Eigenvalue bounds for the independence and chromatic number of graph powers

Aida Abiad

Eindhoven University of Technology
Ghent University
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Workshop on Semidefinite and Polynomial Optimization
CWI, 30 August 2022

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Outline

1. Background
2. Eigenvalue bounds: an overview
3. New inertia and ratio-type bounds and optimization
4. Relating the inertia and the ratio-type bounds
5. Concluding remarks

Background

The beginning

Adjacency matrix and closed walks

Adjacency matrix $A = (a_{ij})$

Power adjacency matrix $A^k = (a_{ij}^k)$

$$a_{ij}^k = \# \text{ walks of length k from i to j}$$

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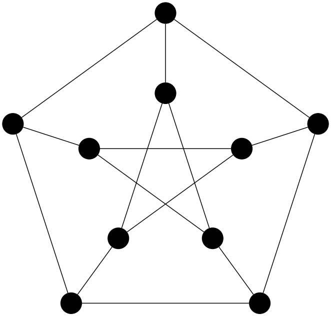
algebraic combinatorics

Eigenvalues

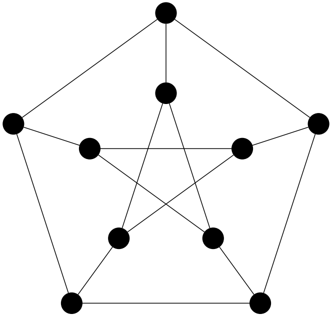
Spectrum: $\lambda_1 \geq \cdots \geq \lambda_n$



$$3, 1, 1, 1, 1, 1, -2, -2, -2, -2$$

Eigenvalues

Spectrum: $\{\theta_0^{m_0}, \ldots, \theta_d^{m_d}\}$



$3^1, 1^5, -2^4$

Walk-regularity and k-partially walk-regularity

- A graph G is **walk-regular** if the number of closed walks of any length from a vertex to itself does not depend on the choice of the vertex (Godsil and McKay 1980).

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  If G is k-partially walk-regular, for any polynomial $p \in \mathbb{R}_k[x]$, the diagonal of $p(A)$ is constant with entries

$$(p(A))_{uu} = \frac{1}{n} \operatorname{tr} p(A) = \frac{1}{n} \sum_{i=1}^{n} p(\lambda_i) \quad \text{for all } u \in V.$$

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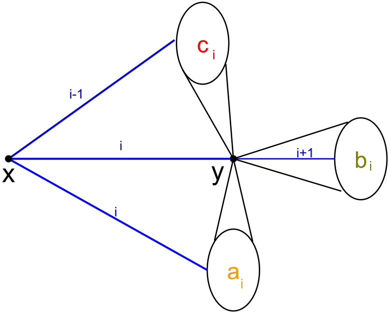
  Every graph is k-partially walk-regular for $k = 0, 1$, and every regular graph is 2-partially walk-regular.

- G is k-partially walk-regular for any k iff G is walk-regular.

Distance-regularity and k-partially distance-regularity

- A graph G is **distance-regular** if there are constants c_i, a_i, b_i such that for all $i = 0, 1, \ldots, D$, and all vertices x and y at distance $i = d(x, y)$, among the neighbors of y, there are c_i at distance $i-1$ from x, a_i at distance i, and b_i at distance $i+1$.



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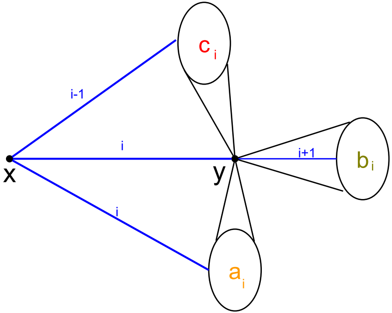

- G is k-**partially distance-regular** if it is distance-regular up to distance k.

Graph powers

The k^{th} **power of a graph** $G = (V, E)$, denoted by G^k, is formed by connecting two vertices if they are at distance at most k.

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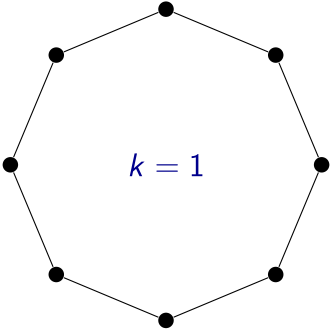

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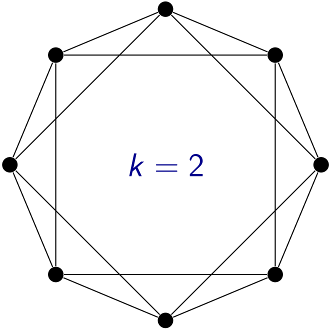

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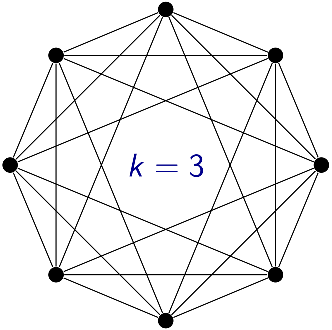

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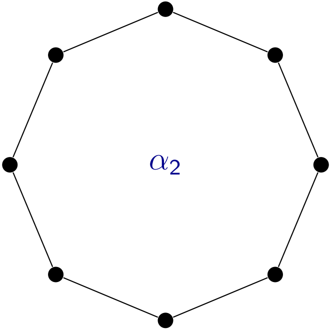

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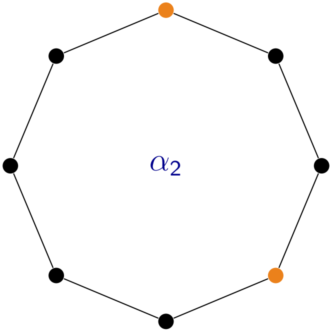

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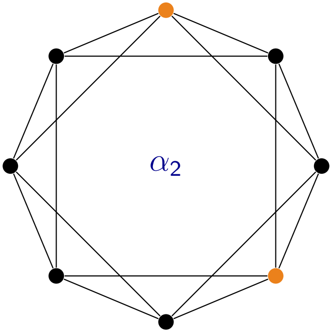

Note: $\alpha_k(G) = \alpha(G^k)$

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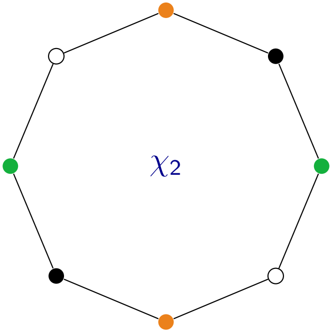

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Upper bounds on α_k give lower bounds on χ_k and vice versa:

$$\chi(G) \geq \frac{|V(G)|}{\alpha(G)}$$

Applications α_k

- Coding theory: codes relate to k-independent sets in Hamming graphs, bounds on α_k used to show the non-existence of perfect codes (Fiol 2020), ...

SPCodingSchool

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  We use MILPs to compute when the new inertial-type bound is tight.
- Related to other graph parameters: α_k has been used to obtain tight lower bounds for the average distance (Firby and Haviland 1997), ...

Note that ...

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Examples:

- average degree (Devos, McDonald and Scheide 2013)
- rainbow number (Basavaraju, Chandran, Rajendraprasad and Ramaswamy 2014)
- eigenvalues
- ...

Motivation α_k

(Kong and Zhao 1993) Computing α_k and χ_k is NP-complete.

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No, in general the spectrum of G^k cannot be derived from G, and vice versa.

$$\Downarrow$$

We will find bounds that only depend on the spectrum of G.

Overall goal

Extend two classic eigenvalue bounds for α to α_k in terms of the eigenvalues of the original graph.

Main tool: interlacing

Let $m < n$.
Sequences $\lambda_1 \geq \cdots \geq \lambda_n$ and $\mu_1 \geq \cdots \geq \mu_m$ **interlace** if

$$\lambda_i \geq \mu_i \geq \lambda_{n-m+i} \qquad (1 \leq i \leq m)$$

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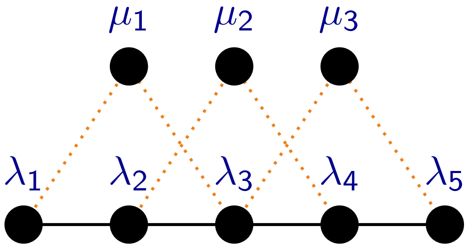

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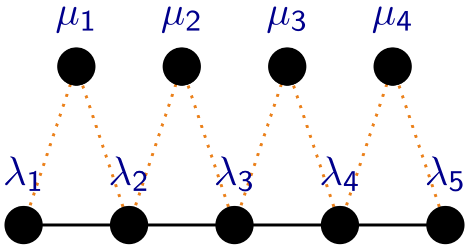

$m = n - 1$

Eigenvalue interlacing

$\lambda_1, \lambda_2, \ldots, \lambda_n$ eigenvalues of a matrix A

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$\mu_1, \mu_2, \ldots, \mu_m$ eigenvalues of a matrix B

First case of eigenvalue interlacing

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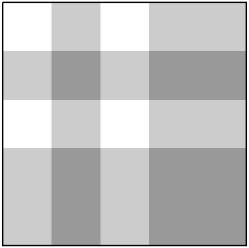
(Cauchy interlacing)
If B is a principal submatrix of A, then the eigenvalues of B interlace those of A.



Second case of eigenvalue interlacing

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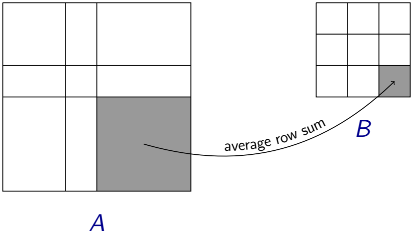
(Haemers interlacing 1995)
If B is the quotient matrix of a partition of A, then the eigenvalues of B interlace the eigenvalues of A.



Eigenvalue bounds: an overview

Classic eigenvalue bounds

Inertia bound (Cvetković 1972)

If G is a graph with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$, then

$$\alpha(G) \leq \min\{|i : \lambda_i \geq 0|, |i : \lambda_i \leq 0|\}.$$

Classic eigenvalue bounds

Ratio bound (Hoffman 1970)

If G is regular with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$, then

$$\alpha(G) \leq n \frac{-\lambda_n}{\lambda_1 - \lambda_n}$$

and if an independent set C meets this bound then every vertex not in C is adjacent to precisely $-\lambda_n$ vertices of C.

⚠ Delsarte proved the ratio bound for SRGs, later Hoffman extended it to regular graphs and Haemers to irregular graphs.

(Lovász 1979)

The Lovász theta number $\vartheta(G)$ is a lower bound for the Hoffman bound.

More on the ratio bound

ELSEVIER

Linear Algebra and its Applications
Volume 617, 15 May 2021, Pages 215-219

Hoffman's ratio bound

In memory of Alan J. Hoffman

Willem H. Haemers

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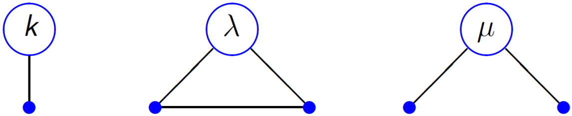
Abstract

Hoffman's ratio bound is an upper bound for the independence number of a regular graph in terms of the eigenvalues of the adjacency matrix. The bound has proved to be very useful and has been applied many times. Hoffman did not publish his result, and for a great number of users the emergence of Hoffman's bound is a black hole. With his note I hope to clarify the history of this bound and some of its generalizations.

Inertia vs ratio bound for some strongly regular graphs



Graph	(n, k, λ, μ)	α	Inertia bound	(Floor of) ratio bound
Cycle C_5	$(5, 2, 0, 1)$	2	2	2
Petersen	$(19, 3, 0, 1)$	4	4	4
Clebsh	$(16, 5, 0, 2)$	5	**5**	**6**
Hoffman-Singleton	$(50, 7, 0, 1)$	15	21	15
Gewirtz	$(56, 10, 0, 2)$	16	20	16
Mesner M_{22}	$(77, 16, 0, 7)$	21	21	21
Higman-Sims	$(100, 22, 0, 6)$	22	**22**	**26**

Some known upper bounds on α_k

- (Firby and Haviland 1997) For connected graphs using average distance.
- (Fiol 1997) For regular graphs using eigenvalues and alternating polynomials.
- (Beis, Duckworth and Zito 2005) For random r-regular graphs.
- (O, Shi and Taoqiu 2019) For r-regular graphs for every $k \geq 2$ and $r \geq 3$.
- (Jou, Lin and Lin 2020) For trees and $k = 2$.
- (Atkinson and Frieze 2003) For random graphs $G_{n,p}$, $p = d/n$ (d a large constant).

Optimization and eigenvalue bounds

Independence number:

- (Delsarte 1973) LP bound on α for distance-regular graphs.
- (Lovász 1979) SDP bound ϑ.
- ...

***k*-independence number:**

Ratio bound	Inertia bound
↓	↓
(Fiol 2019) LP with minor polynomials	?

Line of work

(1) (A., Cioabă and Tait 2016) New bounds on α_k in terms of λ_i^k.

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(3) (A., Coutinho, Fiol, Nogueira and Zeijlemaker 2022) Optimize the bounds over $p \in \mathbb{R}_k[x]$.

Optimization of the new eigenvalue bounds for the independence and chromatic number of graph powers

Joint work with G. Coutinho, M.A. Fiol, B. Nogueira and S. Zeijlemaker




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Graphs with large chromatic number

Question (Alon and Mohar 2000)

What is the largest possible value of the chromatic number $\chi(G^k)$ of G^k, among all graphs G with maximum degree at most d and girth at least g?

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What is the largest possible value of the chromatic number $\chi(G^k)$ of G^k, among all graphs G with maximum degree at most d and girth at least g?

- $k = 1$: long-standing problem by Vizing, settled asymptotically by (Johansson 1996) using the probabilistic method.
- $k = 2$: settled asymptotically by (Alon and Mohar 2002).
- $k \geq 3$: bounds by (Alon and Mohar 2002), (Kang and Pirot 2016), (Kang and Pirot 2018), ...

A lower bound on χ_k

Let $G = (V, E)$ be a graph with spectrum $\theta_0^{m_0}, \ldots, \theta_d^{m_d}$ and consider the inner product

$$\langle f, g \rangle_G = \frac{1}{n} \operatorname{tr}(f(A)g(A)) = \frac{1}{n} \sum_{i=0}^{d} m_i f(\theta_i) g(\theta_i).$$

The **predistance polynomials** $p_0, \ldots, p_d$ are orthogonal polynomials with respect to the above product, with $\operatorname{dgr} p_i = i$, and normalized such that $\|p_i\|_G^2 = p_i(\theta_0)$ (Fiol and Garriga 1997).

A lower bound on χ_k

Let $G = (V, E)$ be a graph with spectrum $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and predistance polynomials $p_0, \ldots, p_d$.
For a given integer $k \leq d$, consider the polynomial
$q_k = p_0 + \cdots + p_k$.

(Fiol 2012)

Let $s_k(u)$ be the number of vertices at distance at most k from u. Then $q_k(\lambda_1)$ is bounded above by

$$q_k(\lambda_1) \leq H_k = \frac{n}{\sum_{i \in V} \frac{1}{s_k(u)}}.$$

Equality occurs if and only if $q_k(A) = I + A(G^k)$.

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$\Rightarrow$ Spectrum of G and G^k are related.

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For a given integer $k \leq d$, consider the polynomial $q_k = p_0 + \cdots + p_k$.

(A., Coutinho, Fiol, Nogueira and Zeijlemaker 2022)

Let $q_k' = q_k - 1$. If G is regular with eigenvalues satisfying $q_k(\lambda_1) = H_k$, then

$$\chi_k \geq \frac{n}{\min\{|\{i : q_k'(\lambda_i) \geq 0\}|, |\{i : q_k'(\lambda_i) \leq 0\}|\}}$$

and

$$\chi_k \geq \frac{n}{1 - \frac{q_k'(\lambda_1)}{\min\{q_k'(\lambda_i)\}}}.$$

A lower bound on χ_k

First spectral bounds for Alon and Mohar question for regular graphs.

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First spectral bounds for Alon and Mohar question for regular graphs. But how do we find the polynomial $q_k = p_0 + \cdots + p_k$?

(A., van Dam and Fiol 2016)

$q_k(A) = A(G^k) + I$ when G is a δ-regular graph with girth g and $k = \lfloor \frac{g-1}{2} \rfloor$. In this case G is k-partially distance-regular, and

$$q_0 = 1,\ q_1 = 1 + x,\ q_{i+1} = xq_i - (\delta - 1)q_{i-1}.$$

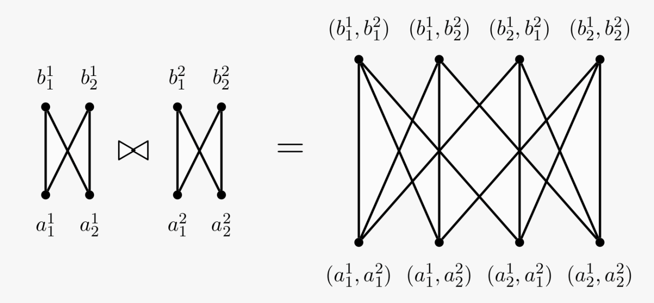
Tight examples

Our bound is tight for several named Sage graphs.

Name	Girth g	$k = \lfloor \frac{g-1}{2} \rfloor$	α_k
Moebius-Kantor graph	6	2	4
Nauru graph	6	2	6
Blanusa First Snark graph	5	2	4
Blanusa Second Snark graph	5	2	4
Brinkmann graph	5	2	3
Heawood graph	6	2	2
Sylvester graph	5	2	6
Coxeter graph	7	3	4
Dyck graph	6	2	8
F26A graph	6	2	6
Flower Snark graph	5	2	5

Tight examples

(Kang and Pirot 2016) used **balanced bipartite products** $\bowtie$ for their lower bound construction.



This product also gives several graphs which attain equality for our bound, for example the products of even cycles $C_8 \bowtie C_8$, $C_8 \bowtie C_{12}$, ...

The spectrum of G^k and G are not related.

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Optimization of inertial type bounds

Why optimization using MILPs?

(i) The quantum k-independence number is upper bounded by the inertial-type bound (A., Elphick and Wocjan 2022):

$$\alpha_k \leq \alpha_{kq} \leq \min\{|i : p(\lambda_i) \geq w(p)|, |i : p(\lambda_i) \leq W(p)|\}.$$

For $k > 1$ we can use the MILPs to compute values of the quantum parameter when the bound is tight. For $k = 1$:

$$\alpha \leq \alpha_q \leq \min\{|i : \lambda_i \geq 0|, |i : \lambda_i \leq 0|\}.$$

(ii) Closed formulas for small k.

(iii) Use the polynomials involved in the MILPs:
inertial-type bound (A., Coutinho, Fiol 2019)
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First inertial-type bound

Let G be a graph with adjacency matrix A and $p \in \mathbb{R}_k[x]$.

$$w(p) := \min_i \ p(A)_{ii}$$

$$W(p) := \max_i \ p(A)_{ii}$$

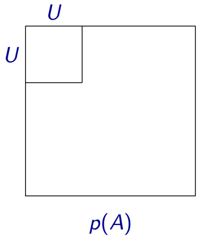
(A., Coutinho, Fiol 2019)

Let $p \in \mathbb{R}_k[x]$, then

$$\alpha_k(G) \leq \min\{|i : p(\lambda_i) \geq w(p)|, |i : p(\lambda_i) \leq W(p)|\}.$$

Proof sketch

Let U be a k-independent set of G with size α_k.



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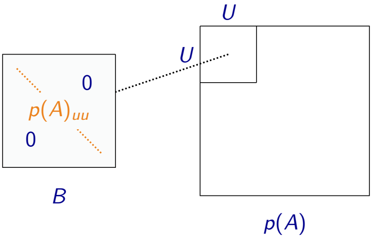

Let μ be the smallest eigenvalue of B.

- Cauchy interlacing ($\lambda_i \geq \mu_i$ for $i = 1, \ldots, m = |U|$): $\geq |U|$ eigenvalues of $p(A)$ are larger than μ
- $\mu \geq w(p)$ by definition of $w(p) = \min_{u \in V}\{(p(A))_{uu}$.

Therefore, $|U| \leq |\{i : p(\lambda_i) \geq w(p)\}|$.

First inertial-type bound: corollary

For $k = 1$,

First inertial-type bound: corollary

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Inertia bound (Cvetković 1972)

If G is a graph, then

$$\alpha(G) \leq \min\{|i : \lambda_i \geq 0|, |i : \lambda_i \leq 0|\}.$$

First inertial-type bound: optimization

$$\alpha_k(G) \leq \min\{|i : p_k(\lambda_i) \geq w(p_k)|, |i : p_k(\lambda_i) \leq W(p_k)|\}$$

Linear?

First inertial-type bound: optimization

$$\alpha_k(G) \leq \min\{|i : p_k(\lambda_i) \geq w(p_k)|, |i : p_k(\lambda_i) \leq W(p_k)|\}$$

Linear?

Invariant under scaling and translation

- may assume $\min\{|i : p_k(\lambda_i) \geq w(p_k)|\}$, otherwise take $-p_k$
- translate: $\min\{|i : p_k(\lambda_i) \geq 0|\}$.

First MILP $\alpha_k \leq \min\{|i : p_k(\lambda_i) \geq 0|\}$

Let G be a graph with spectrum $\{\theta_0^{m_0}, \ldots, \theta_d^{m_d}\}$ and $p_k(x) = a_k x^k + \cdots + a_0$ the polynomial to optimize.

For each $u \in V$, assume $w(p_k) = p_k(A)_{uu}$ and solve

$$
\begin{array}{rl}
\texttt{minimize} & \boldsymbol{m}^T \boldsymbol{b} \\
\texttt{subject to} & \sum_{i=0}^{k} a_i (A^i)_{vv} \geq 0, \quad v \in V(G) \backslash \{u\} \\
& \sum_{i=0}^{k} a_i (A^i)_{uu} = 0 \\
& \sum_{i=0}^{k} a_i {\theta_j}^i - M b_j + \varepsilon \leq 0, \quad j = 0, ..., d \\
& \boldsymbol{b} \in \{0,1\}^{d+1}
\end{array}
$$

with M large, $\varepsilon > 0$ small.

variables: $a_1, \ldots, a_k$, $(b_0, \ldots, b_d)$
parameters: k, $\{\theta_0^{m_0}, \ldots, \theta_d^{m_d}\}$

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Vector b encodes whether $p_k(\theta_i) \geq w(p_k)$: $b_i = 1$ iff $p_k(\theta_i) \geq 0$.

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Vector b encodes whether $p_k(\theta_i) \geq w(p_k)$: $b_i = 1$ iff $p_k(\theta_i) \geq 0$ (if $p_k(\theta_i) = 0$ then we need ε to force $b_i = 1$)

First MILP $\alpha_k \leq \min\{|i : p_k(\lambda_i) \geq 0|\}$

Let G be a graph with spectrum $\{\theta_0^{m_0}, \ldots, \theta_d^{m_d}\}$ and $p_k(x) = a_k x^k + \cdots + a_0$ the polynomial to optimize.

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with M large, $\varepsilon > 0$ small.

⚠Linear combination of the eigenvalues mutiplicities (minimizing the quantity of indices j)

First MILP: results

For large graphs, solving n MILPs takes a lot of time.
However, the first inertial-type bound does not require walk-regularity like in the optimization of the ratio-type bound (Fiol 2020).

Proportion of small irregular graphs for which the optimal solution of the MILP equals α_2:

Number of vertices	4	5	6	7	8	9
Proportion	0.86	0.84	0.76	0.62	0.46	0.27

First MILP: results

Name	Best 2019	ϑ_2	First MILP	α_2
Balaban 10-cage	17	17	19	17
Frucht graph	3	3	3	3
Meredith Graph	14	10	10	10
Moebius-Kantor Graph	4	4	6	4
Bidiakis cube	3	2	4	2
Gosset Graph	2	2	8	2
Gray graph	14	11	19	11
Nauru Graph	6	5	8	6
Blanusa First Snark Graph	4	4	4	4
Pappus Graph	4	3	7	3
Blanusa Second Snark Graph	4	4	4	4
Poussin Graph	-	2	4	2
Brinkmann graph	4	3	6	3
Harborth Graph	12	9	13	10
Perkel Graph	10	5	18	5
Harries Graph	17	17	18	17
Bucky Ball	16	12	16	12
Harries-Wong graph	17	17	18	17
Robertson Graph	3	3	5	3
Heawood graph	3	2	2	2
Herschel graph	-	2	3	2
Hoffman Graph	3	2	5	2
...				

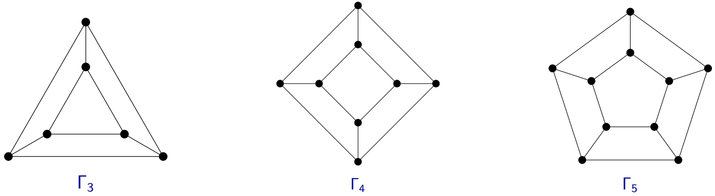
First inertial-type bound: walk-regular graphs

Let G be a k-partially walk-regular. Then $p_k(A)$ has constant diagonal, so we can simplify:

$$
\begin{array}{rl}
\texttt{minimize} & \boldsymbol{m}^T\boldsymbol{b} \\
\texttt{subject to} & \sout{\sum_{i=0}^{k} a_i (A^i)_{vv} \geq 0, \quad v \in V(G)\backslash\{u\}} \\
 & \sout{\sum_{i=0}^{k} a_i (A^i)_{uu} = 0} \\
 & \sum_{i=0}^{d} m_i p_k(\theta_i) = 0 \\
 & \sum_{i=0}^{k} a_i {\theta_j}^i - Mb_j + \varepsilon \leq 0, \quad j = 0, \ldots, d \\
 & \boldsymbol{b} \in \{0,1\}^{d+1}
\end{array}
$$

First inertial-type bound: equality $k = 2$

Prism graphs Γ_n

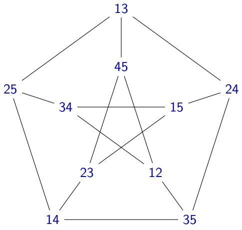


$$\alpha_2(\Gamma_{4i+j}) = \begin{cases} 2i+1 & \text{if } j = 3 \\ 2i & \text{otherwise} \end{cases}$$

These graphs are walk-regular. For $n \neq 2 \mod 4$, the MILP is tight.

First inertial-type bound: equality

Odd graphs O_ℓ: vertices corresponding to the $(\ell-1)$-subsets of a $(2\ell-1)$-set, and the adjacencies are defined by void intersection.



O_3

(A., Coutinho, Fiol, Nogueira and Zeijlemaker 2022)

For $i = 0, \ldots, \ell - 1$, let μ_i and m_i be the eigenvalues and multiplicities of the Odd graph $O_\ell = O_{d+1}$. Then,

$$\alpha_{d-1}(O_{d+1}) \leq \begin{Bmatrix} m_1 & \text{for even } d \\ m_1 + 1 & \text{for odd } d \end{Bmatrix} = \begin{cases} 2d & \text{for even } d, \\ 2d+1 & \text{for odd } d. \end{cases}$$

First inertial-type bound: equality

Odd graph $O_\ell = O_{d+1}$	α_{d-1}	First inertial-type bound
O_2 (K_3)	$\alpha_0 = 3$	$m_0 + m_1 = 3$
O_3 (Petersen)	$\alpha_1 = 4$	$m_1 = 4$
O_4	$\alpha_2 = 7$	$m_0 + m_1 = 7$
O_5	$\alpha_3 = 7$	$m_1 = 8$
O_6	$\alpha_4 = 11$	$m_0 + m_1 = 11$
O_7	$\alpha_5 = 12$	$m_1 = 12$
O_8	$\alpha_6 = 15$	$m_0 + m_1 = 15$
O_9	$\alpha_7 = 15$	$m_1 = 16$
O_{10}	$\alpha_8 = 19$	$m_0 + m_1 = 19$
O_{11}	$\alpha_9 = 19$	$m_1 = 20$
O_{12}	$\alpha_{10} = 23$	$m_0 + m_1 = 23$
O_{14}	$\alpha_{12} = 27$	$m_0 + m_1 = 27$
O_{16}	$\alpha_{14} = 31$	$m_0 + m_1 = 31$

Table: The known exact values of α_{d-1} and the upper bounds for the Odd graphs O_{d+1}.

Second inertial-type bound

Sometimes the first inertial bound can be strenghtened:

(A., Coutinho, Fiol, Nogueira and Zeijlemaker 2022)

Let G be a k-partially walk-regular graph with adjacency matrix eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$, $p_k \in \mathbb{R}_k[x]$ such that $\sum_{i=1}^n p_k(\lambda_i) = 0$. Then,

$$\chi_k \geq 1 + \max\left(\frac{|j : p_k(\lambda_j) < 0|}{|j : p_k(\lambda_j) > 0|}\right).$$

Second MILP

$$\chi_k \geq 1 + \max\left(\frac{|j : p_k(\lambda_j) < 0|}{|j : p_k(\lambda_j) > 0|}\right)$$

variables: $a_1, \ldots, a_k$, $(b_1, \ldots, b_n)$, $(c_1, \ldots, c_n)$
parameters: k, $\lambda_1, \ldots, \lambda_n$

$$\begin{array}{rll}
\texttt{maximize} & 1 + \frac{n - \mathbf{1}^T \boldsymbol{b}}{\ell} & \\
\texttt{subject to} & \sum_{j=1}^{n} \sum_{i=0}^{k} a_i \lambda_j^i = 0 & \\
& \sum_{i=0}^{k} a_i \lambda_j^i - M b_j + \varepsilon \leq 0, & j = 1, \ldots, n \\
& \sum_{i=0}^{k} a_i \lambda_j^i - M c_j \leq 0, & j = 1, \ldots, n \\
& \sum_{i=1}^{n} c_i = \ell & \\
& \boldsymbol{b} \in \{0,1\}^n, \quad \boldsymbol{c} \in \{0,1\}^n &
\end{array}$$

⚠ Now we look at all eigenvalues, including the repeated ones

Second MILP

$$\chi_k \geq 1 + \max\left(\frac{|j : p_k(\lambda_j) < 0|}{|j : p_k(\lambda_j) > 0|}\right)$$

- Trace condition $\operatorname{tr} p_k(A) = 0$
- If $p_k(\lambda_j) \geq 0$, then $b_j = 1$. If $p_k(\lambda_j) > 0$, then $c_j = 1$
- Fix $\sum c_i = \ell$, solve for $\ell = 1, \ldots, n-1$
- Maximize $|j : p_k(\lambda_j) < 0| = n - 1^T \boldsymbol{b}$

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Second MILP: results

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Gray graph	14	11	19	19	11
Nauru Graph	6	5	8	8	6
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Pappus Graph	4	3	7	6	3
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Brinkmann graph	4	3	6	6	3
Harborth Graph	12	9	13	13	10
...					
Klein 7-regular Graph	3	3	9	3	3
...					

Tight families:

- Prism graphs Γ_n with $n \neq 2 \mod 4$
- Incidence graphs of projective planes $PG(2;q)$ (q prime power)

Ratio-type bound

$W(p) := \max_{u \in V}\{(p(A))_{uu}\}$
$w(p) := \min_{u \in V}\{(p(A))_{uu}\}$
$\lambda(p) := \max_{i \in [2,n]}\{p(\lambda_i)\}$

(A., Coutinho, Fiol 2019)

Let G be a regular graph with n vertices and eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. Let $p \in \mathbb{R}_k[x]$ with corresponding parameters $W(p)$ and $\lambda(p)$, and assume $p(\lambda_1) > \lambda(p)$. Then,

$$\alpha_k \leq n\frac{W(p) - \lambda(p)}{p(\lambda_1) - \lambda(p)}.$$

Ratio-type bound: corollary

For $k = 1$,

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Ratio bound (Hoffman 1970)

If G is regular then

$$\alpha(G) \leq n \frac{-\lambda_n}{\lambda_1 - \lambda_n}.$$

Ratio-type bound: optimization

(Fiol 2020)

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For k-partially walk-regular graphs

$$(p(A))_{uu} = \frac{1}{n}\operatorname{tr} p(A) = \frac{1}{n}\sum_{i=1}^{n} p(\lambda_i) \quad \text{for all } u \in V.$$

Why optimization using MILPs?

(i) The quantum k-independence number is upper bounded by the inertial-type bound (A., Elphick and Wocjan 2022):

$$\alpha_k \leq \alpha_{kq} \leq \min\{|i : p(\lambda_i) \geq w(p)|, |i : p(\lambda_i) \leq W(p)|\}.$$

For $k > 1$ we can use the MILPs to compute values of the quantum parameter when the bound is tight. For $k = 1$:

$$\alpha \leq \alpha_q \leq \min\{|i : \lambda_i \geq 0|, |i : \lambda_i \leq 0|\}.$$

(ii) Closed formulas for small k.

(iii) Use the polynomials involved in the MILPs:
inertial-type bound (A., Coutinho, Fiol 2019)
ratio-type bound (Fiol 2020)
to relate both bounds.

Ratio-type bound: best polynomial for k=2

(A., Coutinho, Fiol 2019)

Let G be a δ-regular graph with n vertices and distinct eigenvalues $\theta_0(=\delta) > \theta_1 > \cdots > \theta_d$ with $d \geq 2$. Let θ_i be the largest eigenvalue such that $\theta_i \leq -1$. Then,

$$\alpha_2 \leq n\frac{\theta_0 + \theta_i\theta_{i-1}}{(\theta_0 - \theta_i)(\theta_0 - \theta_{i-1})}.$$

Moreover, this is the best possible bound that can be obtained by choosing a polynomial and applying the ratio-type bound.

Ratio-type bound: best polynomial for k=3

(Neuwman, Sajna and Kavi 2022+)

The optimal polynomial for $k = 3$

Theorem (Newman, Sajna and K)

Let G be δ-regular graph with n vertices, adjacency matrix A, and distinct eigenvalues $\delta = \theta_0 > \theta_1 > \cdots > \theta_d$, with $d \geq 2$.
Let s be the largest index such that $\theta_s \geq -\frac{\theta_0^2 + \theta_0\theta_d - \triangle}{\theta_0(\theta_d + 1)}$, where $\triangle = \max_{u \in V}\{(A^3)_{uu}\}$, twice the largest number of triangles on any vertex in G.
Let $b = -(\theta_s + \theta_{s+1} + \theta_d)$ and $c = \theta_d\theta_s + \theta_d\theta_{s+1} + \theta_s\theta_{s+1}$.
Then $p(x) = x^3 + bx^2 + cx$ is an optimal polynomial for $k = 3$. The corresponding bound on the 3-independence number of G is

$$\alpha_3 \leq n \frac{\triangle - \theta_d^3 + b(\theta_0 - \theta_d^2) - c\theta_d}{(\theta_0^3 - \theta_d^3) + b(\theta_0^2 - \theta_d^2) + c(\theta_0 - \theta_d)}.$$

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Relating the inertia and the ratio-type bounds

Relating the inertia and the ratio-type bounds

Joint work with C. Dalfó, M.A. Fiol and S. Zeijlemaker

Motivation: relating inertia and ratio bounds

Both inertia and ratio bounds have been used to prove a significant number of results in Extremal Combinatorics (like the EKR theorem).

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However, there are also many cases that neither is sharp, and their relationship is not understood!

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Intermediate goal: understand their relationship.

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Both inertia and ratio bounds have been used to prove a significant number of results in Extremal Combinatorics (like the EKR theorem).

However, there are also many cases that neither is sharp, and their relationship is not understood!

Ultimate goal: find a way to unify the bounds (interlacing? polynomials?).

Intermediate goal: understand their relationship.

Our strtegy: use the polynomials from the previous bounds and MILPs.

Inertia and ratio-type bounds: overview

(A., Coutinho, Fiol 2019)
Let G have eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. Let $p \in \mathbb{R}_k[x]$, $\lambda(p) = \min_{i \in [2,n]}\{p(\lambda_i)\}$, $W(p) = \max_{u \in V}\{(p(A))_{uu}\}$, and $w(p) = \min_{u \in V}\{(p(A))_{uu}\}$.

(i) **An inertial-type bound.**

$$\alpha_k \leq \min\{|i : p(\lambda_i) \geq w(p)|, |i : p(\lambda_i) \leq W(p)|\}.$$

(ii) **A ratio-type bound.** Assume that G is regular. Let $p \in \mathbb{R}_k[x]$ such that $p(\lambda_1) > \lambda(p)$.

$$\alpha_k \leq n\frac{W(p) - \lambda(p)}{p(\lambda_1) - \lambda(p)}.$$

Inertia and ratio-type bounds: linear transformations

(A., Coutinho, Fiol 2019)

(i) **An inertial-type bound.** A constant can be added to p making $w(p) = \min_{u \in V}\{(p(A))_{uu}\} = 0$. Moreover, multiplying the resulting polynomial by a (positive or negative) appropriate constant, it is enough to find a polynomial $p \in \mathbb{R}_k[x]$ with fixed value of $\lambda(p)$, say -1, and that minimizes the number of eigenvalues λ_i such that $p(\lambda_i) \geq w(p)$.

$$\alpha_k \leq \min_{i \in [1,n]} |\{i : p(\lambda_i) \geq 0\}|.$$

(ii) **A ratio-type bound.** We can use a polynomial p satisfying $p(\lambda_1) = 1$ and $\lambda(p) = 0$.

$$\alpha_k \leq nW(p) = n \max_i \; p(A)_{ii}.$$

Sign and minor polynomials: k-partially walk-regular case

(i) **An inertial-type bound.** (A., Coutinho, Fiol, Nogueira and Zeijlemaker 2022)
If $s \in \mathbb{R}_k[x]$ is a polynomial satisfying $\operatorname{tr} s(A) = 0$, the best result is obtained by the **sign polynomial** s_k, whose coefficients are the solution of a MILP.

$$\begin{array}{ll} \texttt{minimize} & \sum_{i=0}^{d} m_i b_i \\ \texttt{subject to} & \sum_{i=0}^{d} m_i s_k(\theta_i) = 0 \\ & \sum_{i=0}^{k} a_i \theta_j{}^i - M b_j + \epsilon \le 0, \quad j = 0, \ldots, d \quad (*) \\ & \boldsymbol{b} \in \{0,1\}^{d+1} \end{array}$$

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(ii) **A ratio-type bound.** (Fiol 2020)
If $f \in \mathbb{R}_k[x]$ is a polynomial satisfying $\lambda(f) = 0$ and $f(\theta_0) = 1$, the best result is obtained with the **minor polynomial** f_k that minimizes $\sum_{i=0}^{d} m_i f_k(\theta_i)$. This polynomial f_k is defined by $f_k(\theta_0) = x_0 = 1$ and $f_k(\theta_i) = x_i$, for $i = 1, \ldots, d$, where the vector $(x_1, x_2, \ldots, x_d)$ is a solution of a LP.

$$\begin{array}{ll} \texttt{minimize} & \sum_{i=0}^{d} m_i x_i \\ \texttt{subject to} & f[\theta_0, \ldots, \theta_m] = 0, \quad m = k+1, \ldots, d \\ & x_i \ge 0, \ i = 1, \ldots, d \end{array}$$

Sign and minor polynomials: partially walk-regular graphs

Let G be a k-partially walk-regular graph with spectrum $\{\theta_0^{m_0}, \theta_1^{m_1}, \ldots, \theta_d^{m_d}\}$. Let $h(x)$ be the Heaviside function.

(i) **An inertial-type bound.** (A., Coutinho, Fiol, Nogueira and Zeijlemaker 2022)
Let $p = s \in \mathbb{R}_k[x]$ be a polynomial satisfying $\lambda(s) = \min_{i\in[1,d]}\{s(\theta_i)\} = -1$ and $\operatorname{tr} s(A) = 0$. Then,

$$\alpha_k \leq \sum_{i=0}^{d} m_i h(s(\theta_i)).$$

(ii) **A ratio-type bound.** (Fiol 2020)
Let $p = f \in \mathbb{R}[k]$ be a polynomial satisfying $\lambda(f) = \min_{i\in[1,d]}\{f(\theta_i)\} = 0$ and $f(\theta_0) = 1$. Then,

$$\alpha_k \leq \sum_{i=0}^{d} m_i f(\theta_i).$$

Inertia and ratio type bounds: relationship?

Inertial-type bound - sign polynomials s
Ratio-type bound - minor polynomials f

The following relations between the polynomials s and f hold.

(i) If $s \in_k [x]$ satisfies $\min_{i\in[1,d]}\{s(\theta_i)\} = -1$ and $\operatorname{tr} s(A) = 0$, then $f(x) = \frac{1+s(x)}{1+s(\theta_0)}$ satisfies $\min_{i\in[1,d]}\{f(\theta_i)\} = 0$ and $f(\theta_0) = 1$. From the **ratio-type** bound we get

$$\alpha_k \le \frac{n}{1+s(\theta_0)}.$$

(ii) If $f \in \mathbb{R}[k]$ satisfies $\min_{i\in[1,d]}\{f(\theta_i)\} = 0$ and $f(\theta_0) = 1$, then $s(x) = \frac{n}{\operatorname{tr} f(A)} f(x) - 1$ has $\operatorname{tr} s(A) = 0$ and $\min_{i\in[1,d]}\{s(\theta_i)\} = -1$. From the **inertial-type** bound we get

$$\alpha_k \le \sum_{i=0}^{d} m_i h\left(f(\theta_i) - \frac{1}{n}\operatorname{tr} f(A)\right).$$

Inertia and ratio type bounds: relationship?

It seems interesting to know when, for a given value of k, the k-sign polynomial and the k-minor polynomial are **linearly related**.

Inertia and ratio type bounds: relationship?

It seems interesting to know when, for a given value of k, the k-sign polynomial and the k-minor polynomial are **linearly related**.

Moreover, if both polynomials give the same (inertia and ratio) bounds on α_k, we consider such polynomials to be essentially the same.

k-Cvetković-Hoffman graphs

- We define a **k-Cvetković-Hoffman graph** (or **k-CH graph**) as the graph with the same inertia and ratio bounds on α_k.

k-Cvetković-Hoffman graphs

- We define a **k-Cvetković-Hoffman graph** (or **k-CH graph**) as the graph with the same inertia and ratio bounds on α_k.
- If both bounds are equal and tight, we call the graph a **tight k-CH graph**.

k-Cvetković-Hoffman graphs: the case $k = 1$

(A., Dalfó, Fiol and Zeijlemaker 2022+)

(*i*) For any graph, the 1-sign polynomial and the 1-minor polynomial are linearly related.

(*ii*) Every bipartite regular graph with even number $d + 1$ of different eigenvalues is a tight 1-CH graph.

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(ii) Every bipartite regular graph with even number $d + 1$ of different eigenvalues is a tight 1-CH graph.

(Haemers and Higman 1989)

Let G be a strongly regular graph with maximum independent set $U \subset V$. Then both the inertia and ratio bounds are tight if and only if the graph induced by $\overline{U} = V \setminus U$ is strongly regular.

k-Cvetković-Hoffman graphs: the case $k = 1$

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Examples of tight 1-CH graphs:

- The Kneser graph $K(n, k)$, with $n > 2k$, is a tight 1-CH graph (EKR proofs).
- The Taylor 2-graphs for $U_3(q)$ with $q \in \{3, 5, 7, 9\}$.

k-Cvetković-Hoffman graphs: the case $k = d - 1$

We deal with **maximally independent sets** of vertices.

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(A., Dalfó, Fiol and Zeijlemaker 2022+)
If it exists, the **triangle-free strongly regular** graph $G(n)$ with feasible parameters

$$(n^4 + 5n^3 + 6n^2 - n - 1, n^2(n+2), 0, n^2) \qquad \text{for } n = 1, 2, \ldots$$

is a tight $(d-1)$-CH graph.

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If it exists, the **triangle-free strongly regular** graph $G(n)$ with feasible parameters

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is a tight $(d-1)$-CH graph.

Examples of $(d-1)$-CH graphs:

- $G(1) = P$, the Petersen graph with parameters $(5, 2, 0, 1)$
- $G(2) = M_{22}$, the Mesner graph with parameters $(77, 16, 0, 4)$.
- The Odd graph O_ℓ with even degree ℓ is a $(d-1)$-CH graph.
- The Odd graph O_ℓ is a tight $(d-1)$-CH graph for every even $\ell \in \{2, 3, 4, 6, 8, 10, 12, 14, 16\}$.
- The antipodal distance-regular graphs with odd diameter are tight $(d-1)$-CH graphs.

Concluding remarks

Overview new results

Independence number:

- (Delsarte 1973) LP bound on α for distance-regular graphs.
- (Lovász 1979) SDP bound ϑ.
- ...

***k*-independence number:**

Hoffman ratio bound	Inertia bound
↓	↓
(Fiol 2020) LP with minor polynomials	this talk MILP with sign polynomials

...and new relations between bounds.

Overview new results

- Inertial and ratio-type bounds for α_k that depend on the parameters (eigenvalues) of G and not of G^k.
- Optimization of the inertial-type bounds using MILPs: linear combination of the eigenvalues mutiplicities.
- In some cases our approach yields closed formulas for the optimal polynomial.
- Our inertial-type bounds for α_k and χ_k also hold for the corresponding quantum parameters, thus, when the bounds are tight, the MILPs can be used to compute their exact values.
- Use of the polynomials involved in the MILPs to find new relationships between the two bounds.

Open problems

- Complexity of the MILPs? Does increasing k make the problem easier?
- Use the MILPs for other graphs and values of k, and find more closed formulas for graph families.
- Other relationships between inertial-type and ratio-type bounds via the obtained polynomials from the MILPs.
- **SDP formulation for the inertial-type bound?**

$$\alpha \leq \min\{|i : \lambda_i \geq 0|, |i : \lambda_i \leq 0|\}$$
$$\alpha_k \leq \min\{|i : p(\lambda_i) \geq w(p)|, |i : p(\lambda_i) \leq W(p)|\}$$

Thank you for listening!

Further reading:

A. Abiad, G. Coutinho, M.A. Fiol, B.D. Nogueira and S. Zeijlemaker,
Optimization of eigenvalue bounds for the independence and chromatic number of graph powers
Discrete Math. 345(3) (2022)

A. Abiad, C. Dalfó, M.A. Fiol and S. Zeijlemaker
On inertia and ratio type bounds for the k-independence number of a graph and their relationship
arXiv:2201.04901

Open problems: bounding the Shannon capacity of G^k?

(Fiol 2020)

Let G be a k-partially walk-regular graph with adjacency matrix A and spectrum $\{\theta_0^{m_0}, \ldots, \theta_d^{m_d}\}$. Let $f_k \in \mathbb{R}_k[x]$ be a k-minor polynomial, that is, $f_k(\theta_0) = 1$ and $f_k(\theta_i) \geq 0 \ \forall i \neq 0$. Then,

$$\alpha_k(G) = \alpha(G^k) \leq \Theta(G^k) \leq \operatorname{tr} f_k(A) \sum_{i=0}^{d} m_i f_k(\theta_i) \quad \text{for every } k = 0, \ldots, d-1.$$

(A., Fiol 2022+)

Let $p \in \mathbb{R}_k[x]$ such that $p(A)$ fits G^k (a matrix B that **fits** G^k if has all diagonal non-zero entries and $b_{ij} = 0$ if $\operatorname{dist}(i,j) > k$ in G). Then,

$$\alpha_k(G) = \alpha(G^k) \leq \Theta(G^k) \leq \min_{p(A) \propto G^k} \operatorname{rank}(p(A))$$