Higher-order algebras and coalgebras from parameterized endofunctors

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1 **Basic Definitions**
   - Higher-order & parameterized endofunctors
   - Initial and final suitability

2 **Results**

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Higher-order algebras and coalgebras from parameterized endofunctors
Higher-order endofunctors

Definition

A *higher-order endofunctor* is an endofunctor on a functor category.
Higher-order & parameterized endofunctors

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Functor categories $[\mathcal{B}, \mathcal{C}]$, e.g.
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Functor categories \([\mathcal{B}, \mathcal{C}]\), e.g.

- Category, \(\mathcal{C} \cong [1, \mathcal{C}]\).
- Arrow category, \(\mathcal{C} \rightarrow \cong [2, \mathcal{C}]\).
Higher-order endofunctors

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Functor categories $[B, C]$, e.g.

- Category, $C \cong [1, C]$.
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- Endofunctor category, $\text{End}(C) = [C, C]$. 
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Functor categories $[\mathcal{B}, \mathcal{C}]$, e.g.

- Category, $\mathcal{C} \cong [1, \mathcal{C}]$.
- Arrow category, $\mathcal{C} \rightarrow \cong [2, \mathcal{C}]$.
- Endofunctor category, $\text{End}(\mathcal{C}) = [\mathcal{C}, \mathcal{C}]$.
- Monad category, $\text{Mon}(\mathcal{C})$ (abusing terminology slightly)
Parameterized Endofunctors

Definition

A *parameterized endofunctor* is a bifunctor of the type $\mathcal{B} \times \mathcal{C} \to \mathcal{C}$. 
Parameterized Endofunctors

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A parameterized endofunctor is a bifunctor of the type $\mathcal{B} \times \mathcal{C} \rightarrow \mathcal{C}$.

- By currying, a parameterized endofunctor has type $\mathcal{B} \rightarrow \text{End}(\mathcal{C})$. 
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- By currying, a parameterized endofunctor has type $\mathcal{B} \rightarrow \text{End}(\mathcal{C})$.
- There are more constrained notion of parameterized endofunctors, (e.g. structural actions (Blute-Cockett-Seely ’97), parameterized monads (Uustalu ’03, Atkey ’06), etc.)
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- There are more constrained notion of parameterized endofunctors, (e.g. structural actions (Blute-Cockett-Seely ’97), parameterized monads (Uustalu ’03, Atkey ’06), etc.)
- We use the unconstrained definition studied by Kurz and Pattinson ’00.
Parameterized endofunctors to higher-order endofunctors

Definition

For $F : \mathcal{B} \times \mathcal{C} \rightarrow \mathcal{C}$, let $\hat{F} : [\mathcal{B}, \mathcal{C}] \rightarrow [\mathcal{B}, \mathcal{C}]$ be given by

$$\hat{F}(X)(b) = F(b, Xb)$$

for $X : \mathcal{B} \rightarrow \mathcal{C}$ and $b \in \mathcal{B}$. $\hat{F}$ is the higher-order endofunctor generated by the parameterized endofunctor $F$. 
Parameterized endofunctors to higher-order endofunctors

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**Goal**

Characterize initial algebras and final coalgebras of these higher-order endofunctors in terms of lower-order properties.
Suitable Parameterized Endofunctors

Definition

A parameterized endofunctor $F : \mathcal{B} \times \mathcal{C} \to \mathcal{C}$ is

Initial and final suitability

Basic Definitions Results Applications Conclusions

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Higher-order algebras and coalgebras from parameterized endofunctors
Suitable Parameterized Endofunctors

**Definition**

A parameterized endofunctor $F : \mathcal{B} \times \mathcal{C} \to \mathcal{C}$ is

- *initially suitable* if $F(b, -)$ admits an initial algebra for any $b \in \mathcal{B}$. 
Suitable Parameterized Endofunctors

Definition

A parameterized endofunctor $F : \mathcal{B} \times \mathcal{C} \rightarrow \mathcal{C}$ is

- **initially suitable** if $F(b, -)$ admits an initial algebra for any $b \in \mathcal{B}$.
- **finally suitable** if $F(b, -)$ admits a final coalgebra for any $b \in \mathcal{B}$. 
Suitability to Higher-order Algebras and Coalgebras

An initially suitable parameterized endofunctor $F : \mathcal{B} \times \mathcal{C} \to \mathcal{C}$ induces a $\mathcal{C}$-endofunctor $R_F$:

$$F(x, R_F x) \xrightarrow{r_x} R_F x$$

$$F(x, R_F f) \xrightarrow{\text{\hspace{1cm}} F(f, R_F f) = F(-, R_F -) f} F(f, R_F f)$$

$$F(x, R_F y) \xrightarrow{F(f, R_F y)} F(y, R_F y) \xrightarrow{r_y} R_F y$$

$R_F f$ is induced by initiality.

$\hat{F}^{R_F} = F(-, R_F -) r = \Rightarrow R_F$ is a natural isomorphism.

$\hat{F}^{R_F} = \Rightarrow R_F$ is an $\hat{F}$-algebra!
Suitability to Higher-order Algebras and Coalgebras

An initially suitable parameterized endofunctor $F : \mathcal{B} \times \mathcal{C} \rightarrow \mathcal{C}$ induces a $\mathcal{C}$-endofunctor $\mathcal{R}_F$:

$$F(x, \mathcal{R}_F x) \xrightarrow{r_x} \mathcal{R}_F x$$

$$F(x, \mathcal{R}_F f) \xrightarrow{F(f, \mathcal{R}_F f) = F(-, \mathcal{R}_F -) f} \xrightarrow{\mathcal{R}_F f}$$

$$F(x, \mathcal{R}_F y) \xrightarrow{F(f, \mathcal{R}_F y)} F(y, \mathcal{R}_F y) \xrightarrow{r_y} \mathcal{R}_F y$$

$F(x, \mathcal{R}_F x) \xrightarrow{r_x} \mathcal{R}_F x$ is the initial $F(x, -)$-algebra.
Suitability to Higher-order Algebras and Coalgebras

An initially suitable parameterized endofunctor $F: B \times C \to C$ induces a $C$-endofunctor $\mathcal{R}_F$:

\[
F(x, \mathcal{R}_Fx) \xrightarrow{r_x} \mathcal{R}_Fx
\]

\[
F(x, \mathcal{R}_Ff) \xrightarrow{F(f, \mathcal{R}_Ff)_x} F(\mathcal{R}_Ff_x) = F(\mathcal{R}_F(\mathcal{R}_Ff)_x)
\]

\[
F(x, \mathcal{R}_Fy) \xrightarrow{F(f, \mathcal{R}_Fy)} F(y, \mathcal{R}_Fy) \xrightarrow{r_y} \mathcal{R}_Fy
\]

- $F(x, \mathcal{R}_Fx) \xrightarrow{r_x} \mathcal{R}_Fx$ is the initial $F(x, -)$-algebra.
- $\mathcal{R}_Ff$ is induced by initiality.
Suitability to Higher-order Algebras and Coalgebras

An initially suitable parameterized endofunctor $F : \mathcal{B} \times \mathcal{C} \rightarrow \mathcal{C}$ induces a $\mathcal{C}$-endofunctor $\mathcal{R}_F$:

$F(x, \mathcal{R}_F x) \xrightarrow{r_x} \mathcal{R}_F x$

$F(x, R_F f) \xrightarrow{=} F(f, \mathcal{R}_F f) = F(-, \mathcal{R}_F -) f$

$F(x, \mathcal{R}_F y) \xrightarrow{F(f, \mathcal{R}_F y)} F(y, \mathcal{R}_F y) \xrightarrow{r_y} \mathcal{R}_F y$

- $F(x, \mathcal{R}_F x) \xrightarrow{r_x} \mathcal{R}_F x$ is the initial $F(x, -)$-algebra.
- $\mathcal{R}_F f$ is induced by initiality.
- $\hat{F}(R_F) = F(-, \mathcal{R}_F -) \xrightarrow{r} \mathcal{R}_F$ is a natural isomorphism.
Suitability to Higher-order Algebras and Coalgebras

An initially suitable parameterized endofunctor $F : B \times C \to C$ induces a $C$-endofunctor $R_F$:

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\begin{align*}
F(x, R_F x) &\xrightarrow{r_x} R_F x \\
F(x, R_F f) &\xrightarrow{F(f, R_F f)} F(-, R_F -) f \quad \text{induced by initiality.}
\end{align*}
\]

\[
\begin{align*}
F(x, R_F y) &\xrightarrow{F(f, R_F y)} F(y, R_F y) \\
F(f, R_F y) &\xrightarrow{F(f, -)} R_F f
\end{align*}
\]

- $F(x, R_F x) \xrightarrow{r_x} R_F x$ is the initial $F(x, -)$-algebra.
- $R_F f$ is induced by initiality.
- $\hat{F}(R_F) = F(-, R_F -) \xrightarrow{r} R_F$ is a natural isomorphism.
- $\hat{F}R_F \xrightarrow{r} R_F$ is an $\hat{F}$-algebra!
Suitability to Higher-order Algebras and Coalgebras

An initially suitable parameterized endofunctor $F : \mathcal{B} \times \mathcal{C} \to \mathcal{C}$ induces a $\mathcal{C}$-endofunctor $\mathcal{R}_F$:

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\begin{align*}
F(x, \mathcal{R}_F x) & \xrightarrow{r_x} \mathcal{R}_F x \\
F(x, \mathcal{R}_F f) & \xrightarrow{F(f, \mathcal{R}_F f) = F(-, \mathcal{R}_F -) f} \mathcal{R}_F f \\
F(x, \mathcal{R}_F y) & \xrightarrow{F(f, \mathcal{R}_F y)} F(y, \mathcal{R}_F y) \xrightarrow{r_y} \mathcal{R}_F y
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- $F(x, \mathcal{R}_F x) \xrightarrow{r_x} \mathcal{R}_F x$ is the initial $F(x, -)$-algebra.
- $\mathcal{R}_F f$ is induced by initiality.
- $\hat{F}(\mathcal{R}_F) = F(-, \mathcal{R}_F -) \Rightarrow \mathcal{R}_F$ is a natural isomorphism.
- $\hat{F}\mathcal{R}_F \Rightarrow \mathcal{R}_F$ is an $\hat{F}$-algebra!
- Let $S_F \Rightarrow \hat{F}S_F$ be the $\hat{F}$-coalgebra induced by finally suitable parameterized endofunctors.
The Punchline

Theorem (J. Kim '09)

Let \( \hat{F} \) be a \([B, C]\)-endofunctor generated by a parameterized endofunctor \( F : B \times C \to C \). The following are equivalent:

1. \( F \) is initially (resp. finally) suitable.
2. \( \hat{F} \) admits an initial algebra (resp. final coalgebra).
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Let \( \hat{F} \) be a \([\mathcal{B}, \mathcal{C}]\)-endofunctor generated by a parameterized endofunctor \( F : \mathcal{B} \times \mathcal{C} \to \mathcal{C} \). The following are equivalent:

1. \( F \) is initially (resp. finally) suitable.
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If \( F \) is initially suitable, \( \hat{F}\mathcal{R}_F \xrightarrow{r} \mathcal{R}_F \) is the initial \( \hat{F} \)-algebra.
The Punchline

Theorem (J.Kim ’09)

Let $\hat{F}$ be a $[\mathcal{B}, \mathcal{C}]$-endofunctor generated by a parameterized endofunctor $F: \mathcal{B} \times \mathcal{C} \to \mathcal{C}$. The following are equivalent:

1. $F$ is initially (resp. finally) suitable.
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- If $F$ is initially suitable, $\hat{F}R_F \Rightarrow R_F$ is the initial $\hat{F}$-algebra.
- If $F$ is finally suitable, $S_F \Rightarrow \hat{F}S_F$ is the final $\hat{F}$-coalgebra.
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- If $F$ is initially suitable, $\hat{F}R_F \Rightarrow R_F$ is the initial $\hat{F}$-algebra.
- If $F$ is finally suitable, $S_F \Rightarrow \hat{F}S_F$ is the final $\hat{F}$-coalgebra.
- The result can be specialized to “parameterized monads” $F : \mathcal{B} \to \text{Mon}(\mathcal{C})$. A monad structure can be imposed on the higher-order $[\mathcal{B}, \mathcal{C}]$-endofunctor $\hat{F}$.
An Example of a Parameterized Endofunctor I

Example

Let $G_1 \xrightarrow{\theta} G_0$ be a natural transformation between two $C$-endofunctors. Let $D: 2 \times C \rightarrow C$ be given by

$$D(i, x) = G_i(x) \quad \text{and} \quad D(!, x) = \theta x$$

for $i \in 2, x \in C$. Recall $2 = \begin{array}{c}
0 \\
\downarrow \theta \\
1
\end{array}$.
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for $i \in 2$, $x \in \mathcal{C}$. Recall $2 = \begin{array}{c} 0 \ \leftarrow \ 1 \\ \id_0 \ \ id_1 \end{array}$.

$D$ is initially suitable if $G_0$ and $G_1$ admit initial algebras.
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for $i \in 2$, $x \in \mathcal{C}$. Recall $2 = \begin{array}{c} 0 \\ \downarrow \text{id}_0 \\ 1 \end{array}$.

- $D$ is initially suitable if $G_0$ and $G_1$ admit initial algebras.
- $D$ is finally suitable if $G_0$ and $G_1$ admit final coalgebras.
An Example of a Parameterized Endofunctor I

Example

Let $G_1 \xrightarrow{\theta} G_0$ be a natural transformation between two $\mathcal{C}$-endofunctors. Let $D: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ be given by

$$D(i, x) = G_i(x) \quad \text{and} \quad D(!, x) = \theta_x$$

for $i \in \{0, 1\}, x \in \mathcal{C}$. Recall $2 = \begin{array}{c} \vdots \\ \downarrow \id_0 \\ \downarrow \id_1 \\ \vdots \end{array} 0 \leftarrow ! \rightarrow \begin{array}{c} \vdots \\ \downarrow \id_0 \\ \downarrow \id_1 \\ \vdots \end{array} 1$.

- $D$ is initially suitable if $G_0$ and $G_1$ admit initial algebras.
- $D$ is finally suitable if $G_0$ and $G_1$ admit final coalgebras.
- The initiality version of the theorem generalizes a result by Chuang and Lin ’06, proved for arrow categories.
An Example of a Parameterized Endofunctor II

Example

Let $H$ be a $C$-endofunctor. Let $E : C \times C \to C$ be given by

$$E(a, x) = a + Hx$$

for $a, x \in C$.
An Example of a Parameterized Endofunctor II

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Let $H$ be a $C$-endofunctor. Let $E : C \times C \to C$ be given by

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- $\mathcal{R}_E$ is the free monad generated by $H$. 

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for $a, x \in C$.

- $R_E$ is the free monad generated by $H$.
- $S_E$ is the completely iterative monad generated by $H$. 
An Example of a Parameterized Endofunctor II

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Let \( H \) be a \( C \)-endofunctor. Let \( E : C \times C \to C \) be given by

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- \( R_E \) is the free monad generated by \( H \).
- \( S_E \) is the completely iterative monad generated by \( H \).
- \( E \) is finally suitable \( \iff \) \( E \) is iterable
An Example of a Parameterized Endofunctor II

Example

Let $H$ be a $\mathcal{C}$-endofunctor. Let $E : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ be given by

$$E(a, x) = a + Hx$$

for $a, x \in \mathcal{C}$.

- $\mathcal{R}_E$ is the free monad generated by $H$.
- $S_E$ is the completely iterative monad generated by $H$.
- $E$ is finally suitable $\iff$ $E$ is iterable.
- The finality version of the theorem generalizes a result by Aczel, Adámek, Milius, Velebil ’03, proved for iterable endofunctors.
Example

Let $A, B$ be nonempty sets. Let $F : (\text{Set}^{\text{op}} \times \text{Set}) \times \text{Set} \to \text{Set}$ be given by $F(\langle A, B \rangle, C) = (B \times C)^A$. 
An Example of a Parameterized Endofunctor III

Example

Let $A, B$ be nonempty sets. Let $F : (\text{Set}^{\text{op}} \times \text{Set}) \times \text{Set} \rightarrow \text{Set}$ be given by $F(\langle A, B \rangle, C) = (B \times C)^A$.

- $F$ is finally suitable.
An Example of a Parameterized Endofunctor III

Example

Let $A, B$ be nonempty sets. Let $F : (\text{Set}^{\text{op}} \times \text{Set}) \times \text{Set} \to \text{Set}$ be given by $F(\langle A, B \rangle, C) = (B \times C)^A$.

- $F$ is finally suitable.
- $S_F\langle A, B \rangle = \Gamma_{A,B} = \{ f : A^\omega \to B^\omega : f \text{ causal} \}$. Let $\Gamma_{A,B} \xrightarrow{\gamma_{A,B}} (B \times \Gamma_{A,B})^A$ be given by

  $$\gamma_{A,B}(f)(a) = \langle \text{hd} \circ f \circ c_a, \text{tl} \circ f \circ c_a \rangle$$

  for $f \in \Gamma_{A,B}$, $a \in A$, $c_a(\sigma) = a:\sigma$. (Rutten)
An Example of a Parameterized Endofunctor III

Example

Let $A, B$ be nonempty sets. Let $F: (\text{Set}^{\text{op}} \times \text{Set}) \times \text{Set} \to \text{Set}$ be given by $F(\langle A, B \rangle, C) = (B \times C)^A$.

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for $f \in \Gamma_{A,B}$, $a \in A$, $c_a(\sigma) = a:\sigma$. (Rutten)

- The first coordinate $\text{hd} \circ f \circ c_a$ is a constant in $B$ since $f$ is causal.
map as a higher-order coalgebra morphism I

Recall the final $\hat{F}$-coalgebra
map as a higher-order coalgebra morphism

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$\hat{F}$ is a $[\text{Set}^{\text{op}} \times \text{Set}, \text{Set}]$-endofunctor.
map as a higher-order coalgebra morphism I

Recall the final $\hat{F}$-coalgebra

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- Let $S_F = \Gamma : \text{Set}^{\text{op}} \times \text{Set} \to \text{Set}$ be given by $\Gamma(A, B) = \Gamma_{A,B}$. 
map as a higher-order coalgebra morphism I

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- $\Gamma \xrightarrow{\gamma} \hat{F}(\Gamma) = F(\cdot, \Gamma\cdot)$ is the final $\hat{F}$-coalgebra.
map as a higher-order coalgebra morphism

Recall the final $\hat{F}$-coalgebra

- $\hat{F}$ is a $[\text{Set}^{\text{op}} \times \text{Set}, \text{Set}]$-endofunctor.
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- $\Gamma \mapsto \hat{F}(\Gamma) = F(-, \Gamma -)$ is the final $\hat{F}$-coalgebra.

Define another $\hat{F}$-coalgebra
map as a higher-order coalgebra morphism I

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- $\hat{F}$ is a $[\text{Set}^{\text{op}} \times \text{Set}, \text{Set}]$-endofunctor.
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- $\Gamma \xrightarrow{\gamma} \hat{F}(\Gamma) = F(-, \Gamma-)$ is the final $\hat{F}$-coalgebra.

Define another $\hat{F}$-coalgebra

- Let $\text{Hom} : \text{Set}^{\text{op}} \times \text{Set} \to \text{Set}$ be the usual hom-bifunctor.
map as a higher-order coalgebra morphism I

Recall the final \( \hat{F} \)-coalgebra

- \( \hat{F} \) is a \([\text{Set}^{\text{op}} \times \text{Set}, \text{Set}]\)-endofunctor.
- Let \( S_F = \Gamma : \text{Set}^{\text{op}} \times \text{Set} \to \text{Set} \) be given by \( \Gamma(A, B) = \Gamma_{A,B} \).
- \( \Gamma \xrightarrow{\gamma} \hat{F}(\Gamma) = F(\_ \_, \Gamma\_\_) \) is the final \( \hat{F} \)-coalgebra.

Define another \( \hat{F} \)-coalgebra

- Let \( \text{Hom} : \text{Set}^{\text{op}} \times \text{Set} \to \text{Set} \) be the usual hom-bifunctor.
- Let \( \text{Hom} \xrightarrow{\varepsilon} \hat{F} \text{Hom} = F(\_ \_, \text{Hom}\_\_) \) be a \( \hat{F} \)-coalgebra where the components

\[
\text{Hom}(A, B) \xrightarrow{e_{\langle A,B \rangle}} (B \times \text{Hom}(A, B))^A
\]

is given by

\[
e_{\langle A,B \rangle}(f)(a) = \langle f(a), f \rangle.
\]
By finality of $\gamma$, there is a higher-order coalgebra morphism
\[ \text{Hom} \xrightarrow{m} \Gamma: \]
map as a higher-order coalgebra morphism II

By finality of \( \gamma \), there is a higher-order coalgebra morphism
\[
\Hom \xrightarrow{m} \Gamma:
\]

\[
\begin{array}{c}
\text{Hom}(A, B) \xrightarrow{e^{A,B}} (B \times \text{Hom}(A, B))^A \\
m_{A,B} \\
\Gamma_{A,B} \xrightarrow{\gamma_{A,B}} (B \times \Gamma_{A,B})^A \\
\end{array}
\]
map as a higher-order coalgebra morphism II

By finality of $\gamma$, there is a higher-order coalgebra morphism $\text{Hom} \xrightarrow{m} \Gamma$:

- $\text{Hom}(A, B) \xrightarrow{e_{\langle A, B \rangle}} (B \times \text{Hom}(A, B))^A$

  $m_{A,B} \downarrow \quad \downarrow (B \times m_{A,B})^A$

- $\Gamma_{A,B} \xrightarrow{\gamma_{A,B}} (B \times \Gamma_{A,B})^A$

- $m_{A,B}(f)(\sigma_0, \sigma_1, \sigma_2, \ldots) = (f(\sigma_0), f(\sigma_1), f(\sigma_2), \ldots)$
map as a higher-order coalgebra morphism II

By finality of $\gamma$, there is a higher-order coalgebra morphism $\text{Hom} \xrightarrow{m} \Gamma$:

- $\text{Hom}(A, B) \xrightarrow{e_{\langle A, B \rangle}} (B \times \text{Hom}(A, B))^A$
- $m_{A,B} \downarrow \downarrow$
- $\Gamma_{A,B} \xrightarrow{\gamma_{A,B}} (B \times \Gamma_{A,B})^A$

- $m_{A,B}(f)(\sigma_0, \sigma_1, \sigma_2, \ldots) = (f(\sigma_0), f(\sigma_1), f(\sigma_2), \ldots)$
- $m_{A,B}$ is map!
Overview & Conclusions

Overview

Higher-order endofunctors & parameterized endofunctors

Characterization of initial algebras and final coalgebras for higher-order endofunctors generated by parameterized endofunctors.

Generalization of known results.

Derivation of map as a higher-order coalgebra morphism.

Overview

Algebraic and coalgebraic properties of higher-order endofunctors should be studied.

Particulars of other "constrained" functor categories should be studied.

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Higher-order algebras and coalgebras from parameterized endofunctors
Overview & Conclusions

Overview

1. Higher-order endofunctors & parameterized endofunctors
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2. Characterization of initial algebras and final coalgebras for higher-order endofunctors generated by parameterized endofunctors.
Overview & Conclusions

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Conclusions

1. Algebraic and coalgebraic properties of higher-order endofunctors should be studied.
2. Particulars of other “constrained” functor categories should be studied.