Strong relative monads

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Motivation and contribution

- In programming language theory, we use structures like (strong) monads, (monoidal) comonads, arrows to structure syntax and semantics.
- Some natural structures fail to be monads as if for the only reason that the underlying functor is not an endofunctor.
- E.g., untyped/typed lambda calculus syntax (over finite contexts), finite-dimensional vector spaces etc.
- In FoSSaCS 2010, we defined and studied a relative monads as a generalization of monads.
- Here: strong relative monads.
Relative monads

- Given a category $\mathcal{C}$ and another category $\mathcal{J}$ with a functor $J \in \mathcal{J} \rightarrow \mathcal{C}$.

- A relative monad is given by
  - an object function $T \in |\mathcal{J}| \rightarrow |\mathcal{C}|$,
  - for any object $X \in |\mathcal{J}|$, a map $\eta_X \in \mathcal{C}(J X, T X)$ (unit),
  - for any objects $X, Y \in |\mathcal{J}|$ and map $k \in \mathcal{C}(J X, T Y)$, a map $k^* \in \mathcal{C}(T X, T Y)$ (Kleisli extension)

satisfying

- for any $X, Y \in |\mathcal{J}|$, $k \in \mathcal{C}(J X, T Y)$, $k^* \circ \eta_X = k$,
- for any $X \in |\mathcal{J}|$, $\eta_X^* = \text{id}_{T X} \in \mathcal{C}(T X, T X)$,
- for any $X, Y, Z \in |\mathcal{J}|$, $k \in \mathcal{C}(J X, T Y)$, $\ell \in \mathcal{C}(J Y, T Z)$, $(\ell^* \circ k)^* = \ell^* \circ k^* \in \mathcal{C}(T X, T Z)$.

- $T$ is functorial with $T f = (\eta \circ J f)^*$; $\eta$ and $(-)^*$ are natural.
Relative monads (ctd)

- Ordinary monads arise as the special case where \( \mathcal{J} =_{df} \mathcal{C} \), \( J =_{df} \text{Id}_\mathcal{C} \).
- Can define relative adjunctions between \( J \in \mathcal{J} \to \mathcal{C} \) and \( \mathcal{D} \).
- Every relative adjunction gives rise to a relative monad.
- Every relative monad resolves into a relative adjunction in at least two ways, the Kleisli and E-M adjunctions, which are its initial and final resolutions.
- If \( \text{Lan}_J \in [\mathcal{J}, \mathcal{C}] \to [\mathcal{C}, \mathcal{C}] \) exists, then \([\mathcal{J}, \mathcal{C}]\) has a lax monoidal structure and a relative monad on \( J \) is a lax monoid in it.
- If further conditions on \( J \) hold (in particular, \( J \) is fully faithful), then \([\mathcal{J}, \mathcal{C}]\) is (properly) monoidal and a relative monad on \( J \) is a (proper) monoid in it.
Example

- Given a semiring \((R, 0, +, 1, \times)\).
- Let \(\mathcal{J} = \text{df } \mathbb{F}, \mathcal{C} = \text{df } \textbf{Set}, J = \text{ the inclusion.} \)
- Define
  - a set mapping \(T \in \mathbb{F} \to \textbf{Set} \) by \(T m = \text{df } J m \to R\),
  - for any \(m \in |\mathbb{F}|\), a function \(\eta_m \in J m \to T m\) by
    \[ \eta_m(i \in m) = \text{df } \lambda j \in m. \text{ if } i = j \text{ then } 1 \text{ else } 0 \]
  - for any \(m, n \in |\mathbb{F}|, A \in J m \to T n\), a function
    \[ A^* \in T m \to T n \text{ by } A^* x = \text{df } \lambda j \in n. \sum_{i \in m} i \times A i j \]
- \(T m\) is the space of \(m\)-dimensional vectors, \(\eta_m\) is the diagonal \((m \times m)\)-matrix, and \(A^* x\) is the product of matrix \(A\) with a vector \(x\).

- \((T, \eta, (\cdot)^*)\) is a relative monad.
- \(\text{Kl}(T)\) is the category of finite-dimensional vector spaces and linear transformations.
Weak arrows

- Given a category $\mathcal{J}$, a weak arrow on $\mathcal{J}$ is given by
  - an object function $R \in |\mathcal{J}| \times |\mathcal{J}| \to \textbf{Set}$,
  - for any objects $X, Y \in |\mathcal{J}|$, a function $\text{pure} \in \mathcal{J}(X, Y) \to R(X, Y)$,
  - for any $X, Y, Z \in |\mathcal{J}|$, a function $(\ll) \in R(Y, Z) \times R(X, Y) \to R(X, Z)$

satisfying

- $\text{pure} (g \circ f) = \text{pure} g \ll \text{pure} f$,
- $r \ll \text{pure} \text{id} = r$,
- $\text{pure} \text{id} \ll r = r$,
- $t \ll (s \ll r) = (t \ll s) \ll r$.

- $R$ extends to a functor $\mathcal{J}^{\text{op}} \times \mathcal{J} \to \textbf{Set}$ (an endoprofunctor on $\mathcal{J}$); pure and $\ll$ are natural.
Weak arrows = relative monads on Yoneda

- Assume $\mathcal{J}$ is small. Let $\mathcal{C} = \text{df } [\mathcal{J}^{\text{op}}, \text{Set}], J = Y$ (the Yoneda embedding).

- A weak arrow on $\mathcal{J}$ is a functor $R \in \mathcal{J}^{\text{op}} \times \mathcal{J} \to \text{Set}$ with structure.

- This is the same as a functor $T \in \mathcal{J} \to [\mathcal{J}^{\text{op}}, \text{Set}]$ with structure, in fact, a relative monad on $Y$. 
Monads vs relative monads

- Given any $\mathcal{C}, \mathcal{J}$ and $J \in \mathcal{J} \rightarrow \mathcal{C}$.
- If $T$ is a monad on $\mathcal{C}$, then $T^b =_{df} T \cdot J$ is a relative monad on $\mathcal{J}$.
- If $J$ is well-behaved, then
- If $T$ is a relative monad on $J$, then $T^\# =_{df} \text{Lan}_J T$ is a monad on $\mathcal{C}$.
- The adjunction

$$
\begin{array}{ccc}
\mathcal{C}, \mathcal{C} & \xrightarrow{T} & \mathcal{J}, \mathcal{C} \\
\text{Lan}_J & \downarrow & \\
\mathcal{J}, \mathcal{C} & \xleftarrow{- \cdot J}
\end{array}
$$

lifts to an adjunction (a coreflection, if we require that $J$ is fully-faithful)

$$
\begin{array}{ccc}
\text{Mnd}(\mathcal{C}) & \xrightarrow{T} & \text{Mnd}(J) \\
(\cdot)^b & \downarrow & \\
\text{Mnd}(J) & \xleftarrow{(\cdot)^\#}
\end{array}
$$
Strong relative monads

- Given a monoidal categories \((\mathcal{J}, I, \otimes)\), \((\mathcal{C}, I', \otimes')\) and a monoidal functor \((J, e, m)\) between them.
- A strong relative monad is a relative monad \((T, \eta, (-)^*)\) and, for any \(X, Y \in \mathcal{J}\), a map \(\text{st}_{X, Y} \in \mathcal{C}(TX \otimes' JY, T(X \otimes Y))\), natural in \(X, Y\), with \(T, \eta, (-)^*\) strong wrt \(\text{st}\), so that

\[
\begin{xy}
0;0*
>;<<:*< 0 ; 3.8 cm/0.6 cm,**< (TX \otimes' I') (TX \otimes' JI) (T(X \otimes I))
X;0/< : 0.4 cm
(\eta^*_{TX, JI})_{TX \otimes' JI} = \text{id}_{TX \otimes' JI} \otimes' \text{id}_I = \eta \circ_{TX \otimes' JI} \eta_{TX \otimes I}
\]

\[
\begin{xy}
0;0*
>;<<:*< 0 ; 3.8 cm/0.6 cm,**< ((TX \otimes' JY) \otimes' JZ) (TX \otimes' (JY \otimes' JZ)) (T((X \otimes Y) \otimes Z))
X;0/< : 0.4 cm
(\alpha^*_{TX, TY, TZ})_{TX \otimes' (JY \otimes' JZ)} = \text{id}_{TX \otimes' (JY \otimes' JZ)} \otimes' \text{id}_Z = \alpha \circ_{TX \otimes' (JY \otimes' JZ)} \alpha_{TX, TY, TZ}
\]

\[
\begin{xy}
0;0*
>;<<:*< 0 ; 3.8 cm/0.6 cm,**< (TX \otimes' J(Y \otimes Z)) (T(X \otimes (Y \otimes Z)))
X;0/< : 0.4 cm
(\rho_{TX, J(Y \otimes Z)})_{TX \otimes' J(Y \otimes Z)} = \text{id}_{TX \otimes' J(Y \otimes Z)} \otimes' \text{id}_Z = \rho_{TX, J(Y \otimes Z)} \circ_{TX \otimes' J(Y \otimes Z)} \rho_{TY, T(Z)}
\]
Arrows

- Given a (small) monoidal category $\mathcal{J}, I, \otimes$.
- An arrow on $\mathcal{J}, I, \otimes$ is a weak arrow $(R, \text{pure}, \ll, \ll, \ll)$ on $\mathcal{J}$ with, for any $X, Y, Z \in |\mathcal{J}|$, a map $\text{first}_{X,Y,Z} \in R(X, Y) \to R(X \otimes Z, Y \otimes Z)$ satisfying
  - $\text{pure} (\text{id} \otimes f) \ll \text{first } r = \text{first } r \ll \text{pure } \text{id} \otimes f$
  - $\text{pure } \rho \ll \text{first } r = r \ll \text{pure } \rho$
  - $\text{pure } \alpha \ll \text{first } (\text{first } r) = \text{first } r \ll \text{pure } \alpha$
  - $\text{first } (\text{pure } f) = \text{pure } (f \otimes \text{id})$
  - $\text{first } (s \ll r) = \text{first } s \ll \text{first } r$
- $\text{first}_{X,Y,Z}$ is natural in $X, Y$, dinatural in $Z$. 
Arrows = strong relative monads on Yoneda

- Let $\mathcal{J}$ be small, take $\mathcal{C} = \text{df } [\mathcal{J}^{\text{op}}, \text{Set}]$, $J = \text{df } \mathcal{Y}$ (Yoneda on $\mathcal{J}$).
- A monoidal structure $(I, \otimes)$ on $\mathcal{J}$ induces one on $\mathcal{C}$ via
  - $I'Z = \text{df } \mathcal{J}(Z, I)$,
  - $(F \otimes' G)Z = \text{df } \int^{X,Y \in |\mathcal{J}|} \mathcal{J}(Z, X \otimes Y) \times (FX \times GY)$ (the Day convolution)
- $\mathcal{Y}$ becomes a monoidal functor.
- Consider a strong relative monad $(T, \eta, (-)^*, \text{st})$.
- We have
  \[
  (T X \otimes' \mathcal{Y} Y)Z \\
  = \int^{X',Y' \in |\mathcal{J}|} \mathcal{J}(Z, X' \otimes Y') \times (T X X' \times \mathcal{J}(Y', Y)) \\
  \cong \int^{X' \in |\mathcal{J}|} \mathcal{J}(Z, X' \otimes Y) \times T X X'
  \]
- Hence
  \[
  (\text{st}_{X,Y})_Z \in \int^{X' \in |\mathcal{J}|} \mathcal{J}(Z, X' \otimes Y) \times T X X' \rightarrow T (X \otimes Y) Z
  \]
  which is equivalent to having a map
  $\text{first}_{X',X,Y} \in T X X' \rightarrow T (X \otimes Y)(X' \otimes Y)$
Arrows = strong monads in $\text{Prof}$

- Arrows on a (small) category $\mathcal{J}$ are monoids in the category on the endoprofunctors on $\mathcal{J}$.
- Arrows are monads in the bicategory $\text{Prof}$ of (small) categories and profunctors.
Strong monads vs strong relative monads

- If $T$ is a strong monad on $(\mathbb{C}, I', \otimes')$, then $T^\flat = \text{df } T \cdot J$ is a strong relative monad on $(J, e, m)$.
- If $J$ is well-behaved, then
  if $T$ is a strong relative monad on $(J, e, m)$, then $T^\# = \text{df } \text{Lan}_J T$ is a strong monad on $(\mathbb{C}, I', \otimes')$.
- The adjunction

$$
\begin{array}{ccc}
[\mathbb{C}, \mathbb{C}] & \xrightarrow{T} & [J, \mathbb{C}] \\
\xleftarrow{\text{Lan}_J} & & \xleftarrow{\text{Lan}_J} \\
\end{array}
$$

lifts to an adjunction

$$
\begin{array}{ccc}
\text{StrMnd}(\mathbb{C}, I', \otimes) & \xrightarrow{T} & \text{StrMnd}(J, e, m) \\
\xleftarrow{(-)^\#} & & \xleftarrow{(-)^\#} \\
\end{array}
$$
Conclusions

- Adding strength to relative monads is not difficult.
- Key idea: $J$ must be a monoidal functor.
- Arrows become strong relative monads, are hence a natural structure.
  Hughes, Paterson got the axioms right without deriving arrows as an instance of something more general!
Future work

- Alternative descriptions of strong relative monads.
- Formalization in Agda.
- Arrow metalanguage (cf. Lindley, Wadler, Yallop 2010).