Bell Inequalities:
What do we know about them and why should cryptographers care

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Overview

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1. The weirdness of quantum mechanics:
   Bell inequalities & their violation

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2. Why should cryptographers care?

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3. What do we know about Bell inequalities?

Part 1:

Quantum mechanics:
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- Bell'64: there are other quantum predictions that cannot be reproduced by local-realist models

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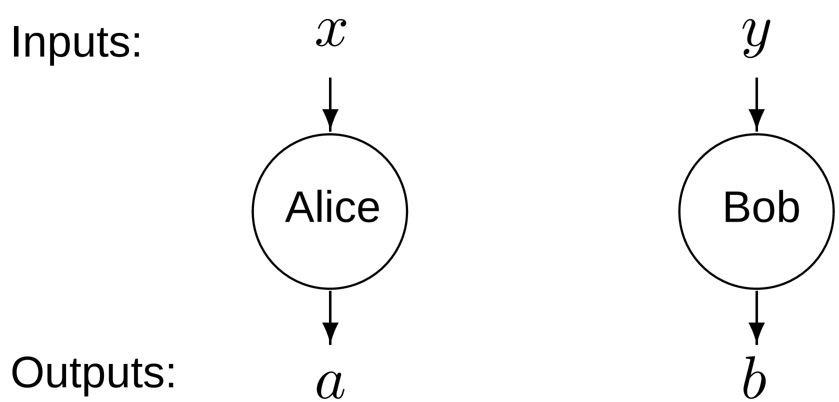

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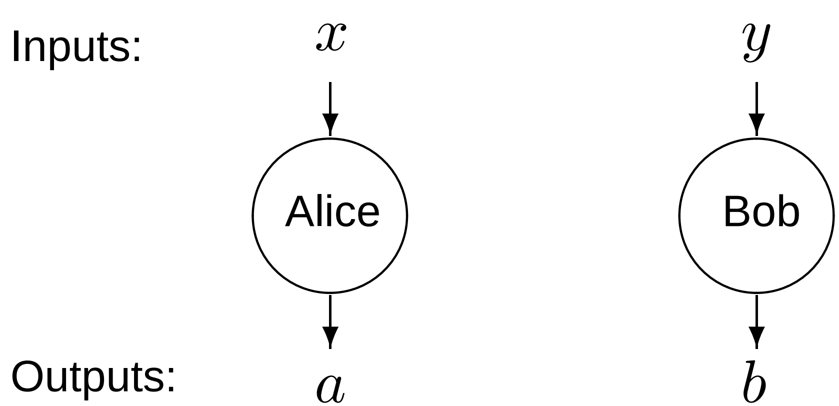

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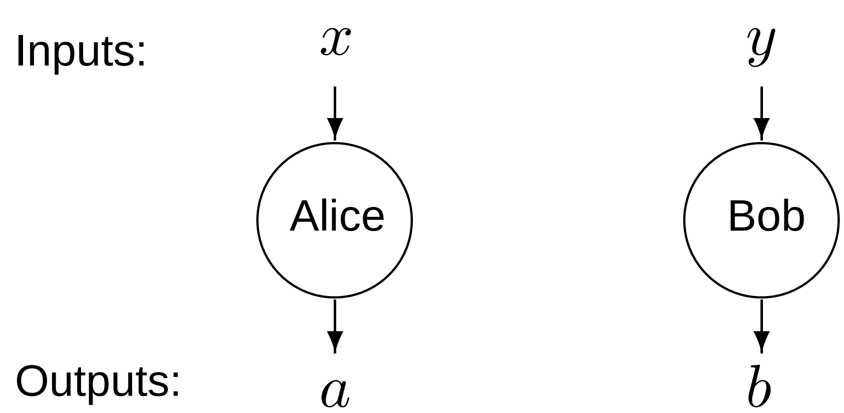

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- Entangled value $\omega^*(G)$: maximal winning probability among quantum protocols (shared entanglement)

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- When implemented, such experiments show that nature is not classical (i.e., not local-realist)

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- Hard to close both loopholes simultaneously:
  to close locality loophole distance between Alice and Bob should be large, but then detection-error goes up

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- Can win with prob 1 using 2 EPR-pairs: $\omega_4^*(G) = 1$

Part 2:

Why should cryptographers care?

Making crypto protocols

Breaking crypto protocols

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- Information-theoretically secure *if* Alice and Bob can trust that they measure qubits in the chosen basis

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- Eve could have a perfect copy without being detected

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- Applications besides QKD: random-number generation, bit commitment and coin flipping

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- Classical hardness-amplification fails here!

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How large can the violation be?

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  - no violation if Alice and Bob share EPR-pairs
  - large violation if they share other, non-maximally entangled state (Junge & Palazuelos'10, Regev'10)

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