SOS and Modal Logic
Revisited

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CMCS, Paphos, 26/03/10

SOS and distributive laws

$$t ::= \texttt{nil} \mid \texttt{a} \mid t \otimes t$$

$$\frac{}{\texttt{a} \xrightarrow{a} \texttt{nil}} \qquad \frac{x \xrightarrow{a} y \quad x' \xrightarrow{a} y'}{x \otimes x' \xrightarrow{a} y \otimes y'}$$

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$$t ::= \texttt{nil} \mid \texttt{a} \mid t \otimes t \qquad\qquad \Sigma X = 1 + A + X^2$$

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$$\lambda_X(\mathtt{nil})(a) = \emptyset$$

$$\lambda_X(\mathtt{a})(b) = \begin{cases} \{\mathtt{nil}\} & \Leftarrow b = a \\ \emptyset & \Leftarrow b \neq a \end{cases}$$

$$\lambda_X(\beta \otimes \beta')(a) = \{y \otimes y' \mid y \in \beta(a),\ y' \in \beta'(a)\}$$

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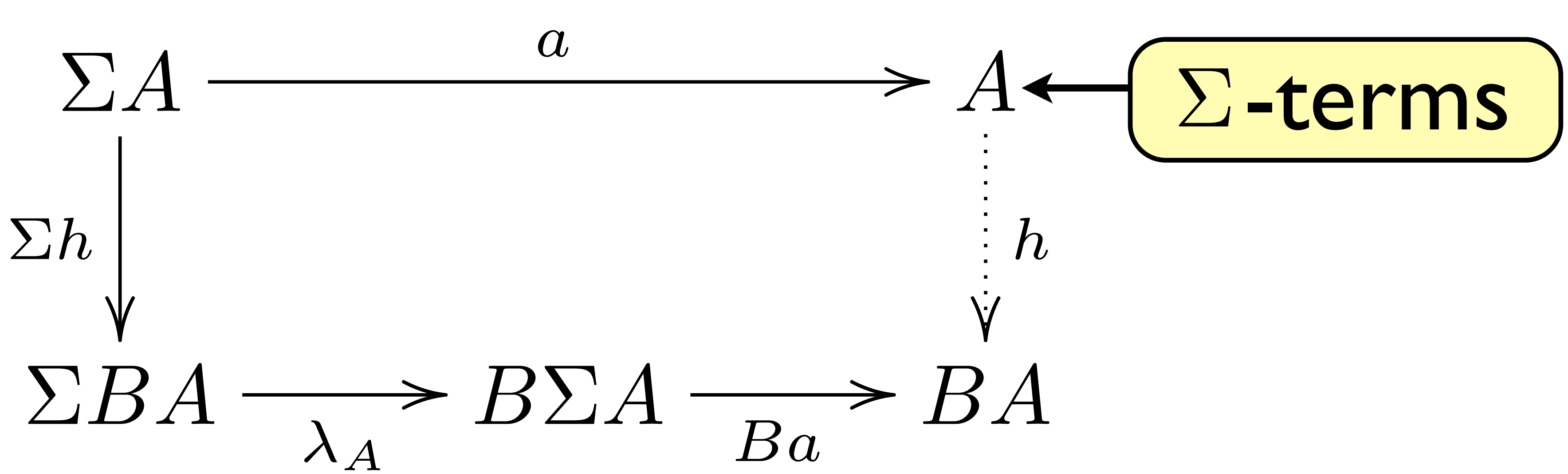

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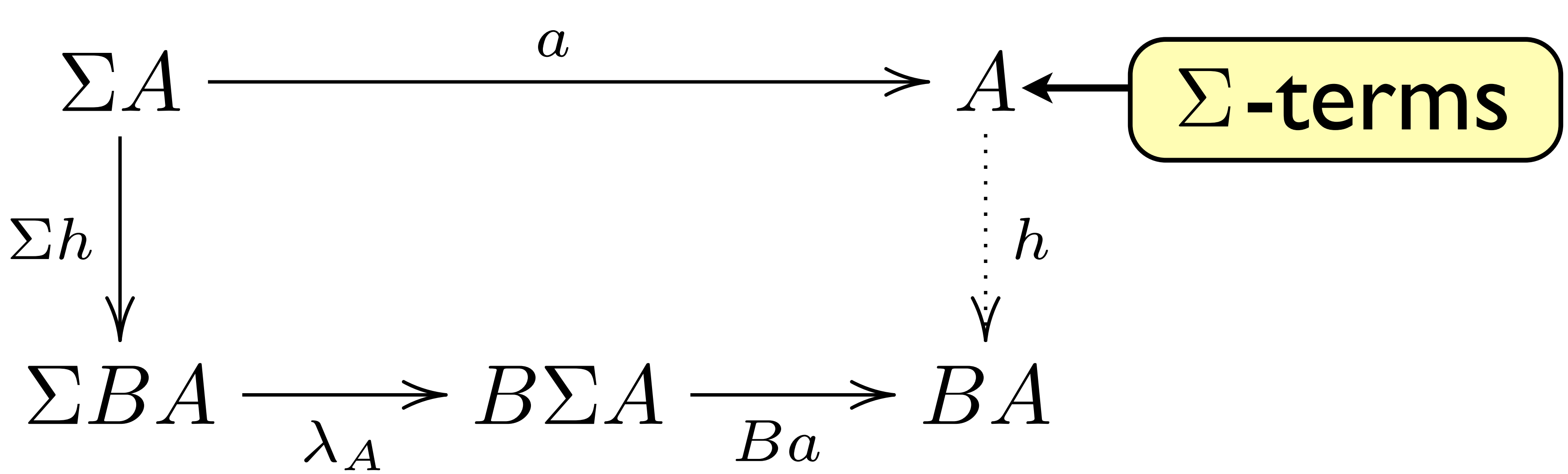

Fact: Bisimilarity is a congruence (for any λ)

Predicate liftings $BX = (\mathcal{P}_\omega X)^A$

Modal logic: $\phi ::= \top \mid \langle a \rangle \phi$

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$x \models \top$ always

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$$\top : B1 \to 2 \qquad \top(\beta) = \mathtt{tt}$$

$$\langle a \rangle : B2 \to 2 \qquad \langle a \rangle(\beta) = \mathtt{tt} \iff \mathtt{tt} \in \beta(a)$$

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$$\langle a \rangle(-_1 \wedge \cdots \wedge -_n) : B2^n \to 2$$

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etc.

Construction of liftings

Variable renaming:

$$\frac{\beta : B2^n \to 2 \qquad f : n \to m}{\beta|_f : B2^m \to 2}$$

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Composition:

$$\frac{\beta : B2^n \to 2 \qquad \left(\sigma_i : \Sigma 2^{m_i} \to 2\right)_{i=1,\ldots,n}}{\beta(\sigma_1, \ldots, \sigma_n) : B\Sigma 2^m \to 2}$$

$$m = \sum_{i=1}^{m} m_i$$

Example

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Valid equations

$$\beta(\sigma_1(x_{11}, \ldots, x_{1n_1}), \ldots, \sigma_m(x_{m1}, \ldots, x_{mn_m}))$$
$$=$$
$$\sigma(\beta_1(y_{11}, \ldots, y_{1k_1}), \ldots, \beta_l(y_{l1}, \ldots, y_{lk_l}))$$

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- RHS+$\uparrow$ define a ΣB-lifting of the same arity.

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The equation is valid

if the two $B\Sigma$-liftings are equal.

Compositionality

Thm. For a family $(\beta_i)_{i\in I}$ of B-liftings,

- find a family $(\sigma_j)_{j\in J}$ of Σ-liftings,
- for every possible LHS:

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s.t. the equation is valid wrt λ.

Then the logical equivalence defined by $(\beta_i)_{i\in I}$ is a congruence on the coalgebra induced by λ.

Example

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$$\phi ::= \top \mid \langle a \rangle \phi$$

B-liftings:

$$\top : B1 \to 2$$

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Σ-liftings:

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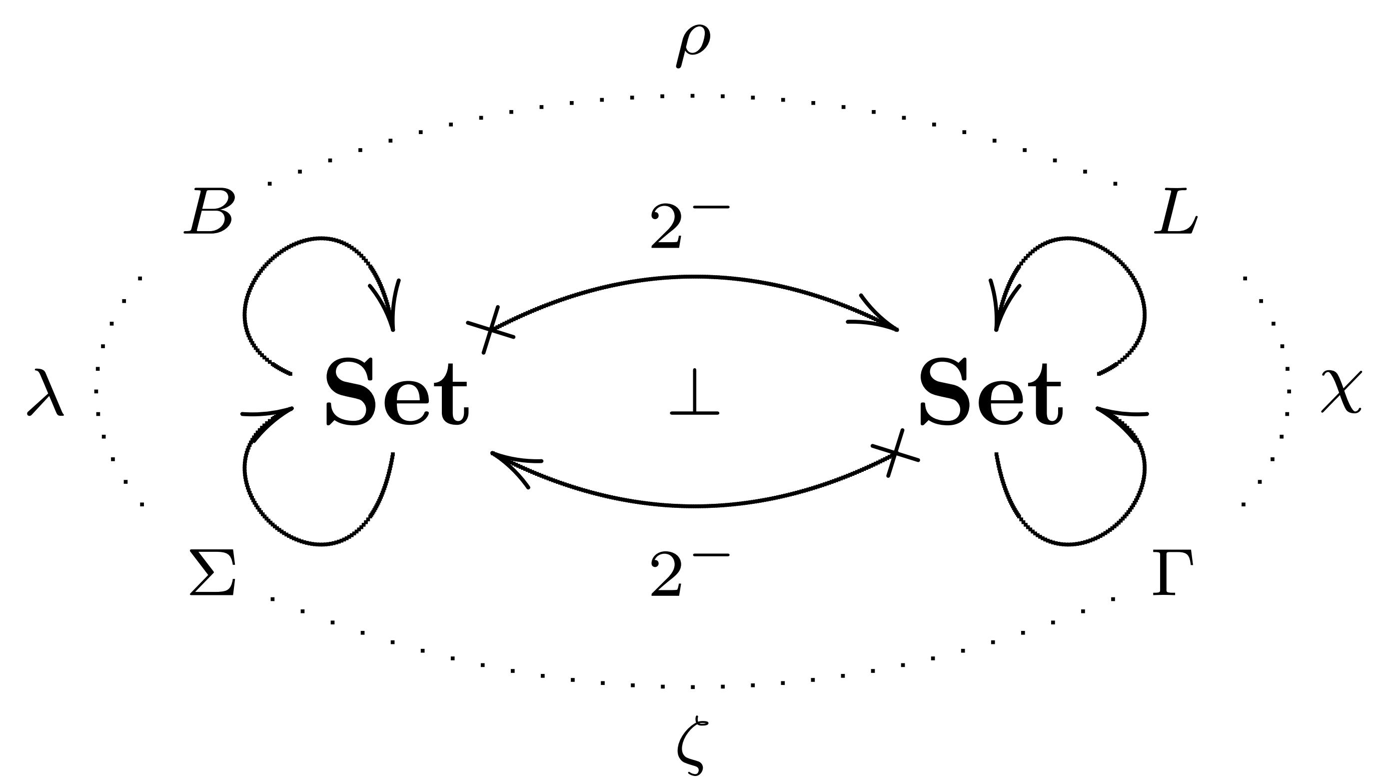
$$\langle a \rangle [\otimes] x = [\otimes] \langle a \rangle x$$

Gobbledygook



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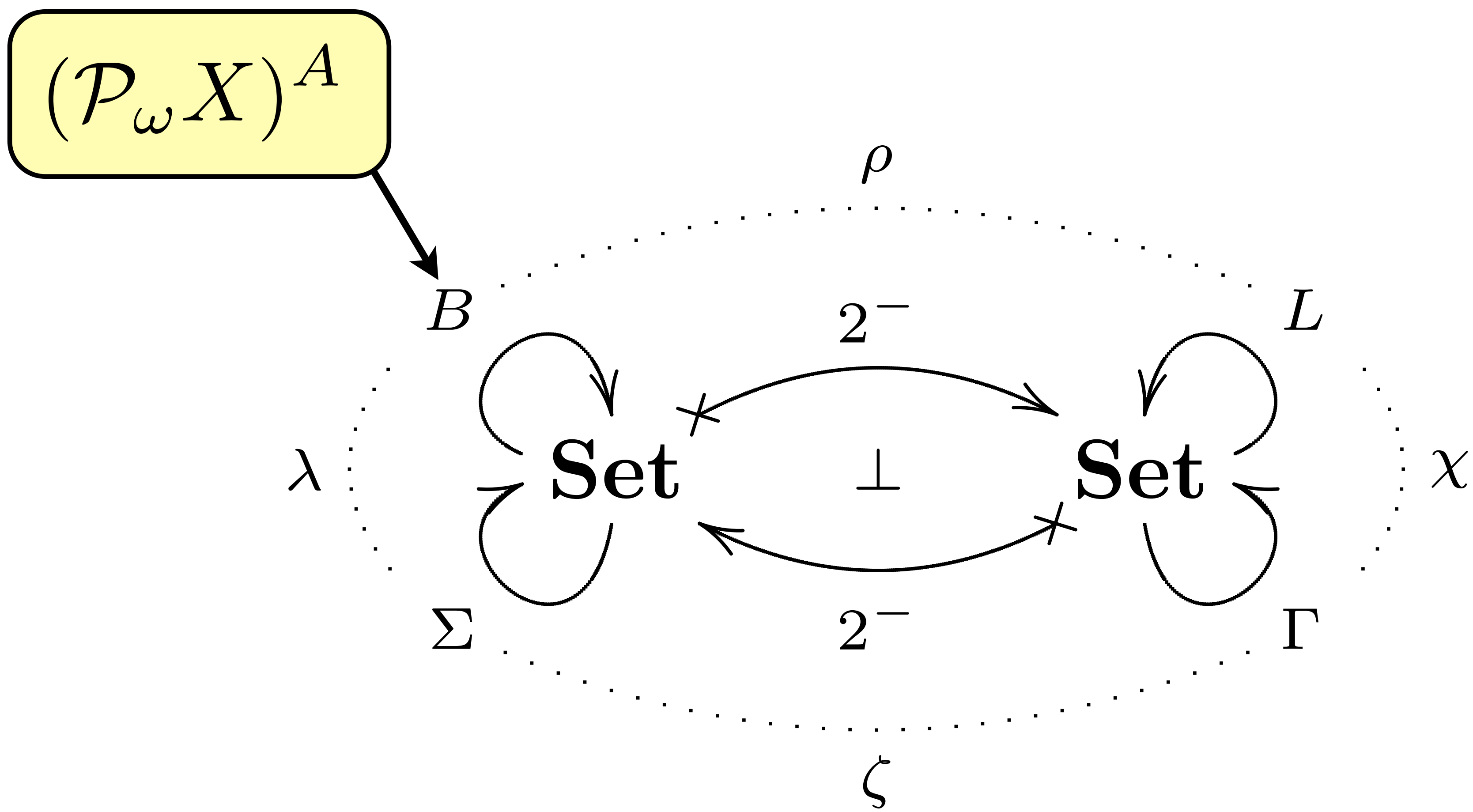

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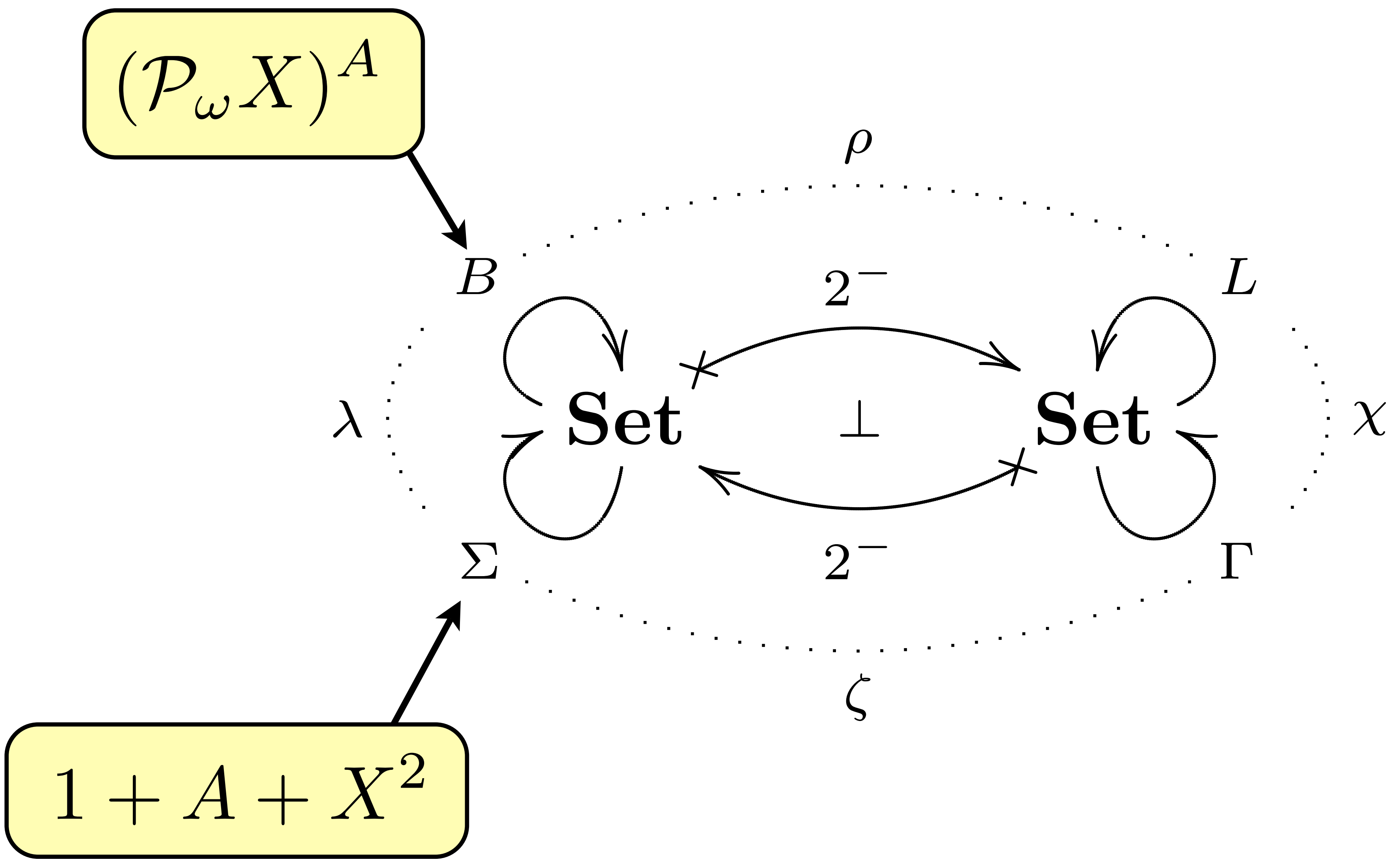

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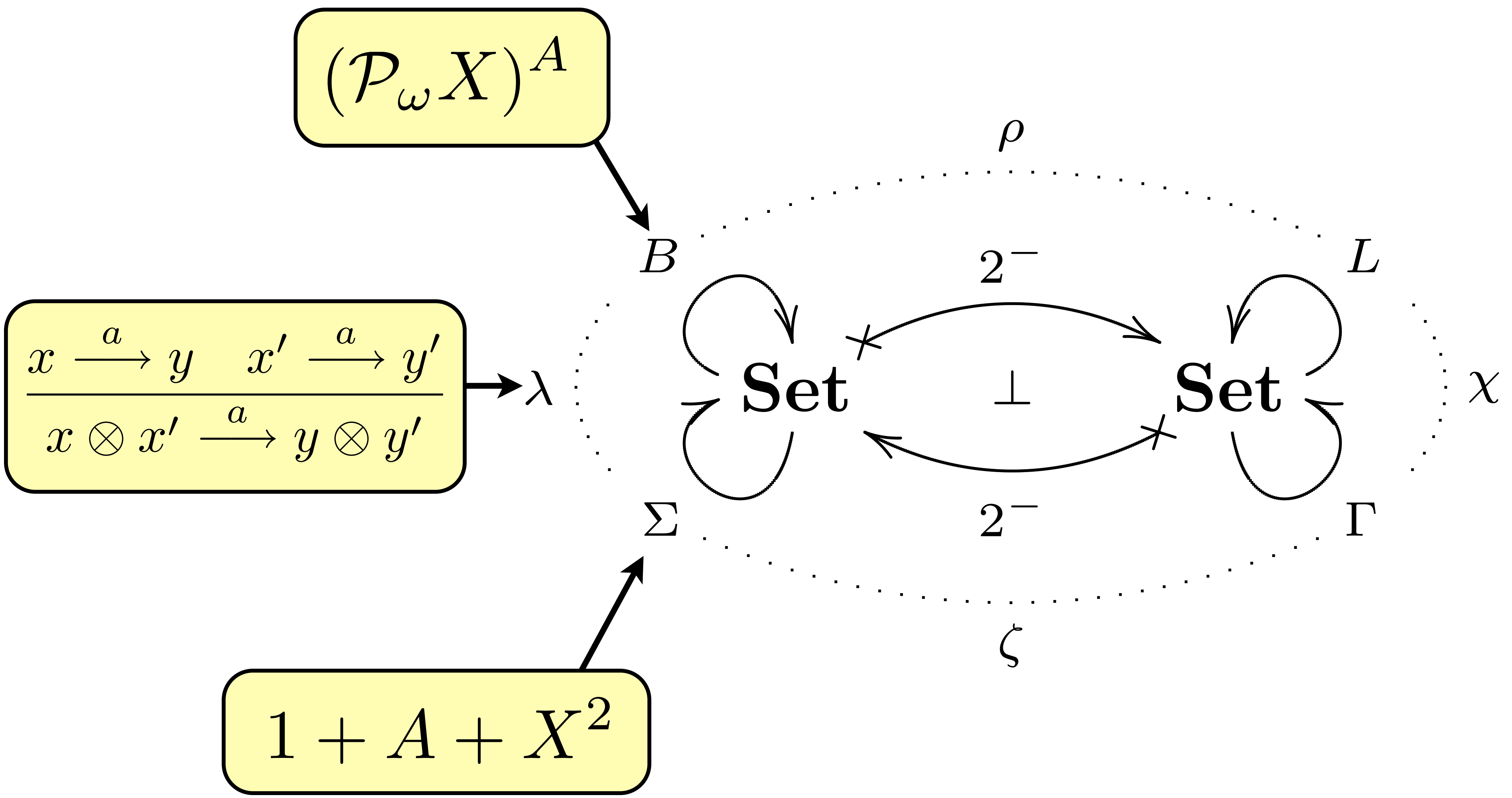

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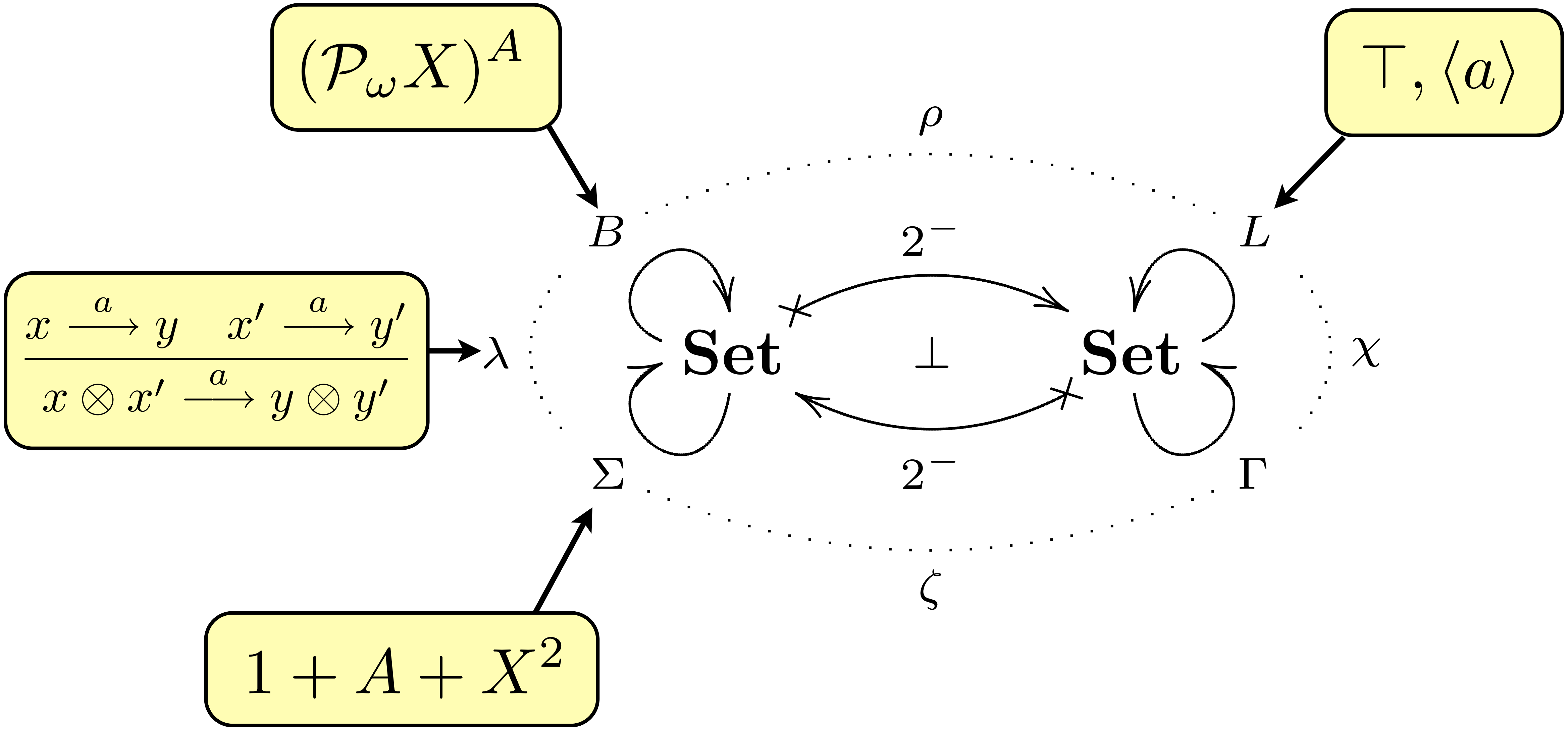

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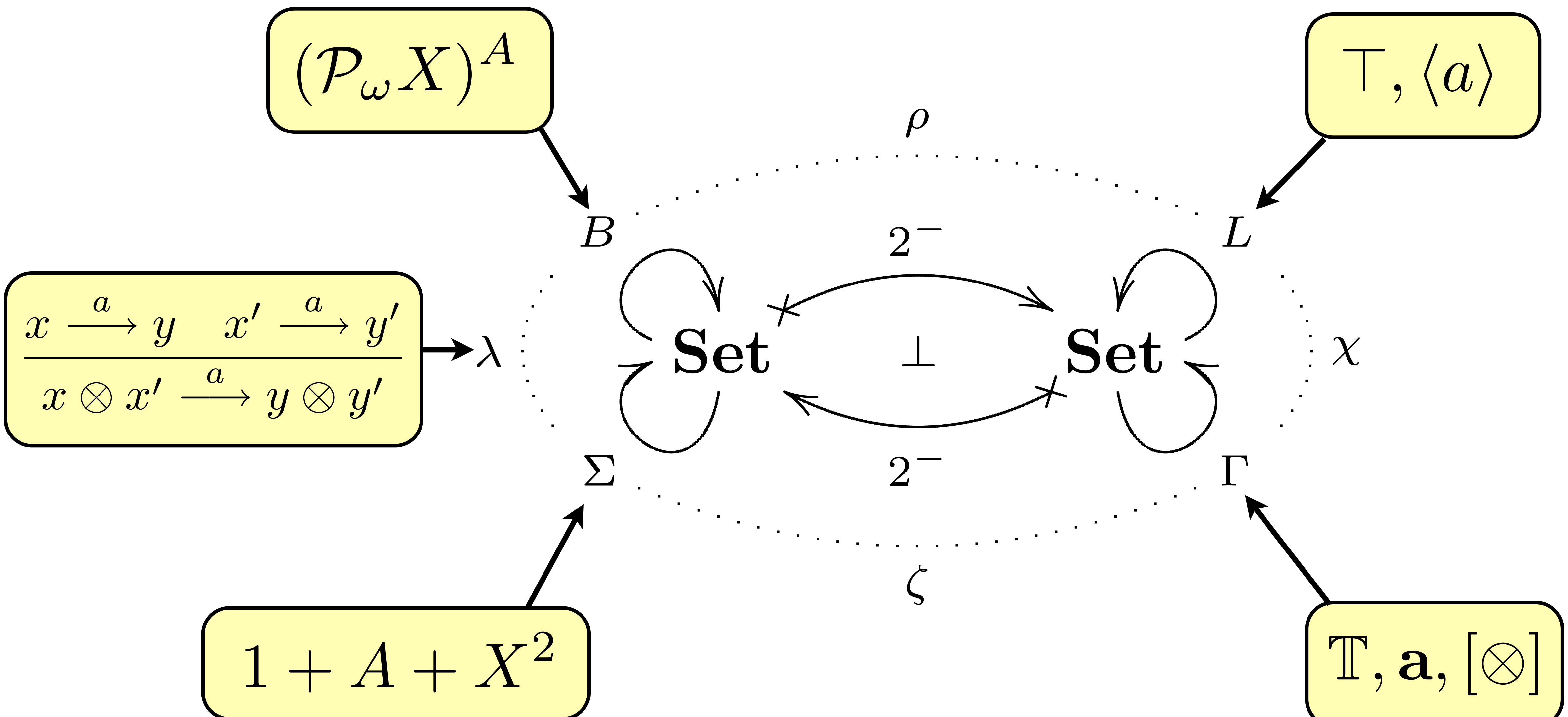

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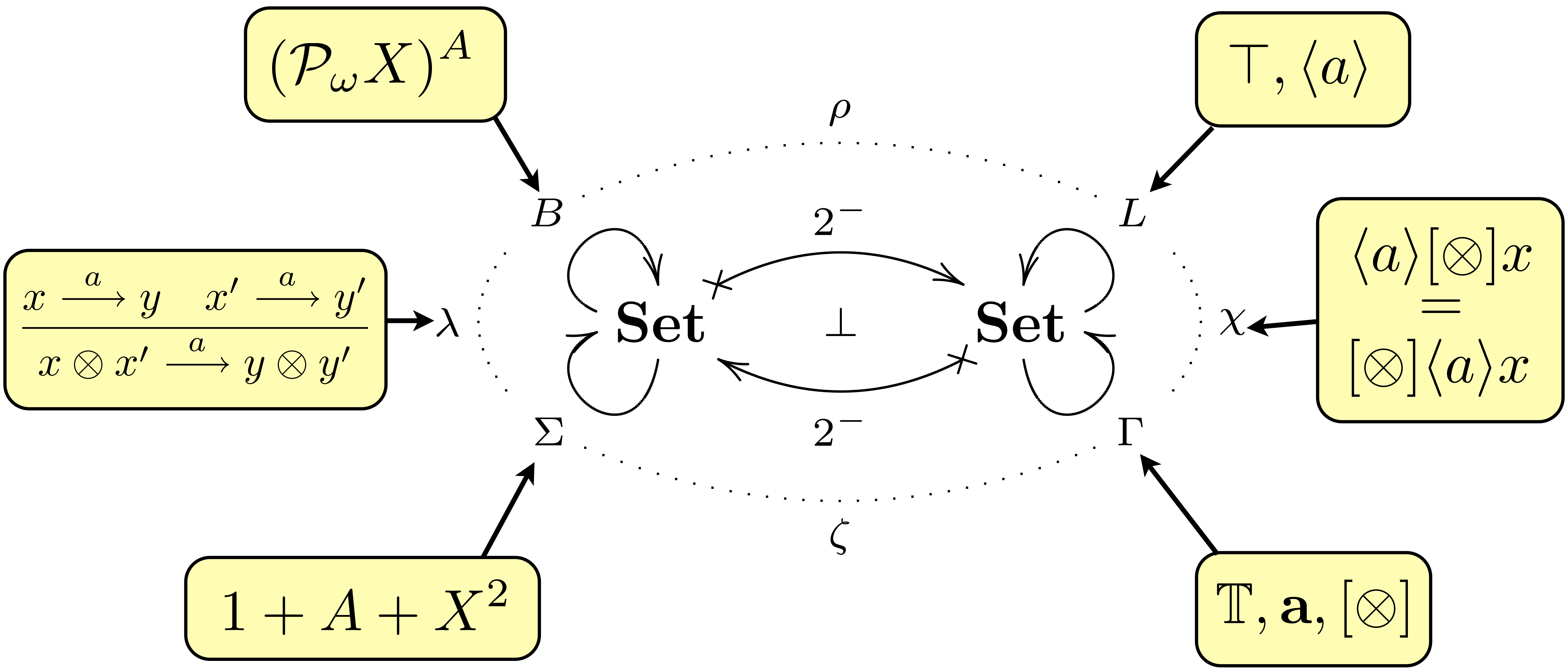

Future work

- Guidelines for choosing syntactic liftings/equations

- Study complex, GSOS-like equations