman, Srivastava 2015

Made use of a connection between the matching polynomial and characteristic polynomial to find bipartite Ramanujan graphs of all degrees.

Godsil, Guo, Kempton, Lippner 2017

Made use of Christoffel-Darboux type identities to find eigenvalues of edge-perturbations of graphs for the purposes of quantum walks.
In this talk

We find a common setting for two different problems in discrete math.
In this talk

We find a common setting for two different problems in discrete math.

We define a generating function for these objects.

We prove basic properties about this polynomial, including the evaluation of a point.
Colourings of graphs
Colourings of graphs

\( k \)-colouring of graphs \{ • • • \}
$k$-list colouring

\{ \text{colours} \}
$k$-list colouring

\{ \}
Colourings of graphs

$k$-list colouring \[ \{ \bullet \quad \bullet \quad \bullet \} \]
$k$-list colouring: $\{\text{red, blue, green}\}$
Colourings of graphs

$k$-correspondence colouring
$k$-correspondence colouring
Colourings of graphs

$k$-correspondence colouring
Colourings of graphs

$k$-correspondence colouring
Colourings of graphs

$k$-correspondence colouring
Colourings of graphs

$k$-correspondence colouring
$k$-correspondence colouring
Colourings of graphs

$k$-correspondence colouring
Colourings of graphs

$k$-correspondence colouring
Colourings of graphs

$k$-correspondence colouring
Colourings of graphs

$k$-correspondence colouring
Historical perspective of colouring

$k$-choosable: any list assignment with lists of size at most $k$ has a proper colouring
Historical perspective of colouring

$k$-choosable: any list assignment with lists of size at most $k$ has a proper colouring

Theorem (Thomassen 1994)

Every planar graph is 5-choosable.
\textbf{Historical perspective of colouring}

\textit{k-choosable}: any list assignment with lists of size at most \textit{k} has a proper colouring

\textbf{Theorem (Thomassen 1994)}

Every planar graph is 5-choosable.

\textbf{Theorem (Dvorak and Postle 2016)}

Every planar graph without cycles of lengths 4, 5, 6, 7, 8 is 3-choosable.
$k$-choosable: any list assignment with lists of size at most $k$ has a proper colouring

Theorem (Thomassen 1994)

Every planar graph is 5-choosable.

Theorem (Dvorak and Postle 2016)

Every planar graph without cycles of lengths 4, 5, 6, 7, 8 is 3-choosable.

They prove it by showing that such a graph is DP-3-colourable under an additional condition.
Historical perspective of colouring

$k$-choosable: any list assignment with lists of size at most $k$ has a proper colouring

Theorem (Thomassen 1994)

Every planar graph is 5-choosable.

Theorem (Dvorak and Postle 2016)

Every planar graph without cycles of lengths 4, 5, 6, 7, 8 is 3-choosable.

They prove it by showing that such a graph is DP-3-colourable under an additional condition.

Theorem (Loeb, Rolek, Liu, Yu 2018)

Every planar graph without cycles of lengths 4, $a$, $b$, 9 for $a, b \in \{6, 7, 8\}$ is DP-3-colourable.
Unique Label Cover
Unique Label Cover
Unique Label Cover
Unique Label Cover
Unique Label Cover
Unique Label Cover
Unique Games Conjecture

Given: a unique label cover instance where there either exists a cover using \((1 - \epsilon) |E|\) edges or there is no cover using more than \(\delta |E|\) edges.
Unique Games Conjecture

Given: a unique label cover instance where there either exists a cover using \((1 - \epsilon)|E|\) edges or there is no cover using more than \(\delta|E|\) edges.

Problem: determine which it is
Unique Games Conjecture

Given: a unique label cover instance where there either exists a cover using $(1 - \epsilon)|E|$ edges or there is no cover using more than $\delta|E|$ edges.

Problem: determine which it is

Unique Games Conjecture (Khot 2002)

The above decision problem is NP-hard.
Unique Games Conjecture

Given: a unique label cover instance where there either exists a cover using \((1 - \epsilon)|E|\) edges or there is no cover using more than \(\delta|E|\) edges.

Problem: determine which it is

Unique Games Conjecture (Khot 2002)

The above decision problem is NP-hard.

Recently, Khot, Minzer and Safra solved the 2-2 Games Conjecture and it is strong evidence that UGC is true.
UGC and hardness of approximation

Many NP-hard problems where there currently exists an $\alpha$-factor approximation algorithm known and, if UGC is true, then the best approximation factor possible is $(\alpha - \epsilon)$. 
UGC and hardness of approximation

Many NP-hard problems where there currently exists an $\alpha$-factor approximation algorithm known and, if UGC is true, then the best approximation factor possible is $(\alpha - \epsilon)$.

Examples:
UGC and hardness of approximation

Many NP-hard problems where there currently exists an $\alpha$-factor approximation algorithm known and, if UGC is true, then the best approximation factor possible is $(\alpha - \epsilon)$.

Examples:

Vertex Cover
UGC and hardness of approximation

Many NP-hard problems where there currently exists an $\alpha$-factor approximation algorithm known and, if UGC is true, then the best approximation factor possible is $(\alpha - \epsilon)$.

Examples:

- Vertex Cover
UGC and hardness of approximation

Many NP-hard problems where there currently exists an $\alpha$-factor approximation algorithm known and, if UGC is true, then the best approximation factor possible is $(\alpha - \epsilon)$.

Examples:

- Vertex Cover
- Max Cut
UGC and hardness of approximation

Many NP-hard problems where there currently exists an $\alpha$-factor approximation algorithm known and, if UGC is true, then the best approximation factor possible is $(\alpha - \epsilon)$.

Examples:

Vertex Cover

Max Cut

Max acyclic subgraph
Covers of Graphs

Graph $G$ with vertices labeled $a$, $b$, $c$, and $d$.
Covers of Graphs

\[ \alpha : E(G) \to S_k \]
Covers of Graphs

\[ \alpha : E(G) \rightarrow S_4 \]

\[ \begin{array}{ccc}
ab & bd & cb \\
1 & 1 & 1 \\
2 & 2 & 2 \\
3 & 3 & 3 \\
4 & 4 & 4 \\
\end{array} \]
Covers of Graphs

\[ \alpha : E(G) \rightarrow S_4 \]

<table>
<thead>
<tr>
<th></th>
<th>ab</th>
<th>bd</th>
<th>cb</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>4</td>
<td>4</td>
</tr>
</tbody>
</table>

cover graph of \( G \) w.r.t. \( \alpha \)

\[ G^\alpha \]
Covers of Graphs

\[ \alpha : E(G) \to S_4 \]

cover graph of \( G \) w.r.t. \( \alpha \)

\[ G^{\alpha} \]
Covers of Graphs

\[ \alpha : E(G) \rightarrow S_4 \]

\begin{align*}
ab & \quad bd & \quad cb \\
1 \quad 1 \quad 1 & \quad 1 \\
2 \quad 2 \quad 2 & \quad 2 \\
3 \quad 3 \quad 3 & \quad 3 \\
4 \quad 4 \quad 4 & \quad 4 \\
\end{align*}

cover graph of \( G \) w.r.t. \( \alpha \)

\[ G^\alpha \]
Covers of Graphs

\[ \alpha : E(G) \rightarrow S_4 \]

cover graph of \( G \) w.r.t. \( \alpha \)

\[ G^\alpha \]
Covers of Graphs

\[ \alpha : E(G) \rightarrow S_4 \]

cover graph of \( G \) w.r.t. \( \alpha \)

\[ G^\alpha \]
Covers of Graphs

\[ \alpha : E(G) \rightarrow S_4 \]

Transversal subgraph:
Induced subgraph using one vertex from each fibre
Covers of Graphs

\[ \alpha : E(G) \to S_4 \]

Transversal subgraph:

Induced subgraph using one vertex from each fibre
Covers of Graphs

\[ \alpha : E(G) \rightarrow S_4 \]

Transversal subgraph:

Induced subgraph using one vertex from each fibre
We define a polynomial whose coefficients count the number of transversal subgraphs with a given number of edges.
We define a polynomial whose coefficients count the number of transversal subgraphs with a given number of edges.

\[ \xi(G^\alpha, t) = \sum_{k=0}^{t} \text{number of transversal subgraphs with } k \text{ edges} \]
We define a polynomial whose coefficients count the number of transversal subgraphs with a given number of edges.

\[ \xi(G^\alpha, t) = \sum_{k=0}^{t} t^k \text{ number of transversal subgraphs with } k \text{ edges} \]

\[ \xi(G^\alpha, t) = \sum_{\text{all transversal subgraphs } H} t|E(H)| \]
Transversal polynomial

We define a polynomial whose coefficients count the number of transversal subgraphs with a given number of edges.

$$\xi(G^{\alpha}, t) = \sum_{k=0}^{t} \text{number of transversal subgraphs with } k \text{ edges} \ t^k$$

$$\xi(G^{\alpha}, t) = \sum_{|E(H)|} t|E(H)|$$

all transversal subgraphs $H$

Constant term is number of correspondence colourings
Transversal polynomial

We define a polynomial whose coefficients count the number of transversal subgraphs with a given number of edges.

$$\xi(G^\alpha, t) = \sum_{k=0}^{t} t^k$$

denotes the number of transversal subgraphs with $k$ edges.

$$\xi(G^\alpha, t) = \sum_{H} t|E(H)|$$

denotes the number of all transversal subgraphs $H$.

Constant term is number of correspondence colourings.

Degree of $\xi$ gives the maximum number of constraints satisfied by any assignment.
Theorem (Godsil, Guo, Royle)

For any $r$-fold cover of a graph $G$ with $n$ vertices,

$$\xi(G^\alpha, -(r - 1)) \equiv 0 \mod r^n.$$
Theorem (Godsil, Guo, Royle)

For any $r$-fold cover of a graph $G$ with $n$ vertices,

$$\xi(G^\alpha, -(r - 1)) \equiv 0 \mod r^n.$$ 

This polynomial cannot be computed in polynomial time (unless $P = NP$), but we can compute this point mod $r^n$. 
Theorem (Godsil, Guo, Royle)

For any \( r \)-fold cover of a graph \( G \) with \( n \) vertices,

\[
\xi(G^\alpha, -(r - 1)) \equiv 0 \mod r^n.
\]

This polynomial cannot be computed in polynomial time (unless \( P = NP \)), but we can compute this point \( \mod r^n \).

The Tutte polynomial is also NP-hard to compute but one can evaluate it at certain points in poly-time. For example, evaluating at \((2, 1)\) gives the number of spanning forests.
A contraction/deletion formula

Deletion
A contraction/deletion formula

Deletion

\[ \begin{align*}
\text{in } G^\alpha: & \quad \begin{array}{c}
1 \quad 1 \\
2 \quad 2 \\
3 \quad 3 \\
4 \quad 4
\end{array} \\
\end{align*} \]
A contraction/deletion formula

Deletion

in $G^\alpha$:

\[
\begin{array}{cccc}
1 & \rightarrow & 1 \\
2 & \times & 2 \\
3 & \times & 3 \\
4 & \rightarrow & 4
\end{array}
\]
A contraction/deletion formula

Deletion

\[ e \]

in \( G^\alpha \):

\[
\begin{array}{ccc}
1 & \hspace{1cm} & 1 \\
2 & \hspace{1cm} \times & 2 \\
3 & \hspace{1cm} \times & 3 \\
4 & \hspace{1cm} & 4 \\
\end{array}
\]

\[ \text{in } G^\alpha \setminus e \]

\[
\begin{array}{ccc}
1 & \hspace{1cm} & 1 \\
2 & \hspace{1cm} & 2 \\
3 & \hspace{1cm} & 3 \\
4 & \hspace{1cm} & 4 \\
\end{array}
\]
A contraction/deletion formula

Deletion in $G^\alpha$:

\begin{align*}
1 & \quad \quad \quad 1 \\
2 & \quad \quad \quad 2 \\
3 & \quad \quad \quad 3 \\
4 & \quad \quad \quad 4 \\
\end{align*}

Contraction in $G^\alpha \setminus e$:

\begin{align*}
1 & \quad \quad \quad 1 \\
2 & \quad \quad \quad 2 \\
3 & \quad \quad \quad 3 \\
4 & \quad \quad \quad 4 \\
\end{align*}
A contraction/deletion formula

Deletion

\[ e \]

in \( G^\alpha \):

\[
\begin{array}{ccc}
1 & 1 \\
2 & 2 \\
3 & 3 \\
4 & 4 \\
\end{array}
\]

\[ \in G^\alpha \setminus e \]

Contraction

\[ e \]

in \( G^\alpha \):

\[
\begin{array}{ccc}
& 1 & \\
2 & 2 \\
3 & 3 \\
4 & 4 \\
\end{array}
\]

\[ \in G^\alpha \setminus e \]
A contraction/deletion formula

Deletion

\[ \begin{array}{c}
\text{in } G^\alpha: \\
1 & \quad 1 \\
2 & \times \quad 2 \\
3 & \times \quad 3 \\
4 & \quad 4
\end{array} \]

\[ \text{in } G^\alpha \setminus e: \\
1 & \quad 1 \\
2 & \quad 2 \\
3 & \quad 3 \\
4 & \quad 4
\]

Contraction

\[ \begin{array}{c}
\text{in } G^\alpha: \\
1 & \quad 1 \\
2 & \times \quad 2 \\
3 & \times \quad 3 \\
4 & \quad 4
\end{array} \]

\[ \text{in } G^\alpha \setminus e: \\
1 & \quad 1 \\
2 & \quad 3 \\
3 & \quad 2 \\
4 & \quad 4
\]
A contraction/deletion formula

Deletion

in $\mathcal{G}^\alpha$:

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\end{array}
\]

\[
\begin{array}{cccc}
e & & & \\
1 & 2 & 3 & 4 \\
\end{array}
\]

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\end{array}
\]

\[
\begin{array}{cccc}
\mathcal{G}^\alpha \setminus e \\
1 & 2 & 3 & 4 \\
\end{array}
\]

Contraction

in $\mathcal{G}^\alpha$:

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\end{array}
\]

\[
\begin{array}{cccc}
e & & & \\
1 & 2 & 3 & 4 \\
\end{array}
\]

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\end{array}
\]

\[
\begin{array}{cccc}
\mathcal{G}^\alpha / e \\
1 & 2 & 3 & 4 \\
\end{array}
\]
A contraction/deletion formula

Deletion

\begin{align*}
\text{in } G^\alpha: & \quad 2 \quad 3 \quad 4 \\
\text{in } G^\alpha \setminus e & \quad 2 \quad 3 \quad 4
\end{align*}

Contraction

\begin{align*}
\text{in } G^\alpha: & \quad 2 \quad 3 \quad 4 \\
\text{in } G^\alpha \setminus e & \quad 2 \quad 3 \quad 4
\end{align*}
A contraction/deletion formula

Deletion

\[ e \]

\begin{align*}
\text{in } G^\alpha: \quad & 1 \quad 1 \\
& 2 \quad 2 \\
& 3 \quad 3 \\
& 4 \quad 4 
\end{align*}

\[ e \quad \text{in } G^\alpha \setminus e \]

Contraction

\[ e \]

\begin{align*}
\text{in } G^\alpha: \quad & 1 \quad 1 \\
& 2 \quad 2 \\
& 3 \quad 3 \\
& 4 \quad 4 
\end{align*}

\[ \quad \leftrightarrow \quad \]

\begin{align*}
\text{in } G^\alpha: \quad & 1 \quad 1 \\
& 2 \quad 2 \\
& 3 \quad 3 \\
& 4 \quad 4 
\end{align*}
A contraction/deletion formula

Deletion

\[ e \]

in \( G^\alpha \):

\[
\begin{array}{cccc}
1 & - & - & 1 \\
2 & - & - & 2 \\
3 & - & - & 3 \\
4 & - & - & 4 \\
\end{array}
\]

\[ \rightarrow \]

in \( G^\alpha \setminus e \):

\[
\begin{array}{cccc}
1 & - & - & 1 \\
2 & - & - & 2 \\
3 & - & - & 3 \\
4 & - & - & 4 \\
\end{array}
\]

Contraction

\[ e \]

in \( G^\alpha \):

\[
\begin{array}{cccc}
1 & - & - & 1 \\
2 & - & - & 2 \\
3 & - & - & 3 \\
4 & - & - & 4 \\
\end{array}
\]

\[ \leftrightarrow \]

in \( G^\alpha / e \):

\[
\begin{array}{cccc}
1 & - & - & 1 \\
2 & - & - & 2 \\
3 & - & - & 3 \\
4 & - & - & 4 \\
\end{array}
\]
A Contraction/Deletion Formula

Theorem (Godsil, Guo, Royle)

\[ \xi(G^{\alpha}, t) = (t - 1)\xi(G^{\alpha}/e, t) + \xi(G^{\alpha} \setminus e, t) \]
Main result

Theorem (Godsil, Guo, Royle)

For any $r$-fold cover of a graph $G$ with $n$ vertices,

\[ \xi(G^\alpha, -(r - 1)) \equiv 0 \mod r^n. \]

Theorem (Godsil, Guo, Royle)

For any 2-fold cover of a graph $G$ with $n$ vertices,

\[ \xi(G^\alpha, 1) = \begin{cases} 2^n, & \text{if } G \text{ is Eulerian;} \\ 0, & \text{otherwise.} \end{cases} \]
Open problems

- Are there other points which we can “evaluate” the transversal polynomial?

- Are there classes of graphs for which we can compute the transversal polynomial in polynomial time?
  
  - For example, choosability is linear time in the number of vertices, for graphs of bounded tree-width.
Thanks!