Drift Theory

and its use for the Analysis of Evolutionary Algorithms

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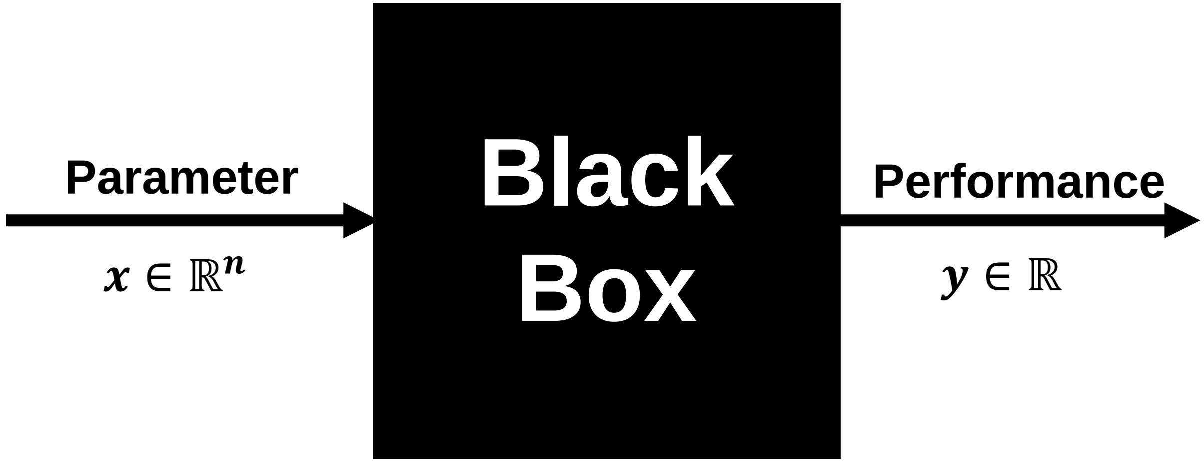
I have always had that drift, that's key for me.
- Yuzvendra Chahal

What is black box optimization?



What is black box optimization?

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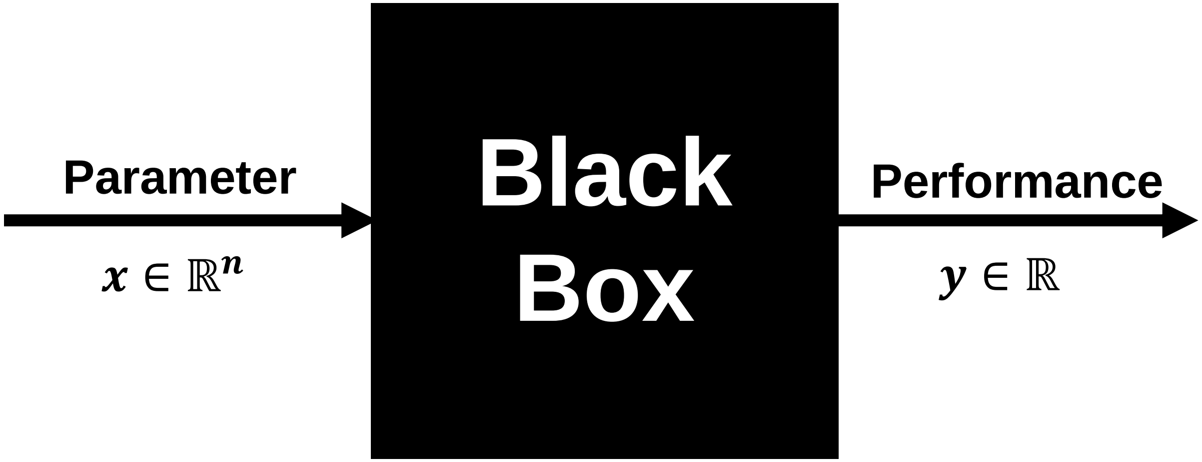

We optimize functions

$$f: \mathbb{R}^n \to \mathbb{R}.$$

Multi-Objective Optimization:

$$f: \mathbb{R}^n \to \mathbb{R}^m.$$

Problems we deal with

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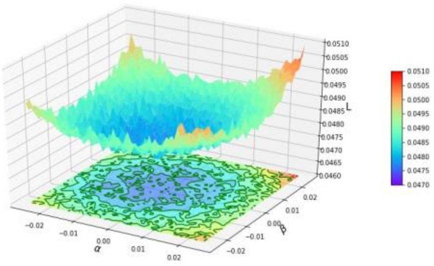
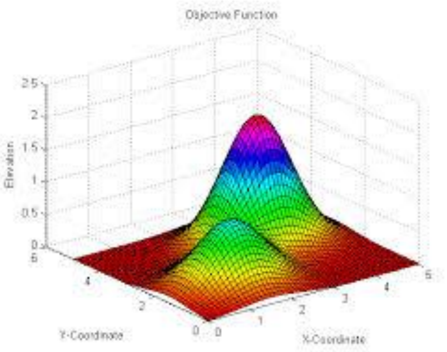
- What hinders search?
  - Ruggedness
  - Local Optima
  - Noise
  - Dynamic Changes
  - Stagnation





Black-Box Algorithms for Theorists

Most Basic Algorithm

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Algorithm 1: Basic Search Algorithm

$x \leftarrow$ random individual;
while true **do**
  $y \leftarrow \texttt{mutate}(x)$;
  if $f(y) \leq f(x)$ **then**
    $x \leftarrow y$;

- Minimization
- Mutation (for example $x \in \{0,1\}^n$)
  - Flip one bit u.a.r.
  - Flip each bit independently with prob. 1/n
- Accept equal; reject worse

More Elaborate Algorithms

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HPI

(μ, λ)-EA

Algorithm 1: $(\mu + \lambda)$-EA

$P \leftarrow \mu$ random individuals;
while `true` **do**
    $P' \leftarrow \emptyset$;
    for λ times **do**
        $x \leftarrow \texttt{select}(P)$;
        $y \leftarrow \texttt{mutate}(x)$;
        $P' \leftarrow P' \uplus \{y\}$;
    $P \leftarrow \texttt{pick}(\mu,$ ~~$P \cup$~~ $P')$;

More Elaborate Algorithms

Algorithm 1: $(\mu + \lambda)$-GA

$P \leftarrow \mu$ random individuals;
while $\texttt{true}$ **do**
    $P' \leftarrow \emptyset$;
    for λ times **do**
        $x \leftarrow \texttt{select}(P)$;
        $y \leftarrow \texttt{select}(P)$;
        $z \leftarrow \texttt{crossover}(x, y)$;
        $P' \leftarrow P' \uplus \{z\}$;
    $P \leftarrow \texttt{pick}(\mu, P \cup P')$;

More Elaborate Algorithms

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(μ, λ)-EA

Algorithm 1: $(\mu + \lambda)$-EA

```
P ← μ random individuals;
while true do
    P' ← ∅;
    for λ times do
        x ← select(P);
        y ← mutate(x);
        P' ← P' ⊎ {y};
    P ← pick(μ, P ∪ P');
```

Algorithm 1: $(\mu + \lambda)$-GA

```
P ← μ random individuals;
while true do
    P' ← ∅;
    for λ times do
        x ← select(P);
        y ← select(P);
        z ← crossover(x, y);
        P' ← P' ⊎ {z};
    P ← pick(μ, P ∪ P');
```

- EA ≅ Mutation only
- GA ≅ Crossover
- Plus-Selection ≅ Keep the best
- Comma-Selection ≅ Only new solutions

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Mathematical Analysis

First Example: OneMax

Algorithm 1: Basic Search Algorithm

```
1 x ← random individual;
2 while true do
3     y ← mutate(x);
4     if f(y) ≤ f(x) then
5         x ← y;
```

Multiplicative Drift Theorem [DJW12]

If

$$E(Y_t - Y_{t+1} \mid X_t) \geq \delta Y_t$$

Then, for $T = min\{t \mid Y_t < 1\}$,

$$E(T) \leq \ln(Y_0)/\delta$$

[DJW12] Doerr, Johannsen, Winzen: *Multiplicative Drift Analysis*, 2012

- Suppose $f(x) = |x|_0$ (called OneMax)
- Defines random process $X_t \in \{0,1\}^n$
  - $E\big(f(X_0)\big) = n/2$
- T = random time till optimum reached
- Know: Step-wise change
  - $E(f(X_t) - f(X_{t+1}) \mid X_t) \geq X_t/(en)$

$$E(T) \leq en \ln(n)$$

Multiplicative Drift

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Multiplicative Drift Theorem [DJW12]

If
$$E(Y_t - Y_{t+1} \mid X_t) \geq \delta Y_t$$
Then, for $T = min\{t \mid Y_t < 1\}$,
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[DJW12] Doerr, Johannsen, Winzen: *Multiplicative Drift Analysis*, 2012

- Prime example: Coupon Collector
- Multi-Drift ≈ Generalized Coupon Collector
- Drift Theorem:
  - Random process ≈ deterministic analogon

Second Example: LeadingOnes

Algorithm 1: Basic Search Algorithm

```
x ← random individual;
while true do
    y ← mutate(x);
    if f(y) ≤ f(x) then
        x ← y;
```

Additive Drift Theorem [HY04]

If

$$E(Y_t - Y_{t+1} \mid X_t) \geq \delta$$

Then, for $T = min\{t \mid Y_t = 0\}$,

$$E(T) \leq Y_0/\delta$$

[HY04] He, Yao: *A study of drift analysis for estimating computation time of evolutionary algorithms*, 2004

- $f(x) =$ #of 1s before first 0 (called LeadingOnes)
  - $E(f(X_0)) \leq n$
- Know: Step-wise change
  - $E(f(X_t) - f(X_{t+1}) \mid X_t) \geq 1/(en)$
- $E(T) \leq en^2$

Additive Drift

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Addtive Drift Theorem
[HY04]

If

$$E(Y_t - Y_{t+1} \mid X_t) \geq \delta$$

Then, for $T = min\{t \mid Y_t = 0\}$,

$$E(T) \leq Y_0/\delta$$

[HY04] He, Yao: *A study of drift analysis for estimating computation time of evolutionary algorithms*, 2004

- Random process ≈ deterministic analogon
- Traps:
  - No overshooting allowed!
  - Unbounded Search spaces?

Conclusion

- Drift theorems turn step-wise gains into expected times

Addtive Drift Theorem [HY04]

If
$$E(Y_t - Y_{t+1} \mid X_t) \geq \delta$$
Then, for $T = min\{t \mid Y_t = 0\}$,
$$E(T) \leq Y_0/\delta$$

Multiplicative Drift Theorem [DJW12]

If
$$E(Y_t - Y_{t+1} \mid X_t) \geq \delta Y_t$$
Then, for $T = min\{t \mid Y_t < 1\}$,
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[DJW12] Doerr, Johannsen, Winzen: *Multiplicative Drift Analysis*, 2012

- Main tool of modern analyzes
- Many more elaborate variants exist
- Approximations also analyzable

Thank You!