Coalgebras and Modal Logics: an Overview

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CMCS 2010, Paphos, Cyprus
Part I: Examples

or:

Why should I care?
A Cook’s Tour Through Modal Semantics

Kripke Frames.

\[ C \rightarrow \mathcal{P}(C) \]

Multigraph Frames.

\[ C \rightarrow \mathcal{B}(C) \]

\[ \mathcal{B}(X) = \{ f : X \rightarrow \mathbb{N} \mid \text{supp}(f) \text{ finite} \} \]

Probabilistic Frames.

\[ C \rightarrow \mathcal{D}(C) \]

\[ \mathcal{D}(X) = \{ \mu : X \rightarrow [0, 1] \mid \sum_{x \in X} \mu(x) = 1 \} \]
Neighbourhood Frames.

\[ C \rightarrow \mathcal{P}\mathcal{P}(C) = \mathcal{N}(C) \]

mapping each world \( c \in C \) to a set of neighbourhoods

Game Frames over a set \( N \) of agents

\[ C \rightarrow \left\{ ((S_n)_{n \in N}, f) \mid f : \prod_n S_n \rightarrow C \right\} = \mathcal{G}(C) \]

associating to each state \( c \in C \) a strategic game with strategy sets \( S_n \) and outcome function \( f \)

Conditional Frames.

\[ C \rightarrow \left\{ f : \mathcal{P}(C) \rightarrow \mathcal{P}(C) \mid f \text{ a function} \right\} = \mathcal{C}(C) \]

where every state yields a selection function that assigns properties to conditions
Coalgebras and Modalities: A Non-Definition

**Coalgebras** are about *successors*. $T$-coalgebras are pairs $(C, \gamma)$ where

$$\gamma : C \to TC$$

maps states to successors. Write $\text{Coalg}(T)$ for the collection of $T$-coalgebras.

<table>
<thead>
<tr>
<th>states = elements $c \in C$</th>
<th>properties of states = subsets $A \subseteq C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>successors = elements $\gamma(c) \in TC$</td>
<td>properties of successors = subsets $\Diamond A \subseteq TC$</td>
</tr>
</tbody>
</table>

**Modal Operators** are about *properties* of successors, so

$$[\phi_1], \ldots, [\phi_n] \subseteq C$$

$$[\Diamond(\phi_1, \ldots, \phi_n)] \subseteq TC$$

with the intended interpretation $c \models \Diamond(\phi_1, \ldots, \phi_n)$ iff $\gamma(c) \in [\Diamond \phi_1, \ldots, \phi_n]$. 
Part II: Approaches to Syntax and Semantics

or:

What’s a modal operator?
Moss’ Coalgebraic Logic: The Synthetic Approach

Idea. ♦ reflects the action of $T$ on sets: ‘import’ semantics into syntax

Concrete Syntax

- $\Phi \subseteq_f L \quad \phi \in L \quad \Phi \in T_\omega L$
- $\land \Phi \in L \quad \neg \phi \in L \quad \nabla \Phi \in L$

Abstract Syntax:

$L \cong F(L) = \mathcal{P}_f(L) + L + T_\omega(L)$

Modal Semantics

$c \models \nabla \Phi \iff (\gamma(c), \Phi) \in T(\models)$

Algebraic Semantics

$$F(L) \longrightarrow F(\mathcal{P}(C))$$

relative to $T$-coalgebra $(C, \gamma : C \rightarrow TC')$ where $T_\omega$ is the finitary part of $T$
Synthetic Semantics Explained

Relation Lifting: from *states* to *successors*

\[
\begin{array}{ccc}
R & \rightarrow & TR \\
\pi_1 & \downarrow & T\pi_1 \\
X & \rightarrow & TX \\
\pi_2 & \downarrow & T\pi_2 \\
Y & \rightarrow & TY
\end{array}
\]

Formal Definition. (Assume \(T\) preserves weak pullbacks to make things work)

\[\overline{TR} = \{(T\pi_1(w), T\pi_2(w)) \mid w \in TR\} \subseteq TX \times TY\]

Modal Semantics. Assume that \(\models\) is already given for ‘ingredients’ of \(\alpha \in TL\)

\[c \models \nabla \alpha \iff (\gamma(c), \alpha) \in \overline{T}(\models)\]

for \(c \in C\) and \((C, \gamma : C \rightarrow TC) \in \text{Coalg}(T)\).

Thm. [Moss, 1999] \(L\) has the Hennessy-Milner Property.
Example: Coalgebraic Logic of Multigraphs

Modal Operators for $BX = \{ f : X \to \mathbb{N} \mid \text{supp}(f) \text{ finite} \}$

\[ \alpha : L \to \mathbb{N} \text{ and supp}(\alpha) \text{ finite} \]
\[ \nabla \alpha \in L \]

Satisfaction. $c \models \nabla \alpha \iff (\gamma(c), \alpha) \in T(\models) \iff$ the ‘magic square’

\[
\begin{array}{cccc|c}
\phi_1 & x_1 & x_2 & \cdots & x_k & \sum \\
\vdots & m_1 & m_2 & \cdots & m_n & w_1 \\
\phi_n & w_n & & & & w_n \\
\hline
\sum & m_1 & m_2 & \cdots & m_n & \end{array}
\]

- $m_j = \gamma(c)(x_j)$ is multiplicity of $x_j$
- $w_i = \alpha(\phi_i)$ is weight of $\phi_i$
- $\phi/\phi$-entry is 0 if $x \not\models \phi$

can be filled according to the rules on the right.
Synthetic Semantics, Algebraically

Syntax as initial algebra. \( L \cong \mathcal{P}_f(L) + LT(L) \)

Semantics as algebra morphism

\[
\begin{array}{c}
\mathcal{P}_f(L) + L + TL \\
\downarrow i \quad \downarrow 1+1+\rho_C \\
\mathcal{P}_f(L) + \mathcal{P}(C') + \mathcal{P}(TC') \\
\downarrow [\mathcal{T},(\cdot)^c,\gamma^{-1}] \\
\mathcal{P}(C)
\end{array}
\]

where \( \rho_C : T\mathcal{P}(C) \to \mathcal{P}(TC) \) is ’lifted membership’, i.e.

\[
\rho_C(\Phi) = \{ t \in TC \mid (t, \Phi) \in \mathcal{T}(\epsilon) \}
\]

where \( \epsilon_C \subseteq C \times \mathcal{P}(C) \) is membership (for \( T = \mathcal{B} \) a ’magic square’ problem)
Logics via Liftings: The Organic Approach

Idea. take $\diamondsuit$ what we want it to mean: grow your own modalities

$T$-Structures then define the semantics of modalities: they assign a nbhd frame translation or, equivalently, a predicate lifting

$$[\diamondsuit]: TC \rightarrow \mathcal{P}(\mathcal{P}(C)^n)$$

$$[\diamondsuit]: \mathcal{P}(C)^n \rightarrow \mathcal{P}(TC)$$

to every modal operator $\diamondsuit$ of the language, parametric in $C$.

Together with a $T$-coalgebra $(C, \gamma)$ this gives (in the unary case) a neighbourhood frame boolean algebra with operator

$$C \xrightarrow{\gamma} TC \xrightarrow{[\diamondsuit]} \mathcal{P}\mathcal{P}(C)$$

$$\mathcal{P}(C) \xrightarrow{[\diamondsuit]} \mathcal{P}(TC) \xrightarrow{\gamma^{-1}} \mathcal{P}(C)$$

Induced Coalgebraic Semantics $[\phi] \subseteq C$ of a modal formula


from a modal perspective equivalent algebraic viewpoint

$$c \in [\diamondsuit \phi] \iff [\phi] \in [\diamondsuit] \circ \gamma([\phi])$$

$$c \in [\diamondsuit \phi] \iff \gamma(c) \in [\diamondsuit]([\phi])$$

May 26, 2010
**Example: The Logic of Multigraphs**

**Modal Operators** for $BX = \{ \mu : X \rightarrow \mathbb{N} \mid \text{supp}(\mu) \text{ finite} \}$

**Our Choice.** $\bigotimes(\phi, \psi)$, intended meaning ‘at least 5 times as much $\phi$’s than $\psi$’s’

**Associated Lifting.**

$$\llbracket \bigotimes \rrbracket_X (A, B) = \{ \mu \in BX \mid \mu(A) \geq 5 \cdot \mu(B) \}$$

where $\mu(A) = \sum_{x \in A} \mu(x)$

**Satisfaction.**

$$c \models \bigotimes(\phi, \psi) \iff \mu(\llbracket \phi \rrbracket) \geq 5 \cdot \mu(\llbracket \psi \rrbracket)$$

where $\mu = \gamma(c)$ is the local weighting as seen from point $c$.

(i.e. one can pick and choose the primitives but has to define their meaning)
Part III: Reasoning in Coalgebraic Logics

or:

What’s a good proof system?
Recall. Semantics as algebra morphism

\[ \mathcal{P}_f(L) + L + TL \xrightarrow{i} \mathcal{P}_f(C') + \mathcal{P}(C') + T\mathcal{P}(C') \]

\[ \downarrow \]

\[ \mathcal{P}_f(C') + \mathcal{P}(C') + \mathcal{T}\mathcal{P}(C') \]

\[ \downarrow \]

\[ [\mathcal{P}, (\cdot)^c, \gamma^{-1}] \]

\[ \mathcal{P}(C') \]

where \( \rho_C : T\mathcal{P}(C) \to \mathcal{P}(TC) \) is \( \rho_C(\Phi) = \{ t \in TC \mid (t, \Phi) \in \overline{T}(\in) \} \)

Slim Redistributions. 'import' the action of \( \rho \) into the proof system.

\[ \Phi \in T\mathcal{P}(X) \text{ redistribution of } A \in \mathcal{P}(TX) \iff A \subseteq \rho_X(\Phi) \]

Call \( \Phi \) slim if \( \Phi \in \mathcal{P}_\omega T_\omega(A) \) (i.e. \( \Phi \) only re-arranges material from \( A \))

Notation. \( \text{SRD}(A) = \{ \Phi \in T\mathcal{P}(A) \mid \Phi \text{ slim redistribution of } A \} \)
Redistributions of Multisets

**Redistributions** of $\mathcal{B}X = \{ f : X \to \mathbb{N} \mid \text{supp}(f) \text{ finite} \}$

$\Phi : \mathcal{P}(X) \to f \mathbb{N} \in \mathcal{B}\mathcal{P}X$ *redistribution* of $A \in \mathcal{P}(X \to f \mathbb{N}) = \mathcal{P}(\mathcal{B}X)$

$A$ only contains $f : X \to f \mathbb{N}$ that allow to fill the 'magic square'

<table>
<thead>
<tr>
<th></th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$\cdots$</th>
<th>$x_k$</th>
<th>$\sum$</th>
</tr>
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<tbody>
<tr>
<td>$S_1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$w_1$</td>
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<tr>
<td>$S_n$</td>
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</tr>
<tr>
<td>$\sum$</td>
<td>$m_1$</td>
<td>$m_2$</td>
<td>$\cdots$</td>
<td>$m_n$</td>
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</table>

- $x/S$-entry is 0 if $x \notin S$
- $m_j$ is $f$-multiplicity of $x_j$
- $w_i$ is $\Phi$-weight of $S_i$

$\Phi$ is *slim* if each nonzero $S_i$ only contains nonzero $x_j$'s relative to some element of $A$
The Synthetic Proof System

Synthetic Proofs.

- judgements are inequalities \( a \leq b \) for \( a, b \in L \)
- propositional logic and cut: from \( a \leq b \) and \( b \leq c \) infer \( a \leq c \)

Modal Proof Rules.

\[
\begin{align*}
(\nabla 1) & \quad \frac{a \leq b}{\nabla \alpha \leq \nabla \beta} \\
(\nabla 4) & \quad \frac{\{a \land \nabla \alpha' \leq \bot \mid \alpha' \in T_\omega(\phi) \setminus \{\alpha\}\}}{a \leq \nabla \alpha} \\
(\nabla 2) & \quad \frac{\{\nabla (T \land)(\Phi) \leq a \mid \Phi \in \text{SRD}(A)\}}{\land\{\nabla \alpha \mid \alpha \in A\} \leq a} \\
(\nabla 3) & \quad \frac{\{\nabla \alpha \leq a \mid (\alpha, \Phi) \in \overline{T}(\in)\}}{\nabla (T \lor) \Phi \leq a}
\end{align*}
\]

where \( a \in L, \alpha, \beta \in T_\omega L, A \in \mathcal{P}_\omega T_\omega(L) \) and \( \Phi \in T_\omega \mathcal{P}_\omega(L) \).

Thm. [Kupke, Kurz, Venema 2009] The synthetic system is sound and complete over \( T \)-coalgebras.
Recall. Language $L$ given by operators $\Diamond$, semantics by $[[\Diamond]] : \mathcal{P}(X) \rightarrow \mathcal{P}(TX)$

**Proof Systems** in terms of sequents: $\Gamma \subseteq L$ with $[[\Gamma]] = \bigcup \{[[A]] \mid A \in \Gamma\}$

**One-step Rules** (*specific* for each choice of $\Diamond$'s)

$\Gamma_1 \ldots \Gamma_n \sim \Gamma_0 \sim \frac{\text{property of states}}{\text{property of successors}} \quad \frac{[[\Gamma_1]] \cap \cdots \cap [[\Gamma_n]] \subseteq X}{[[\Gamma_0]] \subseteq TX}$

where

- $\Gamma_1, \ldots, \Gamma_n \subseteq V \cup \neg V$ are propositional over a set $V$ of variables
- $\Gamma_0 \subseteq \{\Diamond(p_1, \ldots, p_n) \mid \Diamond n\text{-ary}\} \cup \{\neg\Diamond(p_1, \ldots, p_n) \mid \Diamond n\text{-ary}\}$

**Crucial:** need *Coherence Conditions* between proof rules and semantics
Organic Modalities: Coherence Conditions

Consider a set $X$ and a valuation $\tau : V \rightarrow P(X)$.

**Coherence**: matching between rules and semantics at one-step level

**Propositional Sequents** $\Gamma \subseteq V \cup \neg V$

$$\Gamma \ \tau\text{-valid} \iff \llbracket \Gamma \rrbracket_\tau = X \text{ where } \llbracket p \rrbracket_\tau = \tau(p)$$

**Modalised Sequents** $\Gamma \subseteq \{\pm \bigtriangledown (p_1, \ldots, p_n) \mid \bigtriangledown \ n\text{-ary} \}$

$$\Gamma \ \tau\text{-valid} \iff \llbracket \Gamma \rrbracket_\tau = TX \text{ where } \llbracket \bigtriangledown (p_1, \ldots, p_n) \rrbracket_\tau = \llbracket \bigtriangledown \rrbracket(\tau(p_1), \ldots, \tau(p_n))$$

where $\pm$ indicates possible negation.

**Coherence** relates $\tau$-validity of premises with $\tau$-validity of conclusions
Organic Modalities: Coherence Conditions

**One-Step Soundness** of a set $\mathcal{R}$ of one-step rules: for all $\tau : V \rightarrow \mathcal{P}(X)$

\[ \Gamma_1, \ldots, \Gamma_n \text{-valid} \implies \Gamma_0 \text{-valid} \]

for all $\Gamma_1 \ldots \Gamma_n/\Gamma_0 \in \mathcal{R}$

**One-Step Completeness** of a set $\mathcal{R}$ of one-step rules: for all $\tau : V \rightarrow \mathcal{P}(X)$

\[ \Gamma \text{-valid} \implies \exists \frac{\Gamma_1 \ldots \Gamma_n}{\Gamma_0} \in \mathcal{R} \left( \Gamma_i \sigma \text{-valid and } \Gamma_0 \sigma \subseteq \Gamma \right) \]

for some renaming $\sigma : V \rightarrow V$, for all $\Gamma \subseteq_f \{ \pm \bigotimes(p_1, \ldots, p_n) \mid \bigotimes n\text{-ary} \}$.

**Thm.** [P, 2003, Schröder 2007] One-step soundness and one-step completeness imply soundness and (cut-free) completeness, respectively, when augmented with propositional reasoning.
Organic Logics for Multisets

Proof Rules for $\mathcal{B}X = \{\mu : X \rightarrow \mathbb{N} \mid \text{supp}(f) \text{ finite}\}$

Modal Operators

$$\Lambda = \{L_p(c_1, \ldots, c_m) \mid n \in \mathbb{N}, p_1, \ldots, p_m \in \mathbb{Z}\}$$

Intended Meaning.

$$[L_p(c_1, \ldots, c_m)](S_1, \ldots, S_m) = \{\mu \in \mathcal{B}X \mid \sum_{j=1}^{m} c_j \cdot \mu(S_j) \geq p\}$$

Sound and Complete Proof Rules. (subject to arith. side condition)

$$\sum_{i=1}^{n} r_i \cdot \sum_{j=1}^{m_i} c_i^j a_i^j \geq 0$$

$$\{\text{sg}(r_i)L_{p_i}(c_i^1, \ldots, c_{m_i}^i)(a_i^1, \ldots, a_i^{m_i}) \mid i = 1, \ldots, n\}$$

- $\text{sg}(r)A = A$ if $r > 0$ and $\text{sg}(r)A = \neg A$ if $r < 0$

- premise reflects arithmetic of characteristic functions as propositional formula
Part IV: Automated Reasoning in Coalgebraic Logics

or:

How do I mechanise satisfiability?
Idea. Formulas $\phi \leftrightarrow$ Automata $A_\phi$ so that

$$(c, C) \models \phi \iff A \text{ accepts } (c, C)$$

where $C = (C, \gamma)$ is a $T$-coalgebra and $c \in C$.

Satisfiability checking via automata: $\phi$ satisfiable $\iff L(A_\phi) \neq \emptyset$

Coalgebra Automata are tuples $A = (A, a_I, \Delta, \Omega)$ where

- $A$ is a finite set of states and $a_I \in A$ is initial
- $\Delta : A \rightarrow \mathcal{P}(\mathcal{P}(TA))$ is the transition function
- $\Omega : A \rightarrow \mathbb{N}$ is a parity function

(we think of these automata as *alternating* due to layering of $\mathcal{P}$)
Acceptance via Parity Games

**Given.** $A = (A, a_i, \Delta, \Omega)$ and state $c$ of $T$-coalgebra $(C, \gamma)$.

**Acceptance.** $A$ accepts $c$ if $\exists$ has a winning strategy from $(a_I, c)$ on the board

$$B = (A \times C) \cup (TA \times TC) \cup (PTA \times C) \cup P(A \times C)$$

where legal moves are as follows:

<table>
<thead>
<tr>
<th>Position</th>
<th>Player</th>
<th>Moves</th>
<th>Priority</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(a, c) \in A \times C$</td>
<td>$\exists$</td>
<td>${ (\Xi, c) \in P(TA) \times C \mid \Xi \in \Delta(a) } \cup \Omega(a)$</td>
<td></td>
</tr>
<tr>
<td>$(\Xi, c) \in PT(A) \times C$</td>
<td>$\forall$</td>
<td>${ (\xi, \tau) \in TA \times TC \mid \xi \in \Xi, \tau = \gamma(c) } \cup 0$</td>
<td></td>
</tr>
<tr>
<td>$(\xi, \tau) \in TA \times TC$</td>
<td>$\exists$</td>
<td>${ Z \in P(A \times C) \mid (\xi, \tau) \in T(Z) } \cup 0$</td>
<td></td>
</tr>
<tr>
<td>$Z \in P(A \times C)$</td>
<td>$\forall$</td>
<td>$Z$</td>
<td></td>
</tr>
</tbody>
</table>

**Intuition.** (Recall $\Delta : A \rightarrow \mathcal{P}(PTA)$)

- $\Delta(a) \sim$ formula in DNF: \(\exists\) chooses disjunct, \(\forall\) chooses element
- ’modal’ steps lift acceptance relation and attract priorities
Automata and Fixpoint Logic

Modal Language. Positive Logic + $\nabla$ + fixpoint formulas

$$\mu L ::= x \mid \top \mid \bot \mid \phi \land \psi \mid \phi \lor \psi \mid \nabla \alpha \mid \mu x.\phi \mid \nu x.\phi$$

where $\alpha \in T_\omega L$ and $x \in V$ is a variable.

Semantics. As before, with $\mu/\nu$ interpreted as least/greatest fixpoints.

Thm. [Venema, 2008] For every $\phi \in \mu L$ there exists $A_\phi$ such that

$A_\phi$ accepts $(c, C') \iff c \models \phi$

and vice versa. That is: Automata are Formulas are Automata.

Intuition.

- loops in the automaton $\sim$ unfolding of fixpoints
- parity condition: only finite unfoldings of least fixpoints
Organic: Tableau Calculi

Here. Easier to use Tableaux than Sequent Calculi

Formulas.

\[ L \ni A, B ::= p \mid \overline{p} \mid A \land B \mid A \lor B \mid \bigotimes(A_1, \ldots, A_n) \mid \eta p. A \]

where \(\bigotimes\) is \(n\)-ary and \(\eta \in \{\mu, \nu\}\)

Tableau Sequents. Finite sets of formulas \(\Gamma = \{A_1, \ldots, A_n\}\) read conjunctively

Tableau Rules. As before, with modal rules dualised

\[
\begin{align*}
\Gamma; A \land B & \quad \Rightarrow \quad \Gamma; A; B \\
\Gamma; A \lor B & \quad \Rightarrow \quad \Gamma; A \quad \Gamma; B \\
\Gamma; \eta p. A & \quad \Rightarrow \quad \Gamma; A[p := \eta p. A] \\
\Gamma_0 \sigma, \Delta & \quad \Rightarrow \quad \Gamma_1 \sigma \ldots \Gamma_n \sigma \\
\Gamma, A, \overline{A} & \quad \Rightarrow \quad \Gamma
\end{align*}
\]

Remarks.

- Expansion only ever creates finitely many formulas
- No distinction between least and greatest fixpoints

May 26, 2010
As before. Two-Player Parity Games

- every board position \( b \) has a priority \( \Omega(b) \)
- \( \exists \) wins (and \( \forall \) looses) a play if largest infinitely occurring priority is even
- unfolding of \textit{least fixpoints} gives \textit{odd} priorities

**Model Checking Game**
- modal satisfiability game
- played on state/formula pairs
- unfolding of fixpoints

**Tableaux Game**
- played on sequents and rules
- \( \forall \) chooses rule
- \( \exists \) chooses conclusion

**Thm.** [Cîrstea, Kupke, P 2009] A formula is satisfiable if it has a closed tableau.
Part V: Other Aspects of Coalgebraic Logics

or:

What is there that I didn’t comment on?
Other Aspects

Coalgebraic Logics, Categorically.

- Logics via Adjunctions
  [Klin, Kurz, Jacobs, Sokolova]

- Logics via Presentations
  [Bonsangue, Kurz]

Compositionality

- Logics for Composite Functors
  [Cîrstea, P, Schröder]

Proof Theory.

- Sequents for $\nabla$
  [Bílková, Palmigiano, Venema]

- Interpolation [P, Schröder]

Synthetic vs Organic.

- back and forth [Leal]

Complexity.

- via Tableaux
  [Cîrstea, Kupke, Schröder, P]

Extensions of Set-based logics.

- Hybridisation
  [Myers, Kupke, P, Schröder]

- Global Consequence
  [Goré, Kupke, P]

- Path-Based Logics [Cîrstea]
Part VI: Perspectives

or:

What should we think about in the future?
Some Biased Food for Thought

Coalgebraic Logics are Feature-Rich, Compositional and Decidable

Strategic.

• **Implement:** Demonstrate techniques on non-trivial problems

• **Apply:** Use coalgebraic logics in modelling and verification

Technical.

• **Understand:** relationship between Tableaux and Automata

• **Deepen:** (Automated) reasoning with frame conditions

Conceptual.

• **Generalise:** How about e.g. MV-algebras modelling uncertainty?

• **Learn:** Adapt ILP Techniques to enable machine learning
Last Part: Questions

and:

Thanks for your attention!