On the generalized $\vartheta$-number and related problems for highly symmetric graphs

Lennart Sinjorgo
Joint work with Renata Sotirov

Tilburg University

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2 The sequence \((\vartheta_k(G))_{k \in \mathbb{N}}\)

3 Value of \(\vartheta_k(G)\) for specific graphs

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Graph coloring

- Let $G = (V, E)$ be a simple, undirected graph.
- A graph coloring of $G$ is an assignment of colors to the vertices of $G$ such that adjacent vertices are assigned different colors.
- A $k$-multicoloring of $G$ is an assignment of $k$-colors to each vertex of $G$ such that adjacent vertices are assigned disjoint sets of colors.

Figure: Standard graph coloring and multicoloring
Let $G$ be a graph:

- The chromatic number of $G$, $\chi(G)$, is defined as the minimum number of colors required to color the graph.
- The $k$-multichromatic number of $G$, $\chi_k(G)$, is defined as the minimum number of colors required for $k$-multicoloring the graph $G$. 
Independent sets

- An independent set is a set of pairwise non-adjacent vertices.
- The independence number of a graph, $\alpha(G)$, is the size of the largest independent set in $G$.

![Graph](image)

**Figure:** $\alpha(G) = 3$

- An independent set of $G$ is an induced subgraph of $G$, $G'$, satisfying $\chi(G') = 1$.
- The size of the largest induced subgraph of $G$, $G'$, satisfying $\chi(G') \leq k$ is denoted by $\alpha_k(G)$.
Lovász theta function

The LP relaxations for \( \alpha(G) \) and \( \chi(G) \) are rather weak.

Lovász\(^1\) introduced the semidefinite programming (SDP) relaxation, \( \vartheta(G) \), which satisfies the *sandwich inequality*

\[
\alpha(G) \leq \vartheta(G) \leq \chi(\overline{G}),
\]

where \( \overline{G} \) is the complement graph of \( G \).

Can \( \vartheta(G) \) be generalized to bound \( \alpha_k(G) \) and \( \chi_k(\overline{G}) \)?

The generalized $\vartheta$-number

- Narasimhan and Manber\textsuperscript{2} introduce $\vartheta_k(G)$, for $k \in \mathbb{N}$.
- Let $\lambda_1(A) \geq \lambda_2(A) \geq \ldots$ be the eigenvalues of a symmetric matrix $A$. We have

$$\vartheta(G) = \min_{A \in \mathcal{A}(G)} \lambda_1(A),$$

and

$$\vartheta_k(G) = \min_{A \in \mathcal{A}(G)} \sum_{i=1}^{k} \lambda_i(A),$$

for

$$\mathcal{A}(G) = \{ A \in \mathbb{R}^{n \times n} \mid A = A^\top, \text{ and } A_{ij} = 1 \text{ for } \{i, j\} \notin E \}. $$

- The graph parameter $\vartheta_k$ satisfies the generalized sandwich inequality

$$\alpha_k(G) \leq \vartheta_k(G) \leq \chi_k(\overline{G}).$$

The $\vartheta_k$-number in theory

The value $\vartheta_k(G)$ is the optimal value of an SDP. We can study it analytically. In particular, we study

- The sequence $(\vartheta_k(G))_{k \in \mathbb{N}}$.
- Values of $\vartheta_k(G)$ for specific graphs $G$.
- Bounds for $\vartheta_k(G_1 \ast G_2)$, in terms of $\vartheta_k(G_1)$ and $\vartheta_k(G_2)$, where $\ast$ denotes some graph product.
Introduction

The sequence \( \psi_k(G) \) \( k \in \mathbb{N} \)

Value of \( \psi_k(G) \) for specific graphs

A Nordhaus-Gaddum type inequality for \( \chi_k(G) \)
The sequence \((\varphi_k(G))_{k \in \mathbb{N}}\)

Let \(G\) be a graph on \(n\) vertices.

- The sequence \((\varphi_k(G))_k\) is increasing and bounded. That is,

\[
1 \leq \varphi_1(G) \leq \ldots \leq \varphi_n(G) = n.
\]

- Define \(\triangle_k(G) := \varphi_k(G) - \varphi_{k-1}(G)\), for \(k \geq 2\), and \(\triangle_1(G) := \varphi_1(G)\). Then

\[
\triangle_k(G) \geq \triangle_{k+1}(G)
\]
Lemma (L.S. and R. Sotirov)

\[ 1 \leq \vartheta_1(G) \leq \ldots \leq \vartheta_n(G) = n. \]

Proof sketch:
We have for \( G = (V, E), |V| = n \), (dual of previous formulation):

\[
\vartheta_k(G) = \max_{Y \in \mathbb{R}^{n \times n}} \langle J, Y \rangle \\
\text{s.t.} \quad 0 \preceq Y \preceq I, \ Y_{ij} = 0 \ \forall (i, j) \in E, \ Tr(Y) = k.
\]

Let \( Y \) be optimal for the above SDP and \( k < n \). Take

\[
Z := \left(1 - \frac{1}{n-k}\right)Y + \frac{1}{n-k}I.
\]

Matrix \( Z \) is feasible for the SDP defining \( \vartheta_{k+1} \) and thus

\[
\vartheta_{k+1}(G) \geq \langle J, Z \rangle \geq \langle J, Y \rangle = \vartheta_k(G).
\]
Define for $k \geq 2$, $\triangle_k(G) := \vartheta_k(G) - \vartheta_{k-1}(G)$, and $\triangle_1(G) := \vartheta_1(G)$.

Theorem (L.S. and R. Sotirov)

$\triangle_k(G) \geq \triangle_{k+1}(G)$. 
Limiting behaviour of $\triangle_k(G)$

Recall that

$$\triangle_k(G) := \vartheta_k(G) - \vartheta_{k-1}(G).$$

- For strictly positive values of $\triangle_k(G)$, is there a lower bound?
- Can we bound $\triangle_{k+1}(G)/\triangle_k(G)$? (assuming that both $\triangle_{k+1}(G)$ and $\triangle_k(G)$ differ from 0.)
Theorem (L.S. and R. Sotirov)
For all $\varepsilon > 0$, there exists a graph $G$ and an integer $k$, such that

$$0 < \triangle_k(G) < \varepsilon$$

Proof sketch:
Consider the complete graph on $n - 1$ vertices plus a vertex connected by a single edge, denoted $\mathcal{H}_n$:

(The figure shows $\mathcal{H}_9$.) We computed explicitly

$$\vartheta_{n-2}(\mathcal{H}_n) = n - 2 + \frac{2}{n-3} \sqrt{(n-2)(n-4)} < n.$$  

Then $\lim_{n \to \infty} \vartheta_{n-2}(\mathcal{H}_n) - n = 0$. As $\vartheta_n(\mathcal{H}_n) = n$, this proves the theorem.

\[\square\]
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Cycle graphs

The chromatic number of a cycle graph follows

$$\chi(C_n) = \begin{cases} 2 & \text{for even } n, \\ 3 & \text{for odd } n. \end{cases}$$

By Lovász,

$$\vartheta(C_n) = \begin{cases} n/2 & \text{for even } n, \\ \frac{n \cos(\pi/n)}{1+\cos(\pi/n)} & \text{for odd } n. \end{cases}$$
For $k$-multicoloring cycle graphs, we have

$$\chi_k(C_n) = \begin{cases} 2k \\
2k + 1 + \left\lfloor \frac{k-1}{(n-1)/2} \right\rfloor \end{cases}$$

for even $n$ (simple proof),

and

$$\chi_k(\overline{C_n}) = \left\lfloor kn/2 \right\rfloor,$$

see Campêlo et al.$^4$

We derived

$$\vartheta_2(C_n) = 2\vartheta(C_n) = \begin{cases} n \\
\frac{2n \cos (\pi/n)}{1+\cos (\pi/n)} \end{cases}$$

for even $n$,

for odd $n$.

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The Kneser graph

The Kneser graph $K(n, m)$, for integers $1 \leq m \leq n/2$, has as vertices all the $m$-sized subsets of $\{1, 2, \ldots, n\}$. Two vertices are adjacent if their intersection is empty.

We show that

$$\vartheta_k(K(n, m)) = \min \left\{ k \binom{n-1}{m-1}, \binom{n}{m} \right\} = \min \left\{ k \vartheta(G), \binom{n}{m} \right\}$$

Figure: Graph $K(5, 2)$
The Johnson graph

The Johnson graph $J(n, m, f)$, for integers $1 \leq m \leq n/2$, $0 \leq f < m$, has as vertices all the $m$-sized subsets of $\{1, 2, \ldots, n\}$. Two vertices are adjacent if they have $f$ common elements. Note that $K(n, m) = J(n, m, 0)$.

![Graph J(5, 2, 1)](image)

**Figure:** Graph $J(5, 2, 1)$

We show that

$$\vartheta_k(J(n, m, f)) = \min \left\{ k\vartheta(J(n, m, f)), \binom{n}{m} \right\}.$$
Introduction

The sequence $(\vartheta_k(G))_{k \in \mathbb{N}}$

Value of $\vartheta_k(G)$ for specific graphs

A Nordhaus-Gaddum type inequality for $\chi_k(G)$
### Multichromatic number bounds

#### Nordhaus-Gaddum inequality\(^5\)

For any graph \(G\) of order \(n\), we have

\[
2 \sqrt{n} \leq \chi(G) + \chi(\overline{G}) \leq n + 1,
\]

\[
n \leq \chi(G) \chi(\overline{G}) \leq \left(\frac{n + 1}{2}\right)^2.
\]

#### Theorem (L.S. and R. Sotirov)

For any graph \(G\) of order \(n\), we have

\[
2k \sqrt{n} \leq \chi_k(G) + \chi_k(\overline{G}) \leq k(n + 1),
\]

\[
k^2 n \leq \chi_k(G) \chi_k(\overline{G}) \leq k^2 \left(\frac{n + 1}{2}\right)^2.
\]

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Theorem (L.S. and R. Sotirov)

For any graph $G$ of order $n$, we have

\[ 2k\sqrt{n} \leq \chi_k(G) + \chi_k(\overline{G}) \leq k(n+1), \]

\[ k^2 n \leq \chi_k(G)\chi_k(\overline{G}) \leq k^2 \left( \frac{n+1}{2} \right)^2. \]

Proof sketch:

For any graph and any integer $k$, we have $\chi_k(G) \leq k\chi(G)$ (Stahl\textsuperscript{7}). Then

\[ \chi_k(G)\chi_k(\overline{G}) \leq k^2 \chi(G)\chi(\overline{G}) \leq k^2 \left( \frac{n+1}{2} \right), \]

and similarly for the upper bound on $\chi_k(G) + \chi_k(\overline{G})$. The lower bounds follow along the same lines of the original proof, adapted for multicoloring.

Discussed topics

- The sequence \((\vartheta_k(G))_{k \in \mathbb{N}}\).
- Values of \(\vartheta_k(G)\) for specific graphs, and a general bound on \(\vartheta_k(G)\).
- A Nordhaus-Gaddum type inequality for the multichromatic number \(\chi_k(G)\).
- More results on Hamming graphs, strongly regular graphs and orthogonality graphs in the paper.\(^8\)


Appendix A: Proof of $\alpha_k(G) \leq \vartheta_k(G)$

For any symmetric matrix $A$, with eigenvalues $\lambda_1(A) \geq \lambda_2(A) \geq \ldots$, we have

$$\sum_{i=1}^{k} \lambda_i(A) = \max_{X \in \mathbb{R}^{n \times k}} \{ \langle A, XX^\top \rangle, \text{ s.t. } X^\top X = I \}.$$ 

This is a theorem by Fan, and follows from the Courant–Fischer–Weyl min-max principle. Let $G$ be a graph of order $n$ and recall

$$\mathcal{A}(G) = \{ A \in \mathbb{R}^{n \times n} | A = A^\top, \text{ and } A_{ij} = 1 \text{ for } \{i, j\} \notin E \}.$$ 

Let $A \in \mathcal{A}(G)$. There exists a matrix $Z \in \mathbb{R}^{n \times k}$ such that $Z^\top Z = I$ and $\langle A, ZZ^\top \rangle = \alpha_k(G)$ (details omitted). Then

$$\alpha_k(G) = \langle A, ZZ^\top \rangle \leq \max_{X \in \mathbb{R}^{n \times k}} \{ \langle A, XX^\top \rangle, \text{ s.t. } X^\top X = I \} = \sum_{i=1}^{k} \lambda_i(A).$$
Appendix B: Proving the tightness

Consider the complete multipartite graph $K_{3,3,3}$ on $n = 9$ vertices. Clearly, $\chi(K_{3,3,3}) = 3$ and it can be shown that $\chi_k(K_{3,3,3}) = 3k$.
Appendix B: Proving the tightness (cont.)

Consider the complete multipartite graph $K_{3,3,3}$ on $n = 9$ vertices. Clearly, $\chi(K_{3,3,3}) = 3$ and it can be shown that $\chi_k(K_{3,3,3}) = 3k$. Moreover, we have that $\chi_k(K_{3,3,3}) = 3k$. 

Graph $K_{3,3,3}$

Graph $\overline{K_{3,3,3}}$
Appendix B: Proving the tightness (cont.)

Theorem (L.S. and R. Sotirov)

For any graph \( G \) of order \( n \), we have

\[
2k\sqrt{n} \leq \chi_k(G) + \chi_k(\overline{G}) \leq k(n + 1),
\]

\[
k^2n \leq \chi_k(G)\chi_k(\overline{G}) \leq k^2\left(\frac{n + 1}{2}\right)^2.
\]

Let \( G = K_{3,3,3} \), on \( n = 9 \) vertices. We have \( \chi(G) = 3k \) and \( \chi_k(\overline{G}) = 3k \).
Then

\[
\chi_k(G) + \chi_k(\overline{G}) = 6k = 2k\sqrt{n},
\]

\[
\chi_k(G)\chi_k(\overline{G}) = 9k^2 = k^2n.
\]

Thus \( G \) attains both lower bounds.