Recursive Program Schemes with Effects

Daniel Schwencke, 28th March 2010
Outlines

1. Introduction
2. Preliminaries and Definitions
3. A Solution Theorem
4. Future Work
Idea: define new operations using given operations and recursion

**Definition (RPS without effects, classical)**

- disjoint finite sets $F$ – given operation symbols
  - $\Phi$ – new operation symbols
  - $X$ – variables
- $\phi(x_1, \ldots, x_n) \approx t^\phi(x_1, \ldots, x_n)$ for all $\phi \in \Phi_n$, $t^\phi$ term in $F \cup \Phi$
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Example ([Milius Moss 06])

\[
\phi(x) \approx f(x, \phi(gx))
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\psi(x) \approx f(\phi(gx), ggx)
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Generalising category-theoretic approach in [Ghani Lüth de Marchi 03, Milius Moss 06]
ND-RPSs

Idea: add non-deterministic choice on rhs of formal equations

- special binary operation symbol $or \not\in F \cup \Phi$
- terms $t^\phi$ in $F \cup \Phi \cup \{or\}$
- see [Arnold Nivat 77]
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pow(x) \approx x \ or \ (x \cdot pow(x))
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Example

$$\text{pow}(x) \approx x \text{ or } (x \cdot \text{pow}(x))$$

More generally: RPSs with effects

- partiality
- non-determinism
- probabilism
Assumptions

- $(M, \eta^M, \mu^M)$ monad on $\text{Set}$
- $H, V$ finitary $\text{Set}$-functors
- distributive laws $\lambda : HM \to MH$ and $\nu : VM \to MV$

$\Rightarrow$ induced distributive law $\rho : (H + V)M \to M(H + V)$
## A Starting Point

### Assumptions

- \((M, \eta^M, \mu^M)\) monad on \textbf{Set}
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- distributive laws \(\lambda : HM \to MH\) and \(\nu : VM \to MV\)

\[\Rightarrow\text{induced distributive law } \rho : (H + V)M \to M(H + V)\]

**Meaning:**

- \(M\) – effect, e.g. \(+ 1\), \(P\), \(D\)
- \(H, V\) – “signatures” of given/new operations
- \(\lambda, \nu, \rho\) – extension of operations to parameters with effects
A First Lemma

Notation:
- \((F^G, \eta^G, \mu^G)\) free monad on \(G\)
- universal natural transformation \(\kappa^G : G \to F^G\)
- \(T\) monad, \(\sigma : G \to T\). Then
  \(\sigma^\# : F^G \to T\) unique monad morphism such that \(\sigma^\# \cdot \kappa^G = \sigma\)
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Lemma

If \(G\) has free algebras, every distributive law \(\delta : GM \to MG\) induces a distributive law \(\delta' : F^G M \to MF^G\).

\(\Rightarrow\) composite monad \((MF^G, \eta^M F^G \cdot \eta^G, (\mu^M * \mu^G) \cdot M\delta' F^G)\)
RPSs with Effects

Definition

- **M-RPS** \( e : V \rightarrow MF^{H+V} \)
- **guarded** if \( e \equiv V \xrightarrow{e_0} M(HF^{H+V} + \text{Id}) \xrightarrow{\cdots} MF^{H+V} \)
- (uninterpreted) solution of \( e \) \( e^\dagger : V \rightarrow MF^H \) such that \( e^\dagger = \mu^M F^H \cdot M[\eta^M F^H \cdot \eta^H, e^\dagger]^\# \cdot e \)
RPSs with Effects

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**Example**

For $\text{pow}(x) \approx x$ or $(x \cdot \text{pow}(x))$ take $M = \mathcal{P}$, $V = \text{Id}$, $H = \text{Id}^2$

- $e_X(x) = \{x, x \cdot \text{pow}(x)\}$
- guarded since $x \in \text{Id}(X)$ and $x \cdot \text{pow}(x) \in HF^{H+V}X$
- $e^\dagger_X(x) = \{x, x \cdot x, x \cdot (x \cdot x), x \cdot (x \cdot (x \cdot x)), \ldots \}$ is a solution
What We Would Like to Prove... 

Question

Does every guarded M-RPS have a (unique) solution?
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From $H$, $M$, $\lambda$ and $\rho'$ we obtain

- a functor $\mathcal{H} = H \cdot - + \text{Id}$ on $[\text{Set}, \text{Set}]$;
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- a distributive law $\Lambda = [\text{Minl}, \text{Minr}] \cdot (\lambda_+ + \eta^M) : HM \to MH$;
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- a distributive law $\Lambda = \text{[Minl, Minr]} \cdot (\lambda - + \eta^M) : \mathcal{H}M \rightarrow \mathcal{M}\mathcal{H}$;
- equivalently, a lifting $\bar{H}$ of $\mathcal{H}$ to $[\text{Set}, \text{Set}]_M$. 
What We Would Like to Prove. . .

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- the canonical functor $J : [\text{Set}, \text{Set}] \to [\text{Set}, \text{Set}]_\mathcal{M}$. 
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- the canonical functor $J : [\text{Set}, \text{Set}] \to [\text{Set}, \text{Set}]_{\mathcal{M}}$;
- a monad $\mathcal{HF}^{H+V}$ with distributive law over $M$. 

D. Schwencke: Recursive Program Schemes with Effects
Second Order Substitution with Effects

Definition

For a guarded $M$-RPS $e$ let $\bar{e}$ be the unique monad morphism such that the diagram commutes:

\[
H + V \xrightarrow{[\text{Jinl} \cdot H\eta^{H+V}, e_0]} M(HF^{H+V} + \text{Id})
\]

\[
\begin{array}{c}
\kappa^{H+V} \\
F^{H+V}
\end{array}
\]

Remarks

$\bar{e}$ performs second order substitution with effect handling $\bar{e}$ is an $\bar{H}$-coalgebra

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\downarrow^{\kappa^{H+V}} & \\
F^{H+V} & \xrightarrow{\bar{e}} \\
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Remarks

- $\bar{e}$ performs second order substitution with effect handling
- $\bar{e}$ is an $H$-coalgebra
Sufficient Conditions for a Solution

Notation:

\[ \phi^H = \mu^H \cdot \kappa^H F^H : HF^H \to F^H \]
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Two facts:

- \([\phi^H, \eta^H] : \mathcal{H}F^H \to F^H\) is initial \(\mathcal{H}\)-algebra.
Sufficient Conditions for a Solution

Notation:

\[ \phi^H = \mu^H \cdot \kappa^H : HF^H \rightarrow F^H \]

Two facts:

1. \([\phi^H, \eta^H] : \mathcal{H}F^H \rightarrow F^H\) is initial \(\mathcal{H}\)-algebra.
2. If \(J[\phi^H, \eta^H]^{-1} : F^H \rightarrow \tilde{\mathcal{H}}F^H\) is final \(\tilde{\mathcal{H}}\)-coalgebra and the unique \(\tilde{\mathcal{H}}\)-coalgebra homomorphism \(h : F^{H+V} \rightarrow M\mathcal{F}^H\) between \(\tilde{e}\) and \(J[\phi^H, \eta^H]^{-1}\) is a monad morphism then \(h \cdot \kappa^{H+V} \cdot \text{inr} : V \rightarrow M\mathcal{F}^H\) is a solution of \(e\).
A Result for CPO-enriched $\mathbf{Set}_M$

Assumptions

- $\mathbf{Set}_M$ CPO-enriched with strict composition
- $\lambda$ strict
- $\tilde{H}$ locally continuous

Theorem

Under the above assumptions, every guarded $M$-RPS has a solution.
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**Theorem**

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**Proof.**

1. $J[\phi^H, \eta^H]^{-1}$ final $\bar{H}$-coalgebra: use techniques of [Hasuo Jacobs Sokolova 07]
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**Examples ([Milius Palm S 09])**

Monads $\_ + 1$, $\mathcal{P}$ or $\mathcal{D}$ with analytic $H$ and canonical $\lambda$
Future Work

1. uniqueness of solutions
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2. generalise $M$-RPS-definition to allow CIMs
   - [Arnold Nivat 77]-setting category-theoretic
   - environment monad $(\neg)^E$
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1. uniqueness of solutions

2. generalise $M$-RPS-definition to allow CIMs
   - [Arnold Nivat 77]-setting category-theoretic
   - environment monad $(-)^E$

3. interpreted solutions using [Milius Palm S 09]
Literature

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Thank you... for your attention!

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