Interpretations as coalgebra morphisms

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Outline

1. **Starting point**
   - Logics as coalgebras
   - Objectives

2. **Strict refinement revisited**

3. **Category of Logics and interpretations**
   - Logical interpretation
   - The logics induced by the Frege relation
   - Interpretations as coalgebras morphisms

4. **Conclusions**
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4 Conclusions
Abstract definitions of logic

Abstract Logic as a consequence relation

\[ \mathcal{A} = \langle A, \vdash_A \rangle, \]

where \( \vdash_A : \mathcal{P}(A) \times A \) is a consequence relation in \( A \).

Abstract Logic as a closure operator

\[ \mathcal{A} = \langle A, C_A \rangle, \]

where \( C_A \) is a closure operator, i.e., a mapping \( C_A : \mathcal{P}(A) \to \mathcal{P}(A) \) such for that for all \( X, Y \subseteq A \),

1. \( X \subseteq C_A(X) \);
2. \( X \subseteq Y \Rightarrow C_A(X) \subseteq C_A(Y) \);
3. \( C_A(C_A(X)) = C_A(X) \).
Abstract definitions of logic

Abstract Logic as a closure system

\[ \mathcal{A} = \langle A, \mathcal{T}_A \rangle \]

where \( \mathcal{T}_A \) is a closure system on \( A \), i.e., a family \( \mathcal{F} \) of subsets of \( A \) closed under arbitrary intersections (here we consider \( \bigcap \emptyset = A \)).

Theorem

Let \( A \) be a set. For each closure operator \( C_A \) in \( A \) we can associate a closure system \( \mathcal{T}_A \) and, conversely, for each closure system \( \mathcal{T}_A \) a closure operator \( C_A \) in such way that they are mutually inverses of one another:

\[ C_A \mapsto \mathcal{T}_A := \{ X \subseteq A | C_A(X) = X \} \]
\[ \mathcal{T}_A \mapsto C_A(X) := \bigcap \{ T \in \mathcal{T}_A | X \subseteq T \} \]
Logics as coalgebras

Palmigiano shows in [Pal02]

- that an abstract logic can be represented by a coalgebra
- these coalgebras maps a formula into the set of its theories;
- the morphisms on that category correspond exactly to the usual morphisms between logics.
- the class of coalgebras that corresponds to abstract logics of empty signature defines a covariety.
Logics as coalgebras

closure system (contravariant) functor: is the functor that maps a set in the set of the closure systems over it and, each function $f : A \to B$, in the map

$$C(f) : \mathcal{C}(B) \xrightarrow{\mathcal{F}} \mathcal{C}(A) \mapsto \{f^{-1}[T] : T \in \mathcal{F}\}.$$ 

Let $\mathcal{A} = \langle A, \mathcal{T}_A \rangle$.

Fact [Pal02] 

$f$ is a logical morphism between two abstract logics iff it is a morphism between its underlying coalgebras.
Objectives

Logical interpretation on software development

- We introduced in [MMB09a, MMB09b, MMB10] a formalization of refinement on algebraic specifications based on logical interpretations;
- The formalization is suitable to deal with data encapsulation, decomposition of operations in atomic transactions, and on the reuse of specifications;

Aims

- The work aims to frame logical interpretation on the “logics as coalgebras” perspective;
- formalize refinement via interpretation on this setting;
Refinement by interpretation [MMB09a, MMB09b]

Interpretation

\( \tau : \text{Fm}(\Sigma) \rightarrow \mathcal{P}(\text{Fm}(\Sigma')) \) interprets \( SP \) if there is a specification \( SP' \) under \( \Sigma' \) such that:

- \( \forall \varphi \in \text{Fm}(\text{Sig}(SP)), SP \models \varphi \) iff \( SP' \models \tau(\varphi) \)

\( SP' \) is a refinement by the interpretation \( \tau \) of \( SP \) if

- \( \tau \) interprets \( SP \)
- \( \forall \varphi \in \text{Fm}(\text{Sig}(SP)), SP \models \varphi \) implies \( SP' \models \tau(\varphi) \)

Theorem (Characterization)

\( SP \rightarrow_\tau SP' \) if there is an interpretation \( SP^0 \) of \( SP \) such that \( SP^0 \leadsto SP' \).

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Strict refinement revisited

**Definition**

Let $\mathcal{A} = \langle A, C_A \rangle$ and $\mathcal{A}' = \langle A, C_{A'} \rangle$ be two abstract logics. $\mathcal{A} \sim \mathcal{A}'$, if for any $X \cup \{x\} \in A$, $x \in C_A(X) \Rightarrow x \in C_{A'}(X)$.

**Theorem**

$\mathcal{A} \sim \mathcal{A}'$ if $T_{\mathcal{A}'} \subseteq T_\mathcal{A}$.

**First intuition**

$$
A \xrightarrow{i} A \\
\downarrow \xi \downarrow \xi' \\
C(A) \xleftarrow{C(i)} C(A)
$$

However, this implies that $T_{\mathcal{A}'} = T_\mathcal{A}$ and we just need the first inclusion!
Definition (Forward morphism)

A forward morphism between \( \langle A, \alpha \rangle \) and \( \langle B, \beta \rangle \) with respect to a pre-order \( \sqsubseteq \), is a map \( h : A \to B \) such that \( C h \circ \beta \circ h \sqsubseteq \alpha \).

Theorem

\( \mathcal{A}' \) is a strict refinement of \( \mathcal{A} \) iff the inclusion map is a forward morphism from \( \langle A, \xi \rangle \) to \( \langle A, \xi' \rangle \) wrt \( \subseteq \).

Theorem

The tuple \( \langle \text{Log}, \text{ref}, i, \circ \rangle \), where

- \( \text{Log} \) is the class of \( C \)-coalgebras induced by abstract logics;
- \( \text{ref} \) is the class of its inclusion forward morphisms wrt \( \subseteq \);
- \( i \) is the class of identical maps;
- \( \circ \) is the composition of functions, defines a category.
Relating logics: Morphisms & Interpretations

**Definition (Logical morphism)**

A logical morphism between the logics $\mathcal{A} = \langle A, \mathcal{T}_A \rangle$ and $\mathcal{B} = \langle B, \mathcal{T}_B \rangle$ consists of an (algebraic) morphism $h : A \to B$ such that

$$\{ h^{-1}[T'] \mid T' \in \mathcal{T}_B \} = \mathcal{T}_A.$$
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Some preliminaries

Let $f : A \nrightarrow B$ be a multifunction

- **image:** $f[X] = \bigcup \{ f(a) | a \in X \}$;
- **inverse image:** $f^{-1}[Y] = \{ a \in A | f(a) \subseteq Y \}$

Let $\mathcal{A} = \langle A, C_A \rangle$ and $\mathcal{B} = \langle B, C_B \rangle$ two abstract logics. The multifunction $f : A \nrightarrow B$ is said to be

- **continuous** wrt $\mathcal{A}$ and $\mathcal{B}$ if for every $X \subseteq A$, $f[C_A(X)] \subseteq C_B(f[X])$
- **closed** if maps closed set wrt $\mathcal{A}$ in closed sets wrt $\mathcal{B}$;
The category of logics and interpretations

**Definition (Interpretation)**

Let $\mathcal{A} = \langle A, C_\mathcal{A} \rangle$ and $\mathcal{B} = \langle B, C_\mathcal{B} \rangle$ be two abstract logics. A multifunction $f : A \Rightarrow B$ is an interpretation, if for any $\{x\} \cup X \subseteq A,$

$$x \in C_\mathcal{A}(X) \Leftrightarrow f(x) \subseteq C_\mathcal{B}(f[X]).$$

**Lemma**

$f$ is an interpretation iff for any $X \subseteq A,$ $C_\mathcal{A}(X) = f^{-1}[C_\mathcal{B}(f[X])].$

**Lemma**

Let $\mathcal{A} = \langle A, C_\mathcal{A} \rangle$ and $\mathcal{B} = \langle B, C_\mathcal{B} \rangle$ be two abstract logics and $f : A \Rightarrow B$ a closed and continuous multifunction wrt $\mathcal{A}$ and $\mathcal{B}.$ TFAE:

1. $f$ is an interpretation from $\mathcal{A}$ into $\mathcal{B};$
2. for any closed set $T$ wrt $\mathcal{A},$ $T = f^{-1}[C_\mathcal{B}(f[T])].$
The category of logics and interpretations

**Theorem**

The tuple $\langle \text{Log}, \text{Int}, i, \circ \rangle$, where

- **Log** is the class of abstract logics;
- **Int** is the class of its interpretations;
- $i$ is the class of identical maps (for each abstract logic $\langle A, C_A \rangle$ the identical map $i_A : A \Rightarrow A$);
- $\circ$ is the composition of multifunctions,

defines a category.
Logic induced by the Frege relation

The abstract logic co-induced by \( f \) and \( A \) in \( B \) is defined as the abstract logic \( B = \langle B, C_f \rangle \), where \( C_f \) is such that \( \text{Th}B = \{ T | f^{-1}[T] \in \text{Th}A \} \)

- Frege relation: \( \sim_A = \{ \langle a, b \rangle \in A^2 | C_A(a) = C_A(b) \} \);
- Canonical epimorphism \( e : A \cong A/\sim, \text{such } e\sim(a) = [a]_\sim \).
- \( A_\sim := \langle A/\sim, C_e_\sim \rangle \);

Lemma

*For any abstract logic \( A = \langle A, C_A \rangle \), the multifunction \( e : A \cong A_\sim \) is an interpretation from \( A \) to \( A_\sim \).*

Theorem

*Let \( A = \langle A, C_A \rangle \) and \( B = \langle B, C_B \rangle \) be two abstract logics. Then there exists an interpretation \( f : A \cong B \) iff there exists an interpretation \( f^* : A_\sim \cong B_\sim \).*
Frame interpretation on the coalgebraic view

For $\mathcal{A} = \langle A, \mathcal{T}_A \rangle$:

In [Pal02]:

$\eta(a) = \{ T \in \mathcal{T}_A | a \in T \}$

Our aim:

$\xi(X) = \{ T \in \mathcal{T}_A | X \subseteq T \}$
Frame interpretation on the coalgebraic view

**Category \( \mathbf{Pw} \)**

Let \( \mathbf{Pw} \) be the category with

- \( \text{Obj}(\mathbf{Pw}) = \{ \mathcal{P}(X) | X \in \text{Obj}(\mathbf{Set}) \}; \)
- \( \text{Arrow}(\mathbf{Pw}) \) are the functions between \( \mathbf{Pw} \) objects.

**\( \bar{C} : \mathbf{Pw} \to \mathbf{Pw} \)**

\[
\bar{C}(X) := \{ S \subseteq X | S \text{ is a closure system} \}
\]

\[
\bar{C}(f) : \bar{C}(B) \quad \rightarrow \quad \bar{C}(A)
\]

\[
\mathcal{F} \quad \mapsto \quad \{ f^{-1}[T] : T \in \mathcal{F} \}.
\]

**Power-function**

A multifunction \( f : A \rightrightarrows B \) induces a function

\[
f^* : \mathcal{P}(A) \quad \rightarrow \quad \mathcal{P}(B)
\]

\[
X \quad \mapsto \quad \bigcup_{x \in X} f(x).
\]
Theorem

Let $\mathcal{A} = \langle A, \mathcal{T}_A \rangle$ and $\mathcal{B} = \langle B, \mathcal{T}_B \rangle$ be two abstract logics and $f : A \leadsto B$ an interpretation. Then, $\mathcal{T}_A = \{ f^{-1}[T] : T \in \mathcal{T}_B \}$. 

Corollary

Let $\mathcal{A} = \langle A, C_A \rangle$ and $\mathcal{B} = \langle B, C_B \rangle$ be two abstract logics and $\langle A, \xi \rangle, \langle B, \eta \rangle$ the coalgebras induced by them. Hence, if $f : A \leadsto B$ is an interpretation, then $f^*$ is a coalgebraic morphism between its logics, i.e., $f^*$ makes the following diagram to commute:

\[
\begin{array}{c}
\mathcal{A} \\
\downarrow \xi \downarrow \nu \downarrow \mathcal{C} A
\\
\mathbf{f}^* \\
\downarrow \mathcal{C}(f^*) \\
\mathcal{B} \\
\downarrow \eta \\
\mathbf{\bar{C}} B
\end{array}
\]
Theorem

Let $A = \langle A, T_A \rangle$ and $B = \langle B, T_B \rangle$ be two abstract logics and $f : A \supseteq B$ an interpretation. Then, $T_A = \{f^{-1}[T] : T \in T_B\}$.

Corollary

Let $A = \langle A, C_A \rangle$ and $B = \langle B, C_B \rangle$ be two abstract logics and $\langle A, \xi \rangle$, $\langle B, \eta \rangle$ the coalgebras induced by them. Hence, if $f : A \supseteq B$ is an interpretation, then $f^*$ is a coalgebraic morphism between its logics, i.e., $f^*$ makes the following diagram to commute:

$$
\begin{array}{ccc}
A & \xrightarrow{f^*} & B \\
\downarrow{\xi} & & \downarrow{\eta} \\
\bar{C}A & \xleftarrow{f^*} & \bar{C}B
\end{array}
$$

Theorem

Let $A = \langle A, C_A \rangle$ and $B = \langle B, C_B \rangle$ be two abstract logics and $f : A \supseteq B$ a closed and continuous multifunction. Then, $T^A = \{f^{-1}[T] : T \in T^B\}$ implies that $f$ is an interpretation.
Strict refinement

**Theorem (Characterization)**

\[ SP \xrightarrow{\tau} SP' \text{ if there is an interpretation } SP^0 \text{ of } SP \text{ such that } SP^0 \sim SP'. \]

**Strict refinements on** $Pw$

\[ A \xleftarrow{i} C(A) \xrightarrow{\xi} \subseteq \xrightarrow{\eta} C(A) \]

\[ \bar{C}(A) \xleftarrow{\bar{C}(i)} C(A) \]

**for** $A = \mathcal{P}(A)$

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Interpretations as coalgebra morphisms
Refinement via interpretation

**Theorem (Characterization)**

$SP ightarrow_{\tau} SP'$ if there is an interpretation $SP^0$ of $SP$ such that $SP^0 \sim SP'$.

\[ \begin{array}{c}
A \xrightarrow{int'} \cdots \xrightarrow{int} B \xrightarrow{ref} \cdots \xrightarrow{ref'} B \\
\uparrow SP \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow SP' \\
\bar{C}A \xleftarrow{\bar{C}(int')} \cdots \xleftarrow{\bar{C}(int)} \bar{C}B \xleftarrow{\bar{C}(ref)} \cdots \xleftarrow{\bar{C}(ref')} \bar{C}B
\end{array} \]
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Conclusions

- We generalize the coalgebraic perspective of logics presented in [Pal02], capturing the interpretations of logics with coalgebraic morphisms;
- taking this approach, we present an elegant formalization of the refinement via interpretation concept;

Directions to pursue

- An interpretation entails the existence of a bisimulation; what is the logical counterpart to the existence of $\langle \xi, \eta \rangle$-bisimulation?
  > rephrase this work in the relational setting.
- explore in the “logics as coalgebras” perspective
  > finitarity: $C(X) = \{ C(Y) : Y \subseteq X, Y \text{ finite} \}$
  > structurality: by considering the algebraic structure on underlying sets of the logics.

