On the generalized ϑ-number and related problems for highly symmetric graphs

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Graph coloring

- Let $G = (V, E)$ be a simple, undirected graph.
- A graph coloring of G is an assignment of colors to the vertices of G such that adjacent vertices are assigned different colors.
- A k-multicoloring of G is an assignment of k-colors to each vertex of G such that adjacent vertices are assigned disjoint sets of colors.

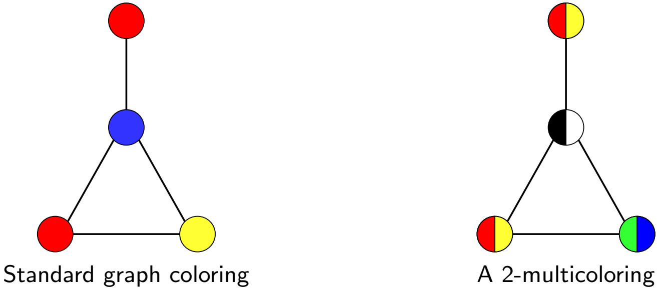

Figure: Standard graph coloring and multicoloring

Multichromatic number

Let G be a graph:

- The chromatic number of G, $\chi(G)$, is defined as the minimum number of colors required to color the graph.
- The k-multichromatic number of G, $\chi_k(G)$, is defined as the minimum number of colors required for k-multicoloring the graph G.

Independent sets

- An independent set is a set of pairwise non-adjacent vertices.
- The independence number of a graph, $\alpha(G)$, is the size of the largest independent set in G.

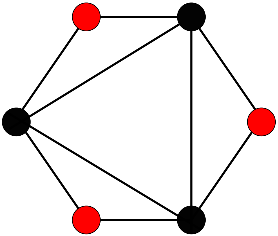

Figure: $\alpha(G) = 3$

- An independent set of G is an induced subgraph of G, G', satisfying $\chi(G') = 1$.
- The size of the largest induced subgraph of G, G', satisfying $\chi(G') \leq k$ is denoted by $\alpha_k(G)$.

Lovász theta function

- The LP relaxations for $\alpha(G)$ and $\chi(G)$ are rather weak.
- Lovász[1] introduced the semidefinite programming (SDP) relaxation, $\vartheta(G)$, which satisfies the *sandwich inequality*

$$\alpha(G) \leq \vartheta(G) \leq \chi(\overline{G}),$$

  where $\overline{G}$ is the complement graph of G.
- Can $\vartheta(G)$ be generalized to bound $\alpha_k(G)$ and $\chi_k(\overline{G})$?

[1] L. Lovász. "On the Shannon capacity of a graph." *IEEE Transactions on Information theory* 25.1 (1979): 1-7.

The generalized ϑ-number

- Narasimhan and Manber[2] introduce $\vartheta_k(G)$, for $k \in \mathbb{N}$.
- Let $\lambda_1(A) \geq \lambda_2(A) \geq \dots$ be the eigenvalues of a symmetric matrix A. We have

$$\vartheta(G) = \min_{A \in \mathcal{A}(G)} \lambda_1(A),$$

and

$$\vartheta_k(G) = \min_{A \in \mathcal{A}(G)} \sum_{i=1}^{k} \lambda_i(A),$$

for

$$\mathcal{A}(G) = \{A \in \mathbb{R}^{n \times n} \,|\, A = A^\top, \text{ and } A_{ij} = 1 \text{ for } \{i,j\} \notin E\}.$$

- The graph parameter ϑ_k satisfies the *generalized sandwich inequality*

$$\alpha_k(G) \leq \vartheta_k(G) \leq \chi_k(\overline{G}).$$

[2] G. Narasimhan, and R. Manber. *A generalization of Lovász's sandwich theorem*. University of Wisconsin-Madison Department of Computer Sciences, 1988.

The ϑ_k-number in theory

The value $\vartheta_k(G)$ is the optimal value of an SDP. We can study it analytically. In particular, we study

- The sequence $(\vartheta_k(G))_{k\in\mathbb{N}}$.
- Values of $\vartheta_k(G)$ for specific graphs G.
- Bounds for $\vartheta_k(G_1 * G_2)$, in terms of $\vartheta_k(G_1)$ and $\vartheta_k(G_2)$, where $*$ denotes some graph product.

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The sequence $(\vartheta_k(G))_{k\in\mathbb{N}}$

Let G be a graph on n vertices.

- The sequence $(\vartheta_k(G))_k$ is increasing and bounded. That is,

$$1 \leq \vartheta_1(G) \leq \ldots \leq \vartheta_n(G) = n.$$

- Define $\triangle_k(G) := \vartheta_k(G) - \vartheta_{k-1}(G)$, for $k \geq 2$, and $\triangle_1(G) := \vartheta_1(G)$. Then

$$\triangle_k(G) \geq \triangle_{k+1}(G)$$

Lemma (L.S. and R. Sotirov)

$$1 \leq \vartheta_1(G) \leq \ldots \leq \vartheta_n(G) = n.$$

Proof sketch:
We have for $G = (V, E)$, $|V| = n$, (dual of previous formulation):

$$\vartheta_k(G) = \max_{Y \in \mathbb{R}^{n \times n}} \quad \langle J, Y \rangle$$
$$\text{s.t.} \quad 0 \preceq Y \preceq I, \; Y_{ij} = 0 \; \forall (i,j) \in E, \operatorname{Tr}(Y) = k.$$

Let Y be optimal for the above SDP and $k < n$. Take

$$Z := \left(1 - \frac{1}{n-k}\right) Y + \frac{1}{n-k} I.$$

Matrix Z is feasible for the SDP defining ϑ_{k+1} and thus

$$\vartheta_{k+1}(G) \geq \langle J, Z \rangle \geq \langle J, Y \rangle = \vartheta_k(G).$$

□

Define for $k \geq 2$, $\triangle_k(G) := \vartheta_k(G) - \vartheta_{k-1}(G)$, *and* $\triangle_1(G) := \vartheta_1(G)$.

Theorem (L.S. and R. Sotirov)

$$\triangle_k(G) \geq \triangle_{k+1}(G).$$

Limiting behaviour of $\triangle_k(G)$

Recall that

$$\triangle_k(G) := \vartheta_k(G) - \vartheta_{k-1}(G).$$

- For strictly positive values of $\triangle_k(G)$, is there a lower bound?
- Can we bound $\triangle_{k+1}(G)/\triangle_k(G)$? (assuming that both $\triangle_{k+1}(G)$ and $\triangle_k(G)$ differ from 0.)

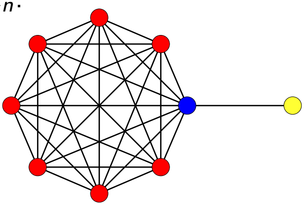
Theorem (L.S. and R. Sotirov)

For all $\varepsilon > 0$, there exists a graph G and an integer k, such that

$$0 < \triangle_k(G) < \varepsilon$$

Proof sketch:

Consider the complete graph on $n-1$ vertices plus a vertex connected by a single edge, denoted $\mathcal{H}_n$:



(The figure shows $\mathcal{H}_9$.) We computed explicitly

$$\vartheta_{n-2}(\mathcal{H}_n) = n - 2 + \frac{2}{n-3}\sqrt{(n-2)(n-4)} < n.$$

Then $\lim_{n\to\infty} \vartheta_{n-2}(\mathcal{H}_n) - n = 0$. As $\vartheta_n(\mathcal{H}_n) = n$, this proves the theorem. □

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Cycle graphs

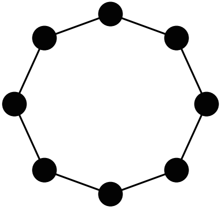

Figure: Cycle graph C_8

- The chromatic number of a cycle graph follows

$$\chi(C_n) = \begin{cases} 2 & \text{for even } n, \\ 3 & \text{for odd } n. \end{cases}$$

- By Lovász,

$$\vartheta(C_n) = \begin{cases} n/2 & \text{for even } n, \\ \frac{n\cos(\pi/n)}{1+\cos(\pi/n)} & \text{for odd } n. \end{cases}$$

Cycle graphs (cont.)

For k-multicoloring cycle graphs, we have

$$\chi_k(C_n) = \begin{cases} 2k & \text{for even } n \text{ (simple proof)}, \\ 2k+1+\left\lfloor \frac{k-1}{(n-1)/2} \right\rfloor & \text{for odd } n>1 \text{ (proof by Stahl).}^3 \end{cases}$$

and

$$\chi_k(\overline{C_n}) = \lfloor kn/2 \rfloor, \text{ see Campêlo et al.}^4$$

We derived

$$\vartheta_2(C_n) = 2\vartheta(C_n) = \begin{cases} n & \text{for even } n, \\ \frac{2n\cos(\pi/n)}{1+\cos(\pi/n)} & \text{for odd } n. \end{cases}$$

[3] S. Stahl. "n-Tuple colorings and associated graphs." *Journal of Combinatorial Theory, Series B* 20.2 (1976): 185-203.

[4] M. Campêlo, R.C. Corrêa, P.F. Moura, and M.C. Santos. "On optimal k-fold colorings of webs and antiwebs." *Discrete Applied Mathematics* 161.1-2 (2013): 60-70.

The Kneser graph

The Kneser graph $K(n, m)$, for integers $1 \leq m \leq n/2$, has as vertices all the m-sized subsets of $\{1, 2, \ldots, n\}$. Two vertices are adjacent if their intersection is empty.

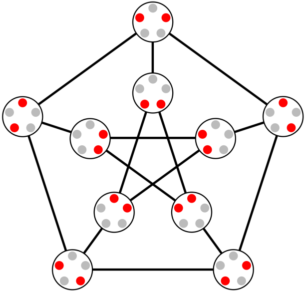

Figure: Graph $K(5, 2)$

We show that

$$\vartheta_k(K(n,m)) = \min\left\{k\binom{n-1}{m-1}, \binom{n}{m}\right\} = \min\left\{k\vartheta(G), \binom{n}{m}\right\}$$

The Johnson graph

The Johnson graph $J(n, m, f)$, for integers $1 \leq m \leq n/2$, $0 \leq f < m$, has as vertices all the m-sized subsets of $\{1, 2, \ldots, n\}$. Two vertices are adjacent if they have f common elements. Note that $K(n, m) = J(n, m, 0)$.

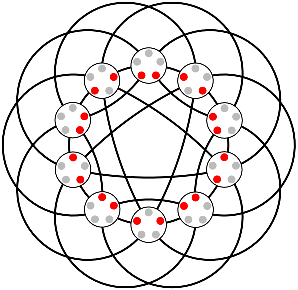

Figure: Graph $J(5, 2, 1)$

We show that

$$\vartheta_k(J(n, m, f)) = \min\left\{k\vartheta(J(n, m, f)), \binom{n}{m}\right\}.$$

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Multichromatic number bounds

Nordhaus-Gaddum inequality[5]

For any graph G of order n, we have

$$2\sqrt{n} \leq \chi(G) + \chi(\overline{G}) \leq n + 1,$$
$$n \leq \chi(G)\chi(\overline{G}) \leq \left(\frac{n+1}{2}\right)^2.$$

Theorem (L.S. and R. Sotirov)

For any graph G of order n, we have

$$2k\sqrt{n} \leq \chi_k(G) + \chi_k(\overline{G}) \leq k(n + 1), \text{ [6]}$$
$$k^2 n \leq \chi_k(G)\chi_k(\overline{G}) \leq k^2\left(\frac{n+1}{2}\right)^2.$$

[5] E.A. Nordhaus, and J. W. Gaddum. "On complementary graphs." *The American Mathematical Monthly* 63.3 (1956): 175-177.

[6] Also in: R.C. Brigham, and R.D. Dutton. "Generalized k-tuple colorings of cycles and other graphs." *Journal of Combinatorial Theory, Series B* 32.1 (1982): 90-94

Theorem (L.S. and R. Sotirov)

For any graph G of order n, we have

$$2k\sqrt{n} \leq \chi_k(G) + \chi_k(\overline{G}) \leq k(n+1),$$
$$k^2 n \leq \chi_k(G)\chi_k(\overline{G}) \leq k^2\left(\frac{n+1}{2}\right)^2.$$

Proof sketch:
For any graph and any integer k, we have $\chi_k(G) \leq k\chi(G)$ (Stahl[7]). Then

$$\chi_k(G)\chi_k(\overline{G}) \leq k^2\chi(G)\chi(\overline{G}) \leq k^2\left(\frac{n+1}{2}\right),$$

and similarly for the upper bound on $\chi_k(G) + \chi_k(\overline{G})$. The lower bounds follow along the same lines of the original proof, adapted for multicoloring.

[7] S. Stahl. "n-Tuple colorings and associated graphs." *Journal of Combinatorial Theory, Series B* 20.2 (1976): 185-203.

Discussed topics

- The sequence $(\vartheta_k(G))_{k\in\mathbb{N}}$.
- Values of $\vartheta_k(G)$ for specific graphs, and a general bound on $\vartheta_k(G)$.
- A Nordhaus-Gaddum type inequality for the multichromatic number $\chi_k(G)$.
- More results on Hamming graphs, strongly regular graphs and orthogonality graphs in the paper.[8]

[8] L. Sinjorgo, and R. Sotirov. "On the generalized ϑ-number and related problems for highly symmetric graphs." *SIAM Journal on Optimization* 32.2 (2022): 1344-1378.

References I

[1] Manoel Campêlo et al. "On optimal k-fold colorings of webs and antiwebs". In: *Discrete Applied Mathematics* 161.1-2 (2013), pp. 60–70.

[2] Ky Fan. "On a theorem of Weyl concerning eigenvalues of linear transformations I". In: *Proceedings of the National Academy of Sciences* 35.11 (1949), pp. 652–655.

[3] László Lovász. "On the Shannon capacity of a graph". In: *IEEE Transactions on Information theory* 25.1 (1979), pp. 1–7.

[4] Giri Narasimhan and Rachel Manber. *A generalization of Lovász's sandwich theorem*. Tech. rep. University of Wisconsin-Madison Department of Computer Sciences, 1988.

[5] Saul Stahl. "n-Tuple colorings and associated graphs". In: *Journal of Combinatorial Theory, Series B* 20.2 (1976), pp. 185–203.

Appendix A: Proof of $\alpha_k(G) \le \vartheta_k(G)$

For any symmetric matrix A, with eigenvalues $\lambda_1(A) \ge \lambda_2(A) \ge \ldots$, we have

$$\sum_{i=1}^{k} \lambda_i(A) = \max_{X \in \mathbb{R}^{n \times k}} \{\langle A, XX^\top \rangle, \text{ s.t. } X^\top X = I\}.$$

This is a theorem by Fan, and follows from the *Courant–Fischer–Weyl min-max principle*. Let G be a graph of order n and recall

$$\mathcal{A}(G) = \{A \in \mathbb{R}^{n \times n} \mid A = A^\top, \text{ and } A_{ij} = 1 \text{ for } \{i,j\} \notin E\}.$$

Let $A \in \mathcal{A}(G)$. There exists a matrix $Z \in \mathbb{R}^{n \times k}$ such that $Z^\top Z = I$ and $\langle A, ZZ^\top \rangle = \alpha_k(G)$ (details omitted). Then

$$\alpha_k(G) = \langle A, ZZ^\top \rangle \le \max_{X \in \mathbb{R}^{n \times k}} \{\langle A, XX^\top \rangle, \text{ s.t. } X^\top X = I\} = \sum_{i=1}^{k} \lambda_i(A).$$

Appendix B: Proving the tightness

Consider the complete multipartite graph $K_{3,3,3}$ on $n = 9$ vertices. Clearly, $\chi(K_{3,3,3}) = 3$ and it can be shown that $\chi_k(K_{3,3,3}) = 3k$.

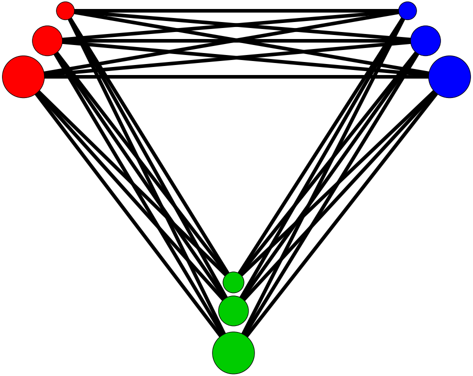
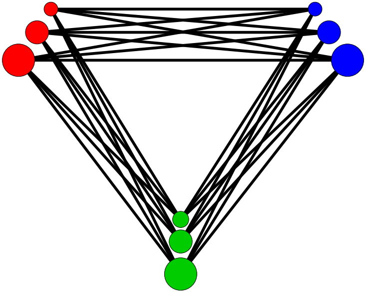


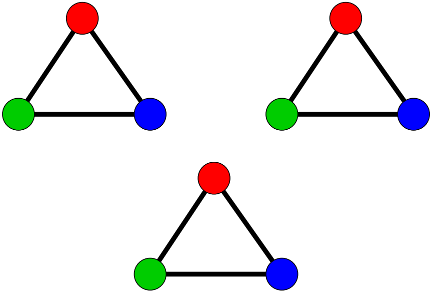
Figure: Graph $K_{3,3,3}$

Appendix B: Proving the tightness (cont.)

Consider the complete multipartite graph $K_{3,3,3}$ on $n = 9$ vertices. Clearly, $\chi(K_{3,3,3}) = 3$ and it can be shown that $\chi_k(K_{3,3,3}) = 3k$. Moreover, we have that $\chi_k(\overline{K_{3,3,3}}) = 3k$.



Graph $K_{3,3,3}$



Graph $\overline{K_{3,3,3}}$

Appendix B: Proving the tightness (cont.)

Theorem (L.S. and R. Sotirov)

For any graph G of order n, we have

$$2k\sqrt{n} \leq \chi_k(G) + \chi_k(\overline{G}) \leq k(n+1),$$
$$k^2 n \leq \chi_k(G)\chi_k(\overline{G}) \leq k^2\left(\frac{n+1}{2}\right)^2.$$

Let $G = K_{3,3,3}$, on $n = 9$ vertices. We have $\chi(G) = 3k$ and $\chi_k(\overline{G}) = 3k$. Then

$$\chi_k(G) + \chi_k(\overline{G}) = 6k = 2k\sqrt{n},$$
$$\chi_k(G)\chi_k(\overline{G}) = 9k^2 = k^2 n.$$

Thus G attains both lower bounds.