Outline

1. Monoidal Traces
2. Coalgebraic traces and iteration
3. Additive and semi-additive monads
4. Results

From Coalgebraic to Monoidal Traces

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Traced monoidal categories

Feedback operator in monoidal category:

\[ X \otimes U \xrightarrow{f} Y \otimes U \]

Satisfying a bunch of requirements.

- Introduced by Joyal-Street-Verity, 1996
- Intuitive string notation possible

Informal distinction

- For “multiplicative” tensor \( \otimes \), wave-style trace
- For “additive” coproduct/biproduct \( \oplus \), particle-style trace

Here we use the latter, in Kleisli categories of monads.

Traces, pictorially

A map \( X \otimes U \xrightarrow{f} Y \otimes U \) becomes:

And its trace \( X \xrightarrow{\text{Tr}(f)} Y \) is:

Standard examples

Usual summation trace in finite-dimensional vector spaces:

\[ \text{Tr}(X \otimes U \xrightarrow{M} Y \otimes U)_{x,y} = \sum_y M_{x\otimes u,y\otimes u} \]

Fixed point in \( \text{Cpos} \) with bottom:

\[ \text{Tr}(X \times U \xrightarrow{f} Y \times U)(x) = \pi_1(\text{fix}(\lambda(y,u). f(x,u))) \]

Iteration in pointed sets / sets with partial functions:

\[ \text{Tr}(X + U \xrightarrow{f} Y + U)(x) = \begin{cases} y & \text{if } \exists u_1, \ldots, u_n \in U. f(x) = u_1 \ldots f(u_i) = u_{i+1} \ldots f(u_n) = y \\ \bot & \text{otherwise} \end{cases} \]

The latter will be generalised to Kleisli categories.
Monoidal T races
Coalgebraic traces and iteration
Additive and semi-additive monads
Results

Basic construction

JSV ’96: for a traced monoidal category \( C \), construct compact closed category \( \text{Int}(C) \) with:

- **Obj**: \( (X^+, X^-) \in C \times C \)
- **Mor**: \( (X^+, X^-) \xrightarrow{f} (Y^+, Y^-) \) are \( X^+ \otimes Y^- \xrightarrow{f} Y^+ \otimes X^- \) in \( C \).

- Composition via trace / feedback
- Applications in linear logic: game semantics / geometry of interaction.

Basic result [HJS’07]

Assuming:
- a monad \( T : C \to C \) whose Kleisli category \( K\ell(T) \) is \( \text{CPO}_\bot \)-enriched
- a functor \( F : C \to C \) with a distributive law \( FT \Rightarrow TF \), giving a lifting \( F \) on \( K\ell(T) \)
- some minor details ...

Initial algebra \( F(A) \Rightarrow A \in C \) forms final coalgebra in \( K\ell(T) \)

Concretely:

\[
\begin{align*}
X & \xrightarrow{c} T(F(X)) \\
X & \xrightarrow{\text{tr}(c)} T(A)
\end{align*}
\]

Question

Is there a connection between coalgebraic and monoidal traces?

Or is this just a coincidence of names?

Towards a monoidal trace

For \( f : X + U \to Y + U \) in \( K\ell(T) \), by finality:

\[
\begin{align*}
Y + (X + U) & \xrightarrow{id + \text{tr}(f)} Y + N \cdot Y \\
\hat{f} & = (id + K\ell) \circ f
\end{align*}
\]

Intuition:

\( \text{tr}(c)(x) = (n, y) \iff \{ y \text{ is reached after } n \text{ iterations of } c, \text{ starting in } x \} \)

Post-composition with codiagonal \( N \cdot Y \xrightarrow{\nabla} Y \) ignores number of iterations, and yields an iteration operation:

\[ K\ell(T)(X, Y + X) \xrightarrow{(-)^\#} K\ell(T)(X, Y) \]


In this CMCS paper: call monad $T$ **partially additive** if $bc$ is cartesian natural transformation:
- each map $bc$: $T(X + Y) → T(X) × T(Y)$ is mono
- naturality squares are pullbacks

**EXAMPLES** Powerset, lift, distribution

For $f: X → T(Y)$ say $1_{\mathcal{H} \delta} f = \nabla \circ b: X → T(Y)$ exists if there is a "bound" $b: X → T(I \cdot Y)$ exists with $p_i \circ b = f_i$.

This makes the Kleisli category partially additive.

Let $T$ be a semi-additive monad whose Kleisli category $\mathcal{K}(T)$ is CPO enriched, on a category $\mathcal{C}$ with (countable) coproducts, then:
- $\mathcal{K}(T)$ is traced monoidal (via $+$, particle style)
- the monoidal trace $\text{Tr}(f)$ can be described via the coalgebraic trace.

**LEMMA** For a monad $T: \mathcal{C} → \mathcal{C}$ the following are equivalent.
- $T(0)$ is final, i.e. $T(0) \cong 1$
- $0 ∈ \mathcal{K}(T)$ is final (and hence zero-object)
- $1 ∈ \text{Alg}(T)$ is initial (and hence zero-object)

**THEOREM** [Coumans & Jacobs 2010]
The following are equivalent
- These $bc$: $T(X + Y) → T(X) × T(Y)$ are isomorphism
- $+$ in $\mathcal{K}(T)$ is product (and hence biproduct)
- $×$ in $\text{Alg}(T)$ is coproduct (and hence biproduct)

In this case we call the monad $T$ **additive**.

**Partial additivity**

Assume $C$ has finite products $(1, \times)$ and coproducts $(0, +)$.

**Main result**

Let $L$ be an additive monad whose Kleisli category $\mathcal{K}(L)$ is CPO enriched, on a category $\mathcal{C}$ with (countable) coproducts, then:
- $\mathcal{K}(L)$ is traced monoidal (via $+$, particle style)
- the monoidal trace $\text{Tr}(f)$ can be described via the coalgebraic trace.

**Additivity I**

Assume $T(0)$ is final, and form “zero-map” in $\mathcal{K}(T)$,

$$0 = \left( X \xrightarrow{1} T(0) \xrightarrow{\mu} T(Y) \right)$$

Form “projections” for $+$ in $\mathcal{K}(T)$,

$$p_1 = \left( X + Y \xrightarrow{[\eta, 0]} T(X) \right) \quad p_2 = \left( X + Y \xrightarrow{[0, \eta]} T(Y) \right)$$

and then a special map, connecting coproducts and products:

$$bc = \left( T(X + Y) \xrightarrow{\mu \circ p_1, \mu \circ p_2} T(X) × T(Y) \right)$$

**Additivity II**

**THEOREM** [Coumans & Jacobs 2010]
The following are equivalent
- These $bc$: $T(X + Y) → T(X) × T(Y)$ are isomorphism
- $+$ in $\mathcal{K}(T)$ is product (and hence biproduct)
- $×$ in $\text{Alg}(T)$ is coproduct (and hence biproduct)

In this case we call the monad $T$ **additive**.

**EXAMPLES**
- $T$ is powerset: explains why categories of relations (Kleisli) and complete lattices (algebras) have biproducts
- $T$ is multiset monad: explains why commutative monoids/groups and modules/vector spaces have biproducts.

**Coalgebraic trace definition** $\text{Tr}(f)$ from previous slide

- Proof of trace properties: steady progress ...
- ... failure with "vanishing II": trace over tensor is double trace

closer inspection showed:
- special property of coproducts in $\mathcal{K}(T)$ was needed
- PhD thesis (Ottawa, 2000) of Estândio Haghverdi covers this: every partially additive category is traced monoidal

Hence: partial additivity in $\mathcal{K}(T)$ is needed (and trace is derived from this partial additivity)

**How this work progressed . . .**

- Partial additivity
- In this CMCS paper: call monad $T$ partially additive if $bc$ is cartesian natural transformation:
  - each map $bc$: $T(X + Y) → T(X) × T(Y)$ is mono
  - naturality squares are pullbacks
- **EXAMPLES** Powerset, lift, distribution

For $f: X → T(Y)$ say $1_{\mathcal{H} \delta} f = \nabla \circ b: X → T(Y)$ exists if there is a “bound” $b: X → T(I \cdot Y)$ exists with $p_i \circ b = f_i$.

This makes the Kleisli category partially additive.
Applications

- Write $\mathcal{B}(\mathcal{T}) = \text{Int}(\mathcal{K}(\mathcal{T}))$ for the category of “bidirectional monadic computation”, with maps $X^+ + Y^- \rightarrow \mathcal{T}(Y^+ + X^-)$

  For lift monad: Abramsky’s category of games.

  Other examples, e.g. probabilistic games for distribution monad, require more study.

- Trace/feedback operator for coalgebraic components: new paper in preparation

Concluding remarks

- Coincidence of names not a coincidence!
- Basic properties of monads identified: (partial) additivity
- New class of examples of traced monoidal categories
- Follow-up work: bidirectional computation & traces for components.